FREQUENCY DEPENDENT CONTRACTION RATES FOR THE BAYESIAN METHOD TO THE INVERSE SOURCE PROBLEM

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Abstract. We consider an inverse source problem for the acoustic waves using a range of frequencies in this paper. The aims of this study are twofold. Firstly, it is known that this inverse problem is severely ill-posed with a logarithmic stability estimate. Nonetheless, the inverse problem becomes more stable when one increases the range of the frequencies. In this paper, we will view the logarithmic stability estimate and the increasing resolution/stability by carefully analyzing the singular values of the forward map. Secondly, in view of the behavior of the singular values, we then study the consistency theorem of the inverse problem by the nonparametric Bayesian method. Due to the ill-posedness, we can show that the posterior distribution contracts around the ground truth at a rate, which consists of two parts: a polynomial rate and a logarithmic rate, depending on the range of frequencies. This phenomenon also reflects the increasing resolution/stability property observed in the PDE inverse source problem. Moreover, such consistency theorems also suggest the amount of measurements taken in account of the experimental quality and the cost of the measurements.

Contents

Acknowledgments. Kow was partially supported by the National Science and Technology Council of Taiwan, NSTC 112-2115-M-004-004-MY3. Wang was partially supported by the National Science and Technology Council of Taiwan, NSTC 112-2115-M-002-010-MY3.

Date: April 1, 2025.

2020 *Mathematics Subject Classification.* Primary: 35P15, 35R25, 35R30, 62G20. Secondary: 35J05, 35J15, 62G05, 62G08.

Key words and phrases. Inverse source problem, Singular values, Bayesian method, Consistency theorem, Contraction rate.

1. Introduction

In this paper, we plan to study an inverse source problem using the nonparametric Bayesian approach. We begin with the description of the scattering problem. Let $n \geq 2$ be an integer and $\kappa > 0$ be a wave number. We consider the time-harmonic acoustic wave with a nontrivial volume source f, which is a distribution on $\mathbb{R}^n$ with a compact support $\mathrm{supp}(f)$:

$$\begin{cases} -(\Delta + \kappa^2)u = f & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} |x|^{\frac{n-1}{2}}(\partial_r u - \mathbf{i}\kappa u) = 0 & \text{uniformly for } \hat{x} = \dfrac{x}{|x|} \in \mathcal{S}^{n-1}. \end{cases} \tag{1.1}$$

The second equation in (1.1) is known as the *Sommerfeld radiation condition*. Without loss of generality, we assume that $\mathrm{supp}(f) \subseteq \overline{B}_1$. We define

$$\Phi_\kappa(x) := \frac{\mathbf{i}\kappa^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}}} |x|^{-\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\kappa|x|),$$

where $H^{(1)}$ is the Hankel function of the first kind. For each $y \in \mathbb{R}^n$, it is known that Φ_κ satisfies the above mentioned Sommerfeld radiation condition and

$$-(\Delta + \kappa^2)\Phi_\kappa = \delta_0 \text{ in distribution sense}$$

where δ_0 is the standard Dirac delta. In other words, Φ_k is the outgoing fundamental solution of the Helmholtz operator. Together with the Rellich uniqueness theorem [CK19, Hör73], we can see that u of (1.1) is written explicitly by

$$u(x) = \Phi_\kappa * f(x), \tag{1.2}$$

which is understood as the convolution of two distributions. Now its *far-field pattern* is defined by

$$u^\infty_\kappa[f](\hat{x}) := \lim_{|x|\to\infty} \gamma^{-1}_{n,\kappa} |x|^{\frac{n-1}{2}} e^{-\mathbf{i}\kappa|x|} u(x) \quad \text{with} \quad \gamma_{n,\kappa} := \frac{e^{-\mathbf{i}\pi\frac{n-3}{4}}}{2(2\pi)^{\frac{n-1}{2}}} \kappa^{\frac{n-3}{2}}.$$

Such choice of $\gamma_{n,k}$ guarantees that

$$\Phi^\infty_\kappa(\hat{x}, y) = e^{-\mathbf{i}\kappa\hat{x}\cdot y} \quad \text{for all } \hat{x} \in \mathcal{S}^{n-1} \text{ and } y \in \mathbb{R}^n, \tag{1.3}$$

see e.g. [KSS24, Section 2] or [Yaf10, Section 1.2.3], therefore u^∞_κ can be represented in terms of Fourier transform:

$$u^\infty_\kappa[f](\hat{x}) = \int_{\mathbb{R}^n} e^{-\mathbf{i}\kappa\hat{x}\cdot y} f(y)\,\mathrm{d}y = \int_{B_1} e^{-\mathbf{i}\kappa\hat{x}\cdot y} f(y)\,\mathrm{d}y = \mathscr{F}[f](\kappa\hat{x}) \quad \text{for all } \hat{x} \in \mathcal{S}^{n-1}. \tag{1.4}$$

We remark that $\mathscr{F}[f]$ is analytic since f is a distribution with compact support.

Here, we are interested in the problem of determining the source f (assumed to be unknown) by the far-field pattern $u^\infty_\kappa(\hat{x})$ for all $\hat{x} \in \mathcal{S}^{n-1}$. In view of the Rellich uniqueness theorem, the far-field measurement $u^\infty_\kappa(\hat{x})$ is equivalent to the near-field measurement $u(\cdot,\kappa)|_\Gamma$, where $\Gamma = \partial B_R$ with $R > 1$. Both types of measurements were considered in literature. Inverse source problems have enormous applications in practice. For example, detection of submarines and of anomalies in various industrial objects like material defects [EV09], [GS17a, GS17b] can be regarded as recovery of acoustic sources from boundary measurements of the pressure (see also [GS18] for similar results for the Maxwell and the elasticity equations). It is known that from the far-field or the near-field data for one single linear differential equation or system (that is, single wave number), it is not possible to find the source uniquely [Isa06, Chap 4]. This non-uniqueness is due to the existence of non-radiating sources [BC77].

It was shown in [EI20] that if one considers the measurements $u(\cdot,\kappa)|_\Gamma$ for all $\kappa \in (0,\mathbf{k})$ with some $\mathbf{k} > 0$, then the uniqueness is true, i.e., f can be uniquely determined by $u(\cdot,\kappa)|_\Gamma$ for all $\kappa \in (0,\mathbf{k})$. The same is true when we measure $u_\kappa^\infty(\hat{x})$ for all $\kappa \in (0,\mathbf{k})$. This is easily seen from the analyticity of $u_\kappa^\infty(\hat{x})$ in κ. On the other hand, a frequency-dependent stability estimate can be derived indicating the increasing resolution phenomena (also known as increasing stability phenomena) as $\mathbf{k}$ increases. For other inverse source problem using multiple frequencies, we refer to, for example, [ABF02, BLT10, BLZ20, CIL16, IL18, IW21, LY17] and references therein. Here, we will view the increasing resolution of identifying the source f by the far-field pattern $u_\kappa^\infty[f](\hat{x})$ for all $\kappa \in (0,\mathbf{k})$ and $\hat{x} \in \mathcal{S}^{n-1}$ as $\mathbf{k}$ increases based on the behavior of the singular values of the forward map.

In this work, we are also interested in the inference of f from $u_\kappa^\infty(\hat{x})$ for $\kappa \in (0,\mathbf{k})$ using the Bayesian approach. We now describe the statistical inference problem in a general form. Consider the observation $Y = Y^{(N)}$ given by Af polluted by a Gaussian white noise with noise level $1/\sqrt{N}$, that is,

$$Y = Af + \frac{1}{\sqrt{N}}W, \tag{1.5}$$

where A is an injective, continuous linear map from a separable Hilbert $\mathbb{H}_1$ to another separable Hilbert space $\mathbb{H}_2$. The Gaussian white noise W is defined as a centered Gaussian process indexed by $\mathbb{H}_2$, that is, $W = (W_h : h \in \mathbb{H}_2)$ is mean-zero Gaussian process with covariance $\mathbb{E}W_hW_{h'} = \langle h, h'\rangle_{\mathbb{H}_2}$. In the inverse source problem considered here, $A = G_{\mathbf{k}} : L^2(B_1) \mapsto L^2(\mathcal{S}_{\mathbf{k}}^{n-1}, \mathrm{d}\kappa\mathrm{d}\hat{x})$ defined by

$$\|G_{\mathbf{k}}f\|^2_{L^2(\mathcal{S}_{\mathbf{k}}^{n-1},\mathrm{d}\kappa\,\mathrm{d}\hat{x})} = \int_0^{\mathbf{k}}\int_{\mathcal{S}^{n-1}} |u_\kappa^\infty[f](\kappa\hat{x})|^2\,\mathrm{d}\hat{x}\,\mathrm{d}\kappa,$$

where $\mathcal{S}_{\mathbf{k}}^{n-1} := (0,\mathbf{k}) \times \mathcal{S}^{n-1}$. The above discussions suggest us to choose $\mathbb{H}_1 = L^2(B_1)$ and $\mathbb{H}_2 = L^2(\mathcal{S}_{\mathbf{k}}^{n-1}, \mathrm{d}\kappa\,\mathrm{d}\hat{x})$.

In the Bayesian inference, we assume that f is a random variable and we assign a prior distribution Π to it. Here Π is a probability distribution defined on the Borel algebra $\mathcal{B}$ of $\mathbb{H}_1$. Let $\Pi_N(\cdot|Y^{(N)})$ be the resulting posterior distribution. The Bayes method is to make inferences of f based on $\Pi_N(\cdot|Y^{(N)})$. Our first aim is to investigate the asymptotic behavior of $\Pi_N(\cdot|Y^{(N)})$ as $N \to \infty$ from the frequentist viewpoint, that is, the data $Y^{(N)}$ is generated by a true parameter $f_0 \in \mathbb{H}_1$. One important question is to study the consistency of $\Pi_N(\cdot|Y^{(N)})$ around f_0 with an explicit contraction rate. In the case of the inverse source problem, we are able to analyze the singular values of the linear map $G_{\mathbf{k}}$. Hence, we will use the singular value decomposition (SVD) approach to prove the consistency of $\Pi_N(\cdot|Y^{(N)})$. For other nonparametric Bayesian inference for linear inverse problems by the SVD, we refer to [Cav08, CGPT02, GP00, KvdVvZ11, Ray13], just to list a few. There is a vast of literature investigating the consistency of the nonparametric Bayes method for both linear and non-linear inverse problems. We refer the reader to [AN19, ALS13, ASZ14, FKW24, GGvdV00, GN20, GvdVY20, Kek22, KSvdVvZ16, MNP19, MNP21, Vol13] and references therein. Furthermore, two nice monographs [GvdV17, Nic23] contain more exhaustive references.

The derivation of contraction rates of $\Pi_N(\cdot|Y^{(N)})$ around f_0 as $N \to \infty$ relies on a testing approach outlined in [GGvdV00, Section 7]. The main idea of this approach is to construct suitable tests for the problem

$$H_0 : f = f_0, \quad H_A : f \in \{f : \|f - f_0\|_{\mathbb{H}_1} \geq \xi_N\} \tag{1.6}$$

with exponentially decaying type-II error for some sequence $\xi_N \to 0$. Following the method as in [AN19, GN11, Ray13], we will use concentration properties of certain centered linear estimators to construct suitable plug-in tests. For the inverse source problem, relying on the singular values of $G_{\mathbf{k}}$, see Section 2, we can construct linear estimators using the band-limited elements as in [LN11, Ray13].

Another key condition in the testing approach is that the prior distributions put sufficient mass near the ground truth f_0. This condition is achieved by establishing lower bounds of the "small-ball problem", i.e., the probability of Af contained in a small-ball centered at Af_0 under the prior. In view of the decaying behavior of singular values of A, a small ball in $\mathbb{H}_2$ is transformed by A^{-1} to a large $\mathbb{H}_1$ ellipsoid whose size is precisely determined by the singular values.

In this paper, we are not only interested in the consistency of the posterior distribution $\Pi_N(\cot|Y^{(N)})$, but also precisely deriving how the contraction rate depends on $\mathbf{k}$. Due to the behavior of the singular values, we can show that the posterior distribution contracts around the ground truth at a rate, which consists of two parts: a polynomial rate and a logarithmic rate, depending on the range of frequencies. This phenomenon also reflects the increasing resolution/stability property observed in the PDE inverse source problem. We believe that our results provide a better way of applying the Bayesian method for ill-posed inverse problems.

This paper is structured as follows. In Section 2, we outline the behaviors of singular values of $G_{\mathbf{k}}$. In Section 3, we will describe the Bayesian method for the inverse source problem and state main consistency theorems using sieve and Gaussian priors. The detailed derivation of the behaviors of singular values of $G_{\mathbf{k}}$ is given in Section 4. In Section 5, we state and prove a general quantitative consistency theorem, which is a refinement of [Ray13, Theorem 2.1] by taking the parameter $\mathbf{k}$ into consideration. The proofs of consistency theorems with $\mathbf{k}$-dependence contraction rates for sieve and Gaussian priors are given in Section 6 and 7, respectively.

Notations. We summarize some notations and function spaces used in this work. Throughout, we shall use the symbol $\lesssim$ and $\gtrsim$ for inequalities holding up to a universal constant. For two real sequences (a_N) and (b_N), we say that $\simeq$ if both $a_N \lesssim b_N$ and $b_N \lesssim a_N$ for all sufficiently large N. In this paper, the universal constants in $\lesssim$, $\gtrsim$ and $\simeq$ are referred as "implied constants", which are all all independent of $\mathbf{k}$ and N. For any $t \geq 0$, $\lceil t \rceil$ denotes the ceiling function t, i.e., $\lceil t \rceil$ is the smallest integer larger than t. All formulas in the Digital Library of Mathematical Functions (`DLMF`, https://dlmf.nist.gov/), which is maintained by the National Institute of Standards and Technology of U.S Department of Commerce (https://www.nist.gov/), can be also found in [OLBC10].

2. Singular values of $G_{\mathbf{k}}$

In this section, we would like to analyze the singular values of $G_{\mathbf{k}}$ as explicitly as possible. Our aim is to understand the decaying behavior of the singular values, accounting the effect of the parameter $\mathbf{k}$.

2.1. **Upper bounds of singular values and an instability result.** By the plane wave expansion DLMF:10.60.E7, we can write

$$e^{-\mathbf{i}\kappa\hat{x}\cdot y} = \sum_{\ell=0}^{\infty}(2\ell+1)\mathbf{i}^{\ell}\mathrm{j}_{\ell}(k|y|)P_{\ell}(\hat{x}\cdot\hat{y}) \quad \text{for all } \hat{x}\in\mathcal{S}^{n-1} \text{ and } y\in B_1\setminus\{0\},$$

where $\hat{y}=y/|y|$, j_ℓ are the spherical Bessel functions and P_ℓ are Legendre polynomials. Consequently, we can use the addition theorem for Legendre polynomials [EF14, Theorem 4.11] to derive the n-dimensional plane wave expansion[1]:

$$e^{-\mathbf{i}\kappa\hat{x}\cdot y} = \Omega_{n-1}\sum_{\ell=0}^{\infty}\frac{(2\ell+1)\mathbf{i}^{\ell}}{N(n,\ell)}\sum_{j=1}^{N(n,\ell)}\mathrm{j}_{\ell}(\kappa|y|)Y_{\ell,j}(\hat{y})Y_{\ell,j}(\hat{x})$$

where $\Omega_{n-1}=\frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of $\mathcal{S}^{n-1}$ [EF14, Proposition 2.3] and $\{Y_{\ell,j}(\hat{x})\}_{j=1}^{N(n,\ell)}$ is the orthonormal set of spherical harmonics with $N(n,\ell)=\frac{2\ell+n-2}{\ell}\binom{\ell+n-3}{\ell-1}$ [EF14, Theorem 4.4]. Here we also point out that

$$N(3,\ell)=2\ell+1 \quad \text{for all } \ell=0,1,2,\cdots. \tag{2.1}$$

Therefore, if $f\in L^2(B_1)$, then from (1.4), it follows that

$$u_\kappa^\infty[f](\hat{x}) = \Omega_{n-1}\sum_{\ell=0}^{\infty}\frac{(2\ell+1)\mathbf{i}^{\ell}}{N(n,\ell)}\sum_{j=1}^{N(n,\ell)}\left(\int_{B_1}\mathrm{j}_{\ell}(\kappa|y|)Y_{\ell,j}(\hat{y})f(y)\,\mathrm{d}y\right)Y_{\ell,j}(\hat{x}) \tag{2.2}$$

which converges in $L^2(\mathcal{S}^{n-1})$.

The inverse source problem aims to recover the source f, which is a distribution supported in B_1, from multifrequency data $\{u_\kappa^\infty[f](\hat{x}) : 0<\kappa<\mathbf{k}, \hat{x}\in\mathcal{S}^{n-1}\}$ for some $\mathbf{k}>0$. From (1.4) we see that such information gives us the value of $\mathscr{F}[f]$ in $B_{\mathbf{k}}$ on the frequency space. Since f is compactly supported, by the analyticity of $\mathscr{F}[f]$, the full data $\mathscr{F}[f]$ on $\mathbb{R}^n$ can be obtained, and finally the Fourier inverse formula recovers the source f. It is also interesting to point out that the set of $\{u_\kappa^\infty[f](\hat{x}) : 0<\kappa<\mathbf{k}, \hat{x}\in\mathcal{S}^{n-1}\}$ is actually identical to the linearized scattering matrix considered in [KSZ24, Lemma 3.1].

Now let us quantify the measurement mentioned above. By using (1.4), we can see that

$$\begin{aligned}\int_0^{\mathbf{k}}\|u_\kappa^\infty[f]\|_{L^2(\mathcal{S}^{n-1})}^2\,\mathrm{d}\kappa &= \int_0^{\mathbf{k}}\int_{|\hat{x}|=1}|\mathscr{F}[f](\kappa\hat{x})|^2\,\mathrm{d}\hat{x}\,\mathrm{d}\kappa \\ &= \int_0^{\mathbf{k}}\int_{|\kappa\hat{x}|=\kappa}\kappa^{1-n}|\mathscr{F}[f](\kappa\hat{x})|^2\,\mathrm{d}(\kappa\hat{x})\,\mathrm{d}\kappa = \int_{B_{\mathbf{k}}}|\xi|^{1-n}|\mathscr{F}[f](\xi)|^2\,\mathrm{d}\xi,\end{aligned} \tag{2.3}$$

which implies that $G_{\mathbf{k}}f$ is isometrically and isomorphically to the operator

$$U_{\mathbf{k}}f(x) = \mathscr{F}^{-1}\left[\chi_{B_{\mathbf{k}}}|\cdot|^{\frac{1-n}{2}}\mathscr{F}[f]\right] = \mathscr{F}^{-1}[\chi_{B_{\mathbf{k}}}]*(-\Delta)^{\frac{1-n}{4}}f \quad \text{in } \mathbb{R}^n, \tag{2.4}$$

where $(-\Delta)^{\frac{1-n}{4}}f = \mathscr{F}^{-1}\left[|\cdot|^{\frac{1-n}{2}}\mathscr{F}[f]\right]$ is the negative power of the Laplacian [Sti19]. In other words, we have

$$\|U_{\mathbf{k}}f\|_{L^2(\mathbb{R}^n)}^2 = \|G_{\mathbf{k}}f\|_{L^2(\mathcal{S}_{\mathbf{k}}^{n-1},\mathrm{d}\kappa\,\mathrm{d}\hat{x})}^2. \tag{2.5}$$

[1]The n-dimensional analogue for the Funk-Hecke formula [CK19, (2.45)] is an immediate consequence of this n-dimensional plane wave.

In the analysis of the singular values, it is more convenient to work with the operator $U_{\mathbf{k}}$. By the Fourier inversion formula, one has

$$\mathscr{F}^{-1}[\chi_{B_{\mathbf{k}}}](x) = (2\pi)^{-n}\mathbf{k}^n \mathscr{F}[\chi_{B_1}](-\mathbf{k}x) = (2\pi)^{-n}\mathbf{k}^n(\mathbf{k}|x|)^{-\frac{n}{2}} J_{n/2}(\mathbf{k}|x|),$$

where J_α is the Bessel function of first kind of order α. In view of (2.4), we rigorously define the linear operator $U_{\mathbf{k}}$ in the following setting:

$$U_{\mathbf{k}} : H^{(1-n)/2}_{\overline{B_1}} \to L^2(\mathbb{R}^n), \tag{2.6}$$

where for each compact set $K \subset \mathbb{R}^n$ and a parameter $s \in \mathbb{R}$, the Sobolev space H^s_K is defined by

$$H^s_K := \{ f \in H^s(\mathbb{R}^n) : \operatorname{supp}(f) \subseteq K \}, \quad \|f\|_{H^s_K} := \|f\chi_K\|_{H^s(\mathbb{R}^n)}.$$

In the following theorem, by the smoothing property of the forward operator and the Courant min-max/max-min principles, we can estimate singular values without using the explicit expression (2.2).

Theorem 2.1. *Let $n \geq 2$ and $\kappa \geq 1$. Then the bounded linear operator (2.6) is compact and injective. In addition, there exist positive constants $C = C(n) > 1$, $c = c(n)$ and $C' = C'(n) > 0$ such that the singular values $\sigma_j\left(U_{\mathbf{k}} : H^{(1-n)/2}_{\overline{B_1}} \to L^2(\mathbb{R}^n)\right)$ of (2.6) satisfy*

$$C^{-1} \leq \sigma_j\left(U_{\mathbf{k}} : H^{(1-n)/2}_{\overline{B_1}} \to L^2(\mathbb{R}^n)\right) \leq C\mathbf{k}^{\frac{n-1}{2}} \quad \text{for } j < C'\mathbf{k}^n, \tag{2.7a}$$

$$\sigma_j\left(U_{\mathbf{k}} : H^{(1-n)/2}_{\overline{B_1}} \to L^2(\mathbb{R}^n)\right) \leq Cj^{\frac{n-1}{2n}} \exp\left(-cj^{\frac{1}{2n}}\mathbf{k}^{-\frac{1}{2}}\right) \quad \text{for } j \geq C'\mathbf{k}^n. \tag{2.7b}$$

Here all the constants C, c, C' are independent of $\mathbf{k}$.

In view of the mechanism explained in [KRS21, KSZ24] (see also [Ray13] in probabilistic settings), the estimates (2.7a) and (2.7b) indicate that the singular values are almost constant (up to a polynomial multiplier $\mathbf{k}^{\frac{n-1}{2}}$) in the stable region $\{j \in \mathbb{N} : j \lesssim \mathbf{k}^n\}$ and begin decay rapidly in the unstable region $\{j \in \mathbb{N} : j \gtrsim \mathbf{k}^n\}$ which led instability in the inverse problem. It is clear that the stable region $\{j \in \mathbb{N} : j \lesssim \mathbf{k}^n\}$ increases and the unstable region $\{j \in \mathbb{N} : j \gtrsim \mathbf{k}^n\}$ decreases as $\mathbf{k} \to \infty$, which demonstrates the increasing resolution phenomena.

We can restrict $U_{\mathbf{k}}$ on $L^2(B_1)$ and thus

$$U^\infty_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^n) \tag{2.8}$$

is compact injective. Using the similar method, we can show that the singular values $\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^n)\right)$ satisfy

$$\begin{aligned} &\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^n)\right) \leq C && \text{for all } j < C'\mathbf{k}^n, \\ &\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^n)\right) \leq C\exp\left(-cj^{\frac{1}{2n}}\mathbf{k}^{-\frac{1}{2}}\right) && \text{for all } j \geq C'\mathbf{k}^n. \end{aligned} \tag{2.9}$$

Following the arguments of the proof in [KSZ24, Theorem 1.5], we can prove that

Corollary 2.2 (Instability of the inverse problem)**.** *If there exists a non-decreasing function $t \in (0,\infty) \mapsto \omega(t) \in (0,\infty)$ such that*

$$\|f\|_{L^2(B_1)} \leq \omega\left(\left(\int_0^{\mathbf{k}} \|u^\infty_\kappa[f]\|^2_{L^2(\mathcal{S}^{n-1})}\,\mathrm{d}\kappa\right)^{1/2}\right)$$

for any $f \in H^1_{B_1}$ *with* $\|f\|_{H^1(B_1)} \le 1$, *then there exist positive constants* $c = c(n)$ *and* $c' = c'(n)$ *such that*

$$\omega(t) \geq c \max\left\{t, \mathbf{k}^{-1}(\log(1/t))^{-2}\right\} \quad \text{for all } \ 0 < t < c'. \tag{2.10}$$

Estimate (2.10) demonstrates that the logarithmic part becomes less dominating when $\mathbf{k}$ increases. This observation reflects the increasing resolution phenomenon proved in literature mentioned in the Introduction.

2.2. **Lower bounds of singular values and a stability result.** Understanding lower bounds of singular values is extremely important in practice, especially in the Bayesian inference method. However, the general method used in deriving the upper bounds obtained in Theorem 2.1 does not seem to be effective. To overcome the difficulty, we will derive lower bounds for (2.9) using the explicit formula (2.2). For definiteness, we restrict our computations for $n = 3$, the most practical case. By (2.1), now (2.2) reads

$$u_\kappa^\infty[f](\hat{x}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=1}^{2\ell+1} \left(\int_{B_1} \mathrm{j}_\ell(\kappa|y|) Y_{\ell,m}(\hat{y}) f(y)\,\mathrm{d}y \right) Y_{\ell,m}(\hat{x}) \tag{2.11}$$

which converges in $L^2(\mathcal{S}^2)$. It is also helpful to see that we can relabel the spherical harmonics as

$$Y_{\ell,m} = Y_{\ell^2+m} \quad \text{for all } \ell = 0, 1, 2, \cdots \text{ and } m = 1, \cdots, 2\ell+1.$$

By the Courant min-max/max-min principles and the expression (2.11), we are able to prove the following theorem.

Theorem 2.3. *Let* $n = 3$ *and* $\mathbf{k} \ge 4$. *The singular values* $\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^3)\right)$ *of* (2.8) *satisfy the lower bound*

$$\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^3)\right) \geq \begin{cases} c\mathbf{k}^{-\frac{4}{3}} & \text{for all } \ j < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ c\mathbf{k}^{-\frac{4}{3}} \exp\left(-3j^{\frac{1}{2}} \log j\right) & \text{for all } \ j \geq \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \end{cases} \tag{2.12}$$

for some positive constant $c > 0$, *which is independent of both* $\mathbf{k}$ *and* j.

Remark 2.4. *Both upper and lower bounds of singular values in* (2.9) *and* (2.12) *can be written for the operator* $G_{\mathbf{k}}$. *In other words, for* $n = 3$ *and* $\mathbf{k} \ge 4$, *we have*

$$\begin{aligned} &\sigma_j\left(G_{\mathbf{k}} : L^2(B_1) \to L^2(\mathcal{S}^2_{\mathbf{k}}, \mathrm{d}\kappa\,\mathrm{d}\hat{x})\right) \leq C && \text{for all } j < C'\mathbf{k}^3, \\ &\sigma_j\left(G_{\mathbf{k}} : L^2(B_1) \to L^2(\mathcal{S}^2_{\mathbf{k}}, \mathrm{d}\kappa\,\mathrm{d}\hat{x})\right) \leq C \exp\left(-cj^{\frac{1}{6}}\mathbf{k}^{-\frac{1}{2}}\right) && \text{for all } j \geq C'\mathbf{k}^3, \end{aligned} \tag{2.13}$$

and

$$\sigma_j\left(G_{\mathbf{k}} : L^2(B_1) \to L^2(\mathcal{S}^2_{\mathbf{k}}, \mathrm{d}\kappa\,\mathrm{d}\hat{x})\right) \geq \begin{cases} c\mathbf{k}^{-\frac{4}{3}} & \text{for all } \ j < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ c\mathbf{k}^{-\frac{4}{3}} \exp\left(-3j^{\frac{1}{2}} \log j\right) & \text{for all } \ j \geq \left(\mathbf{k} - \dfrac{3}{2}\right)^2. \end{cases} \tag{2.14}$$

Combining (2.14) and (2.13), we can derive a stability estimate, which further explains the increasing resolution phenomena of the inverse source problem.

Corollary 2.5 (Stability of the inverse problem)**.** *Let $n = 3$ and $\mathbf{k} \geq 4$. Given any $\mu > \frac{1}{2}$, and we denote $\{\varphi_j\}_{j\in\mathbb{N}}$ the singular basis in $L^2(B_1)$ of (2.8) corresponding to the singular values $\sigma_j\,(u_{\mathbf{k}}^\infty : L^2(B_1) \to L^2(\mathbb{R}^3))$, and define the subspace*[2]

$$\mathcal{K}(\mu) := \left\{ f \in L^2(B_1) : \|f\|_{\mathcal{K}(\mu)}^2 := \sum_{j\in\mathbb{N}} j^\mu |\langle f, \varphi_j \rangle_{L^2(B_1)}|^2 \leq 1 \right\}.$$

There exist a positive constant $C > 0$ and a logarithmic modulus of continuity $\eta_{\log} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ which is a bijective, concave and strictly increasing function (*see* (4.19) *below*)*, both are independent of $\mathbf{k}$ and μ, such that*

$$\begin{aligned} \|f\|_{L^2(B_1)} &\leq \eta_{\log}\left(\|u_{\mathbf{k}}^\infty[f]\|_{L^2(\mathcal{S}_{\mathbf{k}}^2, \mathrm{d}\kappa \mathrm{d}\hat{x})} \right) && \text{for all } f \in \mathcal{K}(\mu, \mathbf{k}; \mathrm{unstable}), \\ \|f\|_{L^2(B_1)} &\leq C\mathbf{k}^{\frac{4}{3}} \|u_{\mathbf{k}}^\infty[f]\|_{L^2(\mathcal{S}_{\mathbf{k}}^2, \mathrm{d}\kappa \mathrm{d}\hat{x})} && \text{for all } f \in \mathcal{K}(\mu, \mathbf{k}; \mathrm{stable}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{K}(\mu, \mathbf{k}; \mathrm{unstable}) &:= \mathcal{K}(\mu) \cap \mathrm{span} \left\{ \varphi_j : j \geq \left(\mathbf{k} - \frac{3}{2}\right)^2 \right\}, \\ \mathcal{K}(\mu, \mathbf{k}; \mathrm{stable}) &:= \mathcal{K}(\mu) \cap \mathrm{span} \left\{ \varphi_j : j < \left(\mathbf{k} - \frac{3}{2}\right)^2 \right\}. \end{aligned}$$

Remark 2.6. *We observe the increasing resolution phenomena in the sense that the cardinality $|\mathcal{K}(\mu, \mathbf{k}; \mathrm{stable})|$ of $\mathcal{K}(\mu, \mathbf{k}; \mathrm{stable})$ satisfies*

$$|\mathcal{K}(\mu, \mathbf{k}; \mathrm{stable})| \nearrow +\infty \quad \text{as } \mathbf{k} \nearrow +\infty.$$

However, since the singular basis $\{\varphi_j\}_{j\in\mathbb{N}}$ the singular basis of (2.8) *depends on $\mathbf{k}$, therefore one should not expect the set inclusion between $\mathcal{K}(\mu, \mathbf{k}_1; \mathrm{stable})$ and $\mathcal{K}(\mu, \mathbf{k}_2; \mathrm{stable})$ for distinct $\mathbf{k}_1$ and $\mathbf{k}_2$.*

3. Consistency of the Bayesian inference

Let us now consider the statistical model (1.5) with A given by $G_{\mathbf{k}}$ defined in (1.5). Here we study the case where $n = 3$. As in the Introduction, we take separable Hilbert spaces $\mathbb{H}_1 = L^2(B_1)$ and $\mathbb{H}_2 = L^2(\mathcal{S}_{\mathbf{k}}^2, \mathrm{d}\kappa\, \mathrm{d}\hat{x})$ here. Also, for simplicity, denote by $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ ($\|\cdot\|_1$, $\|\cdot\|_2$) the inner products (norms) of $\mathbb{H}_1$ and $\mathbb{H}_2$. Let $\{\sigma_j\}_{j\in\mathbb{N}}$ be the singular values of $G_{\mathbf{k}}$ with the corresponding singular vectors $\{\varphi_j\}_{j\in\mathbb{N}}$. As in Corollary 2.5, the set $\{\varphi_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis of $\mathbb{H}_1$. Denote $\{e_j\}_{j\in\mathbb{N}} \subset \mathbb{H}_2$ the conjugate basis of $\{\varphi_j\}_{j\in\mathbb{N}}$, i.e., $G_{\mathbf{k}}\varphi_j = \sigma_j e_j$ for all j. Recall that W is a Gaussian white noise on $\mathbb{H}_2$. In other words, W is described by a mean-zero Gaussian process $W = (W_j := W_{e_j} : j \in \mathbb{N})$ with covariance $\mathbb{E}W_jW_{j'} = \delta_{jj'}$, where $\delta_{jj'}$ is the Kronecker delta function. In this form, (1.5) is interpreted as

$$Y_j = \langle G_{\mathbf{k}} f, e_j \rangle_2 + \frac{1}{\sqrt{N}} W_j, \tag{3.1}$$

[2]See [KRS21, (A.1)].

which is called the *sequence space model* corresponding to (1.5) [Cav08]. Now, we consider $f \in \mathbb{H}_1$ and hence f can be written as

$$f = \sum_j f_j \varphi_j,$$

where $\sum_j$ means $\sum_{j=1}^{\infty}$ and $f_j = \langle f, \varphi_j \rangle_1$. Consequently, Y_j are distributed $\mathcal{N}(\sigma_j f_j, \frac{1}{N})$ independently. So the inverse problem is to estimate $f = \{f_j\}_{j \in \mathbb{N}}$ from the sequence of independent observations $\{Y_j\}_{j \in \mathbb{N}}$. Since each Y_j depends on the noise level $1/\sqrt{N}$, we denote $Y^{(N)} := \{Y_j\}_{j \in \mathbb{N}}$. Next, relying on the Kakutani product martingale theorem [DP06, Theorem 2.7], it was proved in [Ray13] (see (1.4) there) that the posterior distribution conditioned on the observation $Y^{(N)}$ is given by

$$\Pi_N(B|Y^{(N)}) = \frac{\int_B e^{N \sum_j \sigma_j f_j Y_j - \frac{N}{2}\|G_{\mathbf{k}} f\|_2^2} \, \mathrm{d}\Pi(f)}{\int_{\mathcal{S}} e^{N \sum_j \sigma_j f_j Y_j - \frac{N}{2}\|G_{\mathbf{k}} f\|_2^2} \, \mathrm{d}\Pi(f)} \quad \text{for all } B \in \mathcal{B}, \tag{3.2}$$

where $\mathcal{S}$ is the support of the prior Π. For $f \in \mathbb{H}_1$, let $\mathbb{P}_f$ denote the law of the model (1.5). It is know that the family of distributions $(\mathbb{P}_f : f \in \mathbb{H}_1)$ is dominated by the law $\mathbb{P}_{f_0}$ with density

$$\frac{\mathrm{d}\mathbb{P}_f}{\mathrm{d}\mathbb{P}_{f_0}} = \exp\left(N \sum_j \sigma_j f_j Y_j - \frac{N}{2}\|G_{\mathbf{k}} f\|_2^2 \right). \tag{3.3}$$

see [Ray13] or [Ray15, (2.1.3)]. Similar to the priors studied in [Ray13], we consider two types of priors.

3.1. **Sieve priors.** We consider sieve priors in the singular basis $\{\varphi_j\}$ of $G_{\mathbf{k}} : \mathbb{H}_1 \to \mathbb{H}_2$ with truncation level J, i.e.,

$$f = \sum_{j=1}^{J} f_j \varphi_j, \tag{3.4}$$

where J has probability mass function $\mathbf{m}$ on $\mathbb{N}$ with distribution function $\mathcal{J}$. The coefficients f_j are random variables with density $\tau_j^{-1} q(\tau_j^{-1} \cdot)$, for some positive sequence $\{\tau_j\}_{j \in \mathbb{N}}$ to be chosen later, and for fixed density q. Consequently, the prior is given as

$$\Pi = \sum_{j=1}^{\infty} \mathbf{m}(j) \Pi_j \quad \text{where} \quad \Pi_j(x_1, \cdots, x_j) = \prod_{k=1}^{j} \tau_k^{-1} q(\tau_k^{-1} x_k). \tag{3.5}$$

Sieve priors of this type are commonly used in nonparametric Bayesian inference, see, for example, [AGR13, Hua04, SW01, Zha00]. Now we want to make the following assumption on q.

Assumption 1. There exist constants $D > 0, d > 0$, and $\beta \geq 1$ such that the density $q : \mathbb{C} \to [0, \infty)$ satisfies

$$De^{-d|x|^{\beta}} \leq q(x), \quad \text{for all } x \in \mathbb{C}.$$

We first prove a result which is in fact a parametric inference, which holds true for any choice of the positive sequence $\{\tau_j\}_{j \in \mathbb{N}}$.

Theorem 3.1. *Assume that the ground truth* f_0 *takes the form* $f_0 = \sum_{j=1}^{j_0} f_{0,j}\varphi_j$ *for some fixed* $j_0 \in \mathbb{N}$. *Let* q *satisfy* Assumption 1 *and* Π *be the prior defined as* (3.5) *such that there exists a constant* $b > 0$ *and*

$$0 < \mathbf{m}(j) \quad \text{and} \quad \sum_{\ell=j+1}^{\infty} \mathbf{m}(\ell) \lesssim e^{-bj} \quad \text{for all } j \in \mathbb{N}.$$

If $\mathbf{k} > \sqrt{j_0} + \frac{3}{2}$, *then the following holds for any parameter* $\vartheta > 0$: *there exist* $M > 0$ *and* L, *which are independent of* $\mathbf{k}$, *such that*

$$\mathbb{P}_{f_0}\left(\Pi_N\left(f \in L^2(B_1) : \|f - f_0\|_1 \geq M\xi_N | Y^{(N)}\right) \geq \vartheta\right) \lesssim (\log(\mathbf{k}N))^{-1} \tag{3.6}$$

for all sufficiently large $N \gtrsim \log \mathbf{k}$, *where*

$$\xi_N = \mathbf{k}^{\frac{4}{3}} \left(\frac{\log(\mathbf{k}N)}{N}\right)^{1/2} \cdot \begin{cases} 1 & \text{if } \lceil L\log(\mathbf{k}N)\rceil < \left(\mathbf{k} - \frac{3}{2}\right)^2, \\ \exp\left(3\lceil L\log(\mathbf{k}N)\rceil^{\frac{1}{2}} \log(\lceil L\log(\mathbf{k}N)\rceil)\right) & \\ \quad \text{if } \lceil L\log(\mathbf{k}N)\rceil \geq \left(\mathbf{k} - \frac{3}{2}\right)^2, \end{cases} \tag{3.7}$$

and all the implied constants are independent of $\mathbf{k}$.

From (3.6), one sees that

$$\Pi_N\left(f \in L^2(B_1) : \|f - f_0\|_1 \geq M\xi_N | Y^{(N)}\right) \xrightarrow{\mathbb{P}_{f_0}} 0 \quad \text{uniformly in } \mathbf{k}.$$

The form of ξ_N strongly suggests that we consider the "stable region"

$$\mathcal{R}_{\mathbf{k}} := \left\{ N \in \mathbb{R} : \log \mathbf{k} \lesssim N \lesssim \exp\left(\left(\mathbf{k} - \frac{3}{2}\right)^2\right)\right\},$$

and we observe the increasing resolution phenomenon in the sense that the range of $\mathcal{R}_{\mathbf{k}}$ increases as $\mathbf{k}$ increases. If the parameter $\mathbf{k}$ is adaptive to the noise level with a growth rate described in the first part of (3.7), we have a better contraction rate than the case where $\mathbf{k}$ is fixed. Nonetheless, in the parametric case, the contraction rate of the posterior distribution using a sieve prior is $N^{-1/2+\epsilon}$ with any $\epsilon > 0$, regardless the value of $\mathbf{k}$.

We now consider the nonparametric inference problem. In order to obtain bounds which are uniform with respect to the key parameter $\mathbf{k}$, we will consider the ground truth which is "smoother" than the one studied in [Ray13]. For each $s \geq 0$, let $H^s_{\exp} \equiv H^s_{\exp}(\mathbb{H}_1)$ be defined in terms of the basis $\{\varphi_j\}_{j\in\mathbb{N}}$ given by

$$H^s_{\exp} = \left\{ f \in L^2(B_1) : \|f\|^2_{H^s_{\exp}(\mathbb{H}_1)} := \sum_{j\in\mathbb{N}} j^{2s} \exp\left(6j^{\frac{1}{2}} \log j\right) |f_j|^2 < +\infty \right\}$$

where $f_j = \langle f, \varphi_j\rangle_1$. The consideration of such weighted Hilbert space is motivated by the discrepancy of lower and upper bounds of singular values derived in (2.13) and (2.14). The following theorem clearly demonstrates the increasing resolution phenomenon for nonparametric case:

Theorem 3.2. *We assume that the density q is a standard Gaussian, i.e., $\beta = 2$ in Assumption 1 and the scale parameters satisfy*

$$\tau_j \simeq \left(j^{\frac{1}{2}} \log j\right)^{-\frac{3+\delta}{2}} \quad \text{for all } j \in \mathbb{N}. \tag{3.8}$$

for some $\delta > 0$. Assume that the ground truth $0 \not\equiv f_0 \in H^s_{\exp}$ for some $s > \frac{2}{3} + \frac{\delta}{4}$. Let Π be the prior defined as (3.5) in which $\mathbf{m}(j)$ satisfies

$$\exp\left(-b\left(j^{\frac{1}{2}} \log j\right)^3\right) \lesssim \mathbf{m}(j), \quad \sum_{\ell=j+1}^{\infty} \mathbf{m}(\ell) \lesssim \exp\left(-b'\left(j^{\frac{1}{2}} \log j\right)^3\right) \quad \text{for all } j \in \mathbb{N} \tag{3.9}$$

with some $b > 0$ and $b' > 0$. Given any $\alpha > 0$ and $\mathbf{k} > \sqrt{e} + \frac{3}{2}$ satisfying[3]

$$\mathbf{k}^{\frac{8}{3}}\left(\mathbf{k} - \frac{3}{2}\right)^{-4s+\delta}\left(2\log\left(\mathbf{k} - \frac{3}{2}\right)\right)^{\delta} \le 1, \quad \alpha \log \mathbf{k} \le 2\left(\mathbf{k} - \frac{3}{2}\right)\log\left(\mathbf{k} - \frac{3}{2}\right), \tag{3.10}$$

then the following holds true for any parameter $\vartheta > 0$: there exists a constant $M > 1$, which is independent of $\mathbf{k}$, such that

$$\mathbb{P}_{f_0}\left(\Pi_N(f \in L^2(B_1) : \|f - f_0\|_1 \ge M\hat{\xi}_N | Y^{(N)}) \ge \vartheta\right) \lesssim (\log(\mathbf{k}N))^{-3}$$

for all $N \gtrsim (\log \mathbf{k})^{3+\delta}$, where

$$\hat{\xi}_N = \begin{cases} \mathbf{k}^{3\alpha}\left(\dfrac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}} & \text{if } k_N^{\frac{1}{2}} \log k_N < \alpha \log \mathbf{k}, \\ \mathbf{k}^{\frac{8}{3}} k_N^{-\frac{\delta}{4}} & \text{if } k_N^{\frac{1}{2}} \log k_N \ge \alpha \log \mathbf{k}, \end{cases}$$

where k_N is the smallest integer satisfying

$$k_N^{-s} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \simeq \mathbf{k}^{-\frac{4}{3}}\left(\frac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}}. \tag{3.11}$$

Here, all the implied constants are independent of $\mathbf{k}$.

Remark. Since (see (6.12) below)

$$\log \mathbf{k} + \log N = \log(\mathbf{k}N) \simeq k_N^{\frac{1}{2}} \log k_N \quad \text{for all large } N \gtrsim (\log \mathbf{k})^{3+\delta},$$

one can choose a sufficiently large $\alpha > 1$ to see that there exists a constant $\alpha' = \alpha'(\alpha) > 0$ such that

$$\begin{aligned} &\left\{N \in \mathbb{R} : N \gtrsim (\log \mathbf{k})^{3+\delta}, k_N^{\frac{1}{2}} \log k_N < \alpha \log \mathbf{k}\right\} \\ &= \left\{N \in \mathbb{R} : N \gtrsim (\log \mathbf{k})^{3+\delta}, \log \mathbf{k} + \log N \lesssim \alpha \log \mathbf{k}\right\} \\ &= \left\{N \in \mathbb{R} : (\log \mathbf{k})^{3+\delta} \lesssim N \lesssim \mathbf{k}^{\alpha'}\right\}, \end{aligned}$$

which demonstrates the increasing resolution phenomenon.

[3] The mapping $t \in (e, \infty) \mapsto t^{-2s+\frac{\delta}{2}}(\log t)^{\delta}$ is monotone decreasing due to $s > \frac{3}{4}\delta$.

From a frequentist perspective, it is reasonable to calibrate the prior adapted to the forward operator $G_{\mathbf{k}}$, see also [KSvdVvZ16, KvdVvZ11, KvdVvZ13, Ray13] which also make strong use of knowledge of the forward operator through the choice of diagonalizing basis.

The estimate (3.13) should read with a caveat. Even though the contraction rate is dominated by the term $\mathbf{k}^{3\alpha}\left(\frac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}}$ at large $\mathbf{k}$, the constant $\mathbf{k}^{3\alpha}$ there suggests that there may be no significant improvement in the quality of the reconstruction of the source f from large amount of measurements $\{u_\kappa^\infty[f] : \kappa \in (0,\mathbf{k})\}$. In the case when the parameter N is large (i.e. the noise level $1/\sqrt{N}$ is low), we expect that collecting large amount of measurements $\{u_\kappa^\infty[f] : \kappa \in (0,\mathbf{k})\}$ would be helpful. Otherwise, if the data is noisy (i.e. the parameter N is small), collecting large amount of such noisy data should not be too helpful.

3.2. **Gaussian priors.** In this section, we consider a conjugate prior. Let $\Lambda : \mathbb{H}_1 \to \mathbb{H}_1$ be a positive semidefinite, self-adjoint and trace class linear operator, satisfying the following assumption similar as in [Ray13, Condition 3] (see also [KvdVvZ11, KvdVvZ13]).

Assumption 2. Suppose that the eigenvectors of $\Lambda : \mathbb{H}_1 \to \mathbb{H}_1$ are identical to the singular basis $\{\varphi_j\}_{j\in\mathbb{N}}$ of $G_{\mathbf{k}} : \mathbb{H}_1 \to \mathbb{H}_2$, and its corresponding eigenvalues are $\tau_j \simeq j^{-\rho}\exp\left(-3j^{\frac{1}{2}}\log j\right)$ for some constant $\rho > \frac{3}{2}$. Here the implied constants are independent of $\mathbf{k}$.

Let $\mathcal{N}(0,\Lambda)$ be the mean-zero Gaussian distribution on $\mathbb{H}_1$ with covariance operator $\Lambda : \mathbb{H}_1 \to \mathbb{H}_1$, that is, $G \in \mathcal{N}(0,\Lambda)$ if and only if $(\langle G,h\rangle_1 : h \in \mathbb{H}_1)$ is a Gaussian process with

$$\mathbb{E}\langle G,h\rangle_1 = 0, \quad \operatorname{cov}\left(\langle G,h\rangle_1, \langle G,h'\rangle_1\right) = \langle h, \Lambda h'\rangle_1 \quad \text{for all } h,h' \in \mathbb{H}_1.$$

We now take the prior $\Pi = \mathcal{N}(0,\Lambda)$. Recall that a Gaussian distribution has support equal to the closure of its reproducing kernel Hilbert space (RKHS) $\mathbb{H}$, see [vdVvZ08b]. Since the posterior has the same support, consistency is only achievable when $G_{\mathbf{k}}f_0$ is contained in this set.

Theorem 3.3. *Assume that the prior $\Pi = \mathcal{N}(0,\Lambda)$ with $\Lambda : \mathbb{H}_1 \to \mathbb{H}_1$ satisfies Assumption 2. Let the ground truth $f_0 \in H^\gamma_{\rm exp}$ for some number $\gamma \geq \rho$, then there exist constants $M > 1$ and $s > 0$ which are independent of $\mathbf{k}$, such that*

$$\mathbb{P}_{f_0}\left(\Pi_N(f \in L^2(B_1) : \|f - f_0\|_1 \geq M\tilde{\xi}_N | Y^{(N)}) \geq \vartheta\right) \lesssim (\log N)^{-3} \tag{3.12}$$

for all $N \gtrsim 1$, where

$$\tilde{\xi}_N = \mathbf{k}^{\frac{4}{3}} \begin{cases} \left(\dfrac{(\log N)^3}{N}\right)^{\frac{1}{2}} & \text{if } k_N < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ k_N^{-s} & \text{if } k_N \geq \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \end{cases} \tag{3.13}$$

where k_N is the smallest integer satisfying

$$k_N^{-s}\exp\left(-3k_N^{\frac{1}{2}}\log k_N\right) \simeq \left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}}.$$

Again, all the implied constants are independent of $\mathbf{k}$.

From (3.12) we have the following uniform convergence

$$\Pi_N(f \in L^2(B_1) : \|f - f_0\|_1 \geq M\tilde{\xi}_N | Y^{(N)}) \xrightarrow{\mathbb{P}_{f_0}} 0 \quad \text{uniformly in } \mathbf{k}.$$

It is reasonable to choose $\mathbf{k}$ as a function of N, for example, we can take

$$\mathbf{k}(N) = k_N^{\frac{1}{2}} + 2, \tag{3.14}$$

which tends to infinity at a logarithmic rate as $N \to \infty$, and now we obatin the following corollary.

Corollary 3.4. *Under the same assumptions as in Theorem 3.3, if $\mathbf{k} = \mathbf{k}(N)$ is given in (3.14), then*

$$\Pi_N\left(f \in L^2(B_1) : \|f - f_0\|_1 \geq M \left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}} \left(k_N^{\frac{1}{2}} + 2\right)^{\frac{4}{3}} \middle| Y^{(N)} \right) \xrightarrow{\mathbb{P}_{f_0}} 0$$

as $N \to +\infty$.

In other words, Corollary 3.4 suggests the amount of measurements taken in account of the experimental quality and the cost of the measurements. From the mathematical point of view, similar discussions also can be found in our previous work [KW24].

4. Proofs of theorems in Section 2

Before proving Theorem 2.1, we use the compact support condition for f to bound $U_{\mathbf{k}}(f)$.

Lemma 4.1. *There exists a constant $C = C(n) > 0$ such that*

$$\|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^n)} \leq C(Cm\mathbf{k})^{2m} \|f\|_{H^{-2m}_{\overline{B}_1}}$$

for all $\mathbf{k} \geq 1$ and integers $m \geq 0$.

Proof. We will prove our lemma by modifying the ideas in [KSZ24, Lemma 3.2]. From (2.3) and the Placherel formula, we write

$$\|U_{\mathbf{k}}(f)\|^2_{L^2(\mathbb{R}^n)} = I_1 + I_2 \tag{4.1}$$

with

$$I_1 = \int_{B_1} |\xi|^{1-n} |\mathscr{F}[f](\xi)|^2 \,\mathrm{d}\xi, \quad I_2 = \int_{B_{\mathbf{k}} \setminus B_1} |\xi|^{1-n} |\mathscr{F}[f](\xi)|^2 \,\mathrm{d}\xi.$$

By using [KSZ24, (3.4)], we see that there exists a constant $C = C(n) > 0$ such that

$$|\mathscr{F}[f](\xi)| \leq C(Cm + C|\xi|)^{2m} \|f\|_{H^{-2m}_{\overline{B}_1}} \quad \text{for all } \ \xi \in \mathbb{R}^n,$$

and therefore I_1 can be estimated for $m \geq 1$ by

$$I_1 \leq C \sup_{|\xi| \leq 1} |\mathscr{F}[f](\xi)|^2 \leq C(Cm)^{4m} \|f\|^2_{H^{-2m}_{\overline{B}_1}}.$$

We proceed to esitmate I_2 by

$$I_2 \leq C^{2+4m} \left(\sup_{1 \leq |\xi| \leq \mathbf{k}} (m + |\xi|)^{4m} |\xi|^{1-n} \right) \|f\|^2_{H^{-2m}_{\overline{B}_1}} \leq C^2 (Cm\mathbf{k})^{4m} \|f\|^2_{H^{-2m}_{\overline{B}_1}}.$$

Plugging the estimates for I_1 and I_2 into (4.1) implies the lemma. □

For any $m > (n-1)/4$, since the embedding $H_{\overline{B}_1}^{(1-n)/2} \subset H_{\overline{B}_1}^{-2m}$ is compact, the operator (2.6) is compact. It follows from [KSZ24, (3.8) in Lemma 3.4] that there exists a constant $C = C(n) > 0$ such that the singular values of the canonical embedding $\iota : H_{\overline{B}_1}^{(1-n)/2} \to H_{\overline{B}_1}^{-2m}$ satisfies

$$\sigma_j \left(\iota : H_{\overline{B}_1}^{(1-n)/2} \to H_{\overline{B}_1}^{-2m} \right) \le C(Cm)^{2m} j^{-\frac{2m+(1-n)/2}{n}} \quad \text{for any integer } m > (n-1)/4. \tag{4.2}$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. First of all, by using [KRS21, Proposition 2.3(a)] and Lemma 4.1 with integer $m = \lceil (n-1)/4 \rceil + 1$, we immediately obtain the following trivial bound via interpolation (e.g., [LM72, Theorem 5.1])

$$\sigma_j(U_{\mathbf{k}}) \le \sigma_1(U_{\mathbf{k}}) = \|U_{\mathbf{k}}\|_{H_{\overline{B}_1}^{(1-n)/2} \to L^2(\mathbb{R}^n)} \le C\mathbf{k}^{\frac{n-1}{2}} \quad \text{for all } j \in \mathbb{N}, \tag{4.3}$$

where $C = C(n)$. The central idea of the remaining proof utilizes the following Courant's min-max principle:

$$\sigma_j(U_{\mathbf{k}}) = \min_S \max_{\{f \in H_{\overline{B}_1}^{(1-n)/2} : f \perp S, \|f\|_{H_{\overline{B}_1}^{(1-n)/2}} = 1\}} \|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^n)} \tag{4.4}$$

where the minimum is taken over all subspaces $S \subset H_{\overline{B}_1}^{(1-n)/2}$ with $\dim(S) = j-1$. Let $m \in \mathbb{N}$ and $\{\psi_\ell\}_{\ell \in \mathbb{N}}$ be an orthonormal basis of $H_{\overline{B}_1}^{(1-n)/2}$ consisting of singular vectors of the canonical embedding $\iota : H_{\overline{B}_1}^{(1-n)/2} \to H_{\overline{B}_1}^{-2m}$. For each $j \in \mathbb{N}$, we consider the vector space

$$S_j := \operatorname{span} \{\psi_1, \cdots, \psi_{j-1}\}.$$

From (4.2) we see that

$$\|f\|_{H_{\overline{B}_1}^{-2m}} = \|\iota \circ f\|_{H_{\overline{B}_1}^{-2m}} \le C(Cm)^{2m} j^{-\frac{2m+(1-n)/2}{n}} \|f\|_{H_{\overline{B}_1}^{(1-n)/2}} \quad \text{for all } f \perp S_j$$

and consequently using the estimate in Lemma 4.1 for $f \in S_j$, we see that

$$\|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^n)} \le C(Cm\mathbf{k}^{1/2})^{4m} j^{-\frac{2m+(1-n)/2}{n}} \|f\|_{H_{\overline{B}_1}^{(1-n)/2}} \quad \text{for all } f \perp S_j.$$

Applying the Courant's min-max principle (4.4) yields

$$\sigma_j(U_{\mathbf{k}}) \le C(Cm\mathbf{k}^{1/2} j^{-\frac{1}{2n}})^{4m} j^{\frac{n-1}{2n}} \quad \text{for all } j \in \mathbb{N} \text{ and } m \in \mathbb{N}. \tag{4.5}$$

Now, for each $j \in \mathbb{N}$, we consider the function

$$f_j(t) = C(Ct\mathbf{k}^{1/2} j^{-\frac{1}{2n}})^{4t} \quad \text{for all } t > 0.$$

It is not difficult to see that such function f_j is convex and has a global minimum t_0 over $t > 0$ with

$$Ct_0\mathbf{k}^{1/2} j^{-\frac{1}{2n}} = 1/e.$$

Next, for any $j \geq (2Ce\mathbf{k}^{1/2})^{2n}$, we note that $t_0 \geq 2$ and so we can choose $m = \lfloor t_0 \rfloor \geq t_0/2 \geq 1$ in (4.5) to see that

$$\begin{aligned}\sigma_j(U_\mathbf{k}) &= \sigma_j\left(U_\mathbf{k} : H^{(1-n)/2}_{\overline{B}_1} \to L^2(\mathbb{R}^n)\right) \leq j^{\frac{n-1}{2n}} f_j(m) \\ &\leq j^{\frac{n-1}{2n}} f_j(t_0/2) = Cj^{\frac{n-1}{2n}} (Ct_0\mathbf{k}^{1/2} j^{-\frac{1}{2n}}/2)^{2t_0} = Cj^{\frac{n-1}{2n}} (2e)^{-\frac{2}{Ce} j^{1/2n}\mathbf{k}^{-1/2}} \\ &= Cj^{\frac{n-1}{2n}} \exp\left(\log\left((2e)^{-\frac{2}{Ce} j^{1/2n}\mathbf{k}^{-1/2}}\right)\right) = Cj^{\frac{n-1}{2n}} \exp\left(-\frac{2\log(2e)}{Ce} j^{\frac{1}{2n}} \mathbf{k}^{-\frac{1}{2}}\right)\end{aligned} \tag{4.6}$$

for all $j \in \mathbb{N}$. Combining (4.6) with the trivial bound (4.3), we conclude the upper bounds in (2.7a) and (2.7b).

We proceed to prove the lower bound in the stable region. Now we use the Courant max-min principle:

$$\sigma_j\left(U_\mathbf{k} : H^{(1-n)/2}_{\overline{B}_1} \to L^2(\mathbb{R}^n)\right)^2 = \max_X \min_{\{f \in X : \|f\|_{H^{(1-n)/2}_{\overline{B}_1}} = 1\}} \|U_\mathbf{k}(f)\|^2_{L^2(\mathbb{R}^n)} \tag{4.7}$$

where the maximum is taken over all subspaces X of $H^{(1-n)/2}_{\overline{B}_1}$ with $\dim(X) = j$. Since

$$\begin{aligned}\|U_\mathbf{k}(f)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\xi|^{1-n} |\mathscr{F}[f](\xi)|^2 \,\mathrm{d}\xi - \int_{\mathbb{R}^n \setminus B_\mathbf{k}} |\xi|^{1-n} |\mathscr{F}[f](\xi)|^2 \,\mathrm{d}\xi \\ &\geq \|f\|^2_{H^{(1-n)/2}_{\overline{B}_1}} - \mathbf{k}^{1-n} \|f\|^2_{L^2(B_1)},\end{aligned}$$

and from (4.7) it follows

$$\sigma_j\left(U_\mathbf{k} : H^{(1-n)/2}_{\overline{B}_1} \to L^2(\mathbb{R}^n)\right)^2 \geq \max_X \min_{\{f \in X : \|f\|_{H^{(1-n)/2}_{\overline{B}_1}} = 1\}} \left(1 - \mathbf{k}^{1-n} \|f\|^2_{L^2(B_1)}\right). \tag{4.8}$$

We choose

$$X_j = \operatorname{span}\{\psi_1, \cdots, \psi_j\}$$

where $\{\psi_\ell\}_{\ell \in \mathbb{N}}$ is an orthonormal basis of $L^2(B_1)$ consisting of singular vectors of $\iota : L^2(B_1) \to H^{(1-n)/2}_{\overline{B}_1}$. Applying [KSZ24, Lemma 3.4] implies that there exists a positive constant $c_0 = c_0(n)$

$$\|f\|^2_{H^{(1-n)/2}_{\overline{B}_1}} = \|\iota \circ f\|^2_{H^{(1-n)/2}_{\overline{B}_1}} \geq c_0 j^{(1-n)/n} \|f\|^2_{L^2(B_1)}.$$

Then by (4.8) we have that

$$\sigma_j\left(U_\mathbf{k} : H^{(1-n)/2}_{\overline{B}_1} \to L^2(\mathbb{R}^n)\right)^2 \geq \min_{\{f \in X_j : \|f\|_{H^{(1-n)/2}_{\overline{B}_1}} = 1\}} \left(1 - \mathbf{k}^{1-n} \|f\|^2_{L^2(B_1)}\right) \geq 1 - c_0^{-1} \mathbf{k}^{1-n} j^{\frac{n-1}{n}}$$

for all $j \in \mathbb{N}$. Consequently, we conclude that

$$\sigma_j\left(U_\mathbf{k} : H^{(1-n)/2}_{\overline{B}_1} \to L^2(\mathbb{R}^n)\right)^2 \geq \frac{1}{4} \quad \text{for all } j \leq \left(\frac{3c_0}{4}\right)^{\frac{n}{n-1}} \mathbf{k}^n,$$

which gives the lower bound in (2.7a). □

Next, we prove Theorem 2.3 using the explicit formula (2.11).

Proof of Theorem 2.3. Similar to Theorem 2.1, the main idea is to use the Courant max-min principle:

$$\sigma_j\left(U_{\mathbf{k}}: L^2(B_1) \to L^2(\mathbb{R}^3)\right)^2 = \max_X \min_{\{f\in X: \|f\|_{L^2(B_1)}=1\}} \|U_{\mathbf{k}}(f)\|^2_{L^2(\mathbb{R}^3)} \tag{4.9}$$

where the maximum is taken over all subspaces X of $L^2(B_1)$ with $\dim(X) = j$. By (2.11) and DLMF:10.51.E3, we can obtain that

$$\begin{aligned}
u_\kappa^\infty[|y|^\ell \overline{Y_{\ell,m}(\hat{y})}](\hat{x}) &= 4\pi\left(\int_0^1 \mathsf{j}_\ell(\kappa r) r^{2+\ell}\,\mathrm{d}r\right) Y_{\ell,m}(\hat{x}) \\
&= 4\pi\kappa^{-(3+\ell)}\left(\int_0^\kappa \mathsf{j}_\ell(z) z^{2+\ell}\,\mathrm{d}z\right) Y_{\ell,m}(\hat{x}) \\
&= 4\pi\kappa^{-1}\mathsf{j}_{\ell+1}(\kappa) Y_{\ell,m}(\hat{x}).
\end{aligned}$$

The relation (2.5) yields

$$\|U_{\mathbf{k}}(|y|^\ell \overline{Y_{\ell,m}(\hat{y})})\|^2_{L^2(\mathbb{R}^3)} = \int_0^{\mathbf{k}} \|u_\kappa^\infty[|y|^\ell \overline{Y_{\ell,m}(\hat{y})}]\|^2_{L^2(\mathcal{S}^2)}\,\mathrm{d}\kappa = 16\pi^2\int_0^{\mathbf{k}} \kappa^{-2}|\mathsf{j}_{\ell+1}(\kappa)|^2\,\mathrm{d}\kappa,$$

where the convergence of the integral can be easily seen from DLMF:10.7.E3.

For each $j\in\mathbb{N}$, we define

$$\phi_j(y) := \sqrt{2\ell+3}|y|^\ell \overline{Y_{\ell,m}(\hat{y})}$$

where we write $j = \ell^2 + m$ for $\ell = 0,1,2,\cdots$ and $m = 1,\cdots,2\ell+1$ as in (2.2). Note that $\{\phi_j\}_{j\in\mathbb{N}}$ forms an orthonormal set of $L^2(B_1)$. By DLMF:10.47.3 (J_ν denotes the Bessel function of the first kind) and DLMF:10.22.E27, we can compute

$$\begin{aligned}
\|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} &= 16\pi^2(2\ell+3)\int_0^{\mathbf{k}} \kappa^{-2}|\mathsf{j}_{\ell+1}(\kappa)|^2\,\mathrm{d}\kappa \\
&= 8\pi^3(2\ell+3)\int_0^{\mathbf{k}} \kappa^{-3}|\mathsf{J}_{\ell+\frac{3}{2}}(\kappa)|^2\,\mathrm{d}\kappa \\
&\ge 8\pi^3(2\ell+3)\mathbf{k}^{-4}\int_0^{\mathbf{k}} \kappa|\mathsf{J}_{\ell+\frac{3}{2}}(\kappa)|^2\,\mathrm{d}\kappa \\
&= 16\pi^3(2\ell+3)\mathbf{k}^{-4}\sum_{\alpha=0}^\infty\left(\ell+\frac{5}{2}+2\alpha\right)|\mathsf{J}_{\ell+\frac{5}{2}+2\alpha}(\mathbf{k})|^2 \\
&\ge 8\pi^3(2\ell+3)(2\ell+5)\mathbf{k}^{-4}|\mathsf{J}_{\ell+\frac{5}{2}}(\mathbf{k})|^2 \\
&\ge 120\pi^3\mathbf{k}^{-4}|\mathsf{J}_{\ell+\frac{5}{2}}(\mathbf{k})|^2.
\end{aligned} \tag{4.10}$$

Observe that $j \le (\ell+1)^2$. If $j \ge (\mathbf{k}-\frac{3}{2})^2$, then $\ell+\frac{5}{2} \ge \mathbf{k}$, and thus we can use DLMF:10.14.E7 to see that

$$\begin{aligned}
\|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} &\ge 120\pi^3\mathbf{k}^{-4}\left|\mathsf{J}_{\ell+\frac{5}{2}}\left(\left(\ell+\frac{5}{2}\right)\frac{\mathbf{k}}{(\ell+\frac{5}{2})}\right)\right|^2 \\
&\ge 120\pi^3\mathbf{k}^{-4}\left|\left(\frac{\mathbf{k}}{(\ell+\frac{5}{2})}\right)^{\ell+\frac{5}{2}}\mathsf{J}_{\ell+\frac{5}{2}}\left(\ell+\frac{5}{2}\right)\right|^2.
\end{aligned} \tag{4.11}$$

Here we remind the readers that

$$J_\nu(\nu) > 0 \quad \text{for all } \nu > 0, \tag{4.12}$$

see DLMF:10.14.E2. Now we use DLMF:10.19.E8, DLMF:9.2.E3, and DLMF:9.2.E4, to derive that

$$J_\nu(\nu) \sim \frac{2^{1/3}}{\nu^{1/3}} \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} \sum_{i=0}^{\infty} \frac{P_i(0)}{\nu^{2i/3}} - \frac{2^{2/3}}{\nu} \frac{1}{3^{1/3}\Gamma(\frac{1}{3})} \sum_{i=0}^{\infty} \frac{Q_i(0)}{\nu^{2i/3}} \quad \text{as} \quad \nu \to \infty,$$

for some polynomial coefficients P_i and Q_i, where $\sim$ is the Poincaré asymptotic expansion described in DLMF:2.1.iii. From DLMF:10.19.E10, we see that

$$P_0(0) = 1, \quad P_1(0) = P_2(0) = 0, \quad P_3(0) = -\frac{1}{225}, \quad P_4(0) = 0,$$

and from DLMF:10.19.E11 we have

$$Q_0(0) = 0, \quad Q_1(0) = \frac{1}{70}, \quad Q_2(0) = Q_3(0) = 0.$$

Next, DLMF:2.1.E15 implies

$$\lim_{\nu\to\infty} \nu^{\frac{1}{3}} J_\nu(\nu) = \frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})}.$$

On the other hand, by the monotonicity of $\nu^{\frac{1}{3}} J_\nu(\nu)$, see [Wat95, 8.54], there exists a constant $c > 0$, which is independent of ν, such that

$$J_\nu(\nu) \geq c\nu^{-\frac{1}{3}} \quad \text{for all } \ \nu \geq 1. \tag{4.13}$$

Putting together (4.11), (4.12) and (4.13) gives

$$\|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} \geq c^2 \mathbf{k}^{2\ell+1} \left(\ell + \frac{5}{2}\right)^{-(2\ell+\frac{17}{3})} \geq c^2 \mathbf{k}^{2\sqrt{j}+1} (4\ell)^{-8\ell},$$

where we used the fact $\ell \geq \sqrt{j} - 1 \geq \mathbf{k} - 5/2 \geq 1$. Furthermore, from $\ell \leq \sqrt{j}$, we see that there exists a positive constant $c > 0$, which is independent of both j and $\mathbf{k}$, such that

$$\begin{aligned} \|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} &\geq c^2 \mathbf{k}^{2\sqrt{j}+1} (4\sqrt{j})^{-8\sqrt{j}} \\ &= c^2 \mathbf{k}^{2\sqrt{j}+1} \exp\left(-4j^{\frac{1}{2}} \log(16j)\right) \end{aligned} \quad \text{for all } j \geq \left(\mathbf{k} - \frac{3}{2}\right)^2. \tag{4.14}$$

We now consider the case where $j < (\mathbf{k} - \frac{3}{2})^2$. Since $\ell^2 < j$, we have $\ell + \frac{3}{2} < \mathbf{k}$. In view of (4.10) and DLMF:10.22.E7, we can estimate

$$
\begin{aligned}
\|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} &= 16\pi^2(2\ell+3)\int_0^{\mathbf{k}} \kappa^{-2}|\mathsf{j}_{\ell+1}(\kappa)|^2 \,\mathrm{d}\kappa \\
&= 8\pi^3(2\ell+3)\int_0^{\mathbf{k}} \kappa^{-3}|\mathsf{J}_{\ell+\frac{3}{2}}(\kappa)|^2 \,\mathrm{d}\kappa \\
&\geq 8\pi^3(2\ell+3)\int_0^{\ell+\frac{3}{2}} \kappa^{-(2\ell+7)}\kappa^{2\ell+4}|\mathsf{J}_{\ell+\frac{3}{2}}(\kappa)|^2 \,\mathrm{d}\kappa \\
&\geq 8\pi^3(2\ell+3)\left(\ell+\frac{3}{2}\right)^{-(2\ell+7)} \frac{(\ell+\frac{3}{2})^{2\ell+5}}{2(2\ell+4)}\left|\mathsf{J}_{\ell+\frac{3}{2}}\left(\ell+\frac{3}{2}\right)\right|^2 \\
&= 4\pi^3\frac{2\ell+3}{2\ell+4}\left(\ell+\frac{3}{2}\right)^{-2}\left|\mathsf{J}_{\ell+\frac{3}{2}}\left(\ell+\frac{3}{2}\right)\right|^2 \\
&\geq 3\pi^3\left(\ell+\frac{3}{2}\right)^{-2}\left|\mathsf{J}_{\ell+\frac{3}{2}}\left(\ell+\frac{3}{2}\right)\right|^2 .
\end{aligned}
\tag{4.15}
$$

From (4.13) and (4.15), it follows that

$$
\|U_{\mathbf{k}}(\phi_j)\|^2_{L^2(\mathbb{R}^3)} \geq c^2\left(\ell+\frac{3}{2}\right)^{-\frac{8}{3}} \geq c^2\mathbf{k}^{-\frac{8}{3}} \quad \text{for all } \ j < \left(\mathbf{k}-\frac{3}{2}\right)^2. \tag{4.16}
$$

Consequently, gathering (4.14) and (4.16) implies

$$
\|U_{\mathbf{k}}(\phi_j)\|_{L^2(\mathbb{R}^3)} \geq \begin{cases} c\mathbf{k}^{-\frac{4}{3}} & \text{for all } \ j < \left(\mathbf{k}-\frac{3}{2}\right)^2, \\ c\mathbf{k}^{-\frac{4}{3}}\exp\left(-3j^{\frac{1}{2}}\log j\right) & \text{for all } \ j \geq \left(\mathbf{k}-\frac{3}{2}\right)^2. \end{cases} \tag{4.17}
$$

Finally, considering $X_j = \operatorname{span}\{\phi_1, \cdots, \phi_j\}$ in (4.9), (4.17) immediately implies the lower bounds of (2.12). □

Now we would like to prove Corollary 2.5 by modifying the ideas in [KRS21, Proposition A.4].

Proof of Corollary 2.5. By (2.9), we can choose a positive constant $c_0 > 0$, which is independent of j and $\mathbf{k}$, such that

$$
c_0\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^3)\right)^2 \leq \frac{1}{e} \quad \text{for all } j \in \mathbb{N}.
$$

Since $\mu > \frac{1}{2}$, there exists a constant $C' = C'(\mu) > 0$ such that $3j^{\frac{1}{2}}\log j \leq C'j^{\mu}$. Given any $s > 0$, from the second estimate in (2.12), for each $j \geq (\mathbf{k} - \frac{3}{2})^2$, we have

$$
c_0\left(\sigma_j\left(U_{\mathbf{k}} : L^2(B_1) \to L^2(\mathbb{R}^3)\right)/j^s\right)^2 \geq c_0c^2\mathbf{k}^{-\frac{8}{3}}j^{-2s}\exp\left(-2C'j^{\mu}\right).
$$

We further obtain that

$$\begin{aligned}
&\left|\log\left(c_0\left(\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)/j^s\right)^2\right)\right| \\
&= -\log\left(c_0\left(\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)/j^s\right)^2\right) \\
&\le -\log\left(c_0 c^2 \mathbf{k}^{-\frac{8}{3}} j^{-2s}\exp\left(-2C' j^{\mu}\right)\right) = \log\left(c_0^{-1}c^{-2}\mathbf{k}^{\frac{8}{3}} j^{2s}\exp\left(2C'j^{\mu}\right)\right) \\
&\le \log\left(c_0^{-1}c^{-2}\left(j^2+\frac{3}{2}\right)^{\frac{8}{3}} j^{2s}\exp\left(2C'j^{\mu}\right)\right).
\end{aligned}$$

We can choose a positive constant $C = C(\mu, s) > 0$ such that

$$c_0^{-1}c^{-2}\left(j^2+\frac{3}{2}\right)^{\frac{8}{3}} j^{2s}\exp\left(2C'j^{\mu}\right) \le \exp(Cj^{\mu}) \quad \text{for all } j\in\mathbb{N},$$

and thus

$$\left|\log\left(c_0\left(\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)/j^s\right)^2\right)\right|^{-1} \ge C^{-1}j^{-\mu}.$$

Taking $s = \frac{\mu}{2}$ above implies

$$\eta_{\log}\left(\left(\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)/j^{\mu/2}\right)^2\right) \ge j^{-\mu} \quad \text{for all } j \ge \left(\mathbf{k}-\frac{3}{2}\right)^2, \tag{4.18}$$

where $\eta_{\log}: \mathbb{R}_{>0}\to\mathbb{R}_{>0}$ is any bijective, concave and strictly increasing function given by

$$\eta_{\log}(t) = C|\log(c_0 t)|^{-1} \quad \text{for all } \ t\in(0,(c_0 e)^{-1}) \tag{4.19}$$

and

$$t\in\mathbb{R}_{>0}\mapsto \frac{\eta_{\log}(t)}{t} \quad \text{is nonincreasing.} \tag{4.20}$$

Now let

$$f = \sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} (f,\varphi_j)_{L^2(B_1)}\varphi_j \in \mathcal{K}(\mu), \tag{4.21}$$

and define $\lambda := \|f\|_{\mathcal{K}(\mu)} (\le 1)$ and the normalized function $f_\lambda := f/\|f\|_{\mathcal{K}(\mu)}$ and its coefficients

$$a_j := \frac{1}{\|f\|_{\mathcal{K}(\mu)}}(f,\varphi_j)_{L^2(B_1)}.$$

The fact $\sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} j^{\mu}|a_j|^2 = \|f\|_{\mathcal{K}(\mu)}^{-2}\sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} j^{\mu}|(f,\varphi_j)_{L^2(B_1)}|^2 = 1$ suggests that we define $c_j := j^{\mu}|a_j|^2$ and so $\sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} c_j = 1$. This enables us to apply Jenson's inequality to obtain

$$\eta_{\log}^{-1}\left(\|f_\lambda\|_{L^2(B_1)}^2\right) = \eta_{\log}^{-1}\left(\sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} c_j j^{-\mu}\right) \le \sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} c_j \eta_{\log}^{-1}\left(j^{-\mu}\right)$$

because $\eta_{\log}^{-1}$ is convex. Therefore, by (4.18), we have

$$\begin{aligned}
\eta_{\log}^{-1}\left(\|f_\lambda\|_{L^2(B_1)}^2\right) &\le \sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} c_j\left(\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)/j^{\mu/2}\right)^2 \\
&= \sum_{j\ge(\mathbf{k}-\frac{3}{2})^2} |a_j|^2\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)^2 = \|U_{\mathbf{k}}(f_\lambda)\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

Consequently, we can use (4.20) and the fact $\lambda^2 \leq 1$ to derive that

$$\|f\|_{L^2(B_1)}^2 \leq \frac{\eta_{\log}(\|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^3)}^2/\lambda^2)}{1/\lambda^2} \leq \eta_{\log}\left(\|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^3)}^2\right) \tag{4.22}$$

for all f given in (4.21).

Next we consider

$$f = \sum_{j<(\mathbf{k}-\frac{3}{2})^2} (f,\varphi_j)_{L^2(B_1)}\varphi_j \in \mathcal{K}(\mu). \tag{4.23}$$

From the first estimate in (2.12), it is easy to see that

$$\begin{aligned} \|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^3)}^2 &= \sum_{j<(\mathbf{k}-\frac{3}{2})^2} |(f,\varphi_j)_{L^2(B_1)}|^2\sigma_j\left(U_{\mathbf{k}}: L^2(B_1)\to L^2(\mathbb{R}^3)\right)^2 \\ &\geq c^2\mathbf{k}^{-\frac{8}{3}} \sum_{j<(\mathbf{k}-\frac{3}{2})^2} |(f,\varphi_j)_{L^2(B_1)}|^2 = c^2\mathbf{k}^{-\frac{8}{3}}\|f\|_{L^2(B_1)}^2 \end{aligned} \tag{4.24}$$

for all f given in (4.23). Finally, we conclude the result from (4.22) and (4.24) by noting $\|U_{\mathbf{k}}(f)\|_{L^2(\mathbb{R}^3)}^2 = \|u_{\mathbf{k}}^\infty[f]\|_{L^2(\mathcal{S}_{\mathbf{k}}^{(n-1)},\mathrm{d}\kappa\mathrm{d}\hat{x})}^2$. □

5. General quantitative consistency theorem

In order to prove the consistency theorems stated in Section 3, we first establish a general contraction theorem. Since our aim is to study the dependence on the key parameter $\mathbf{k}$, we need to refine [Ray13, Theorem 2.1] or [Ray15, Theorem 2.2.1] by tracking the dependence on $\mathbf{k}$ carefully. We first recall some preliminaries as in [Ray13] in the way that fits into our problem.

Let $\sigma_j(\mathbf{k}) := \sigma_j\left(G_{\mathbf{k}} : L^2(B_1) \to L^2(\mathcal{S}_{\mathbf{k}}^2, \mathrm{d}\kappa\,\mathrm{d}\hat{x})\right)$ for $j\in\mathbb{N}$, be the singular values of $G_{\mathbf{k}}$, with the singular basis $\{\varphi_j\}_{j\in\mathbb{N}}$, which forms an orthonormal basis of $\mathbb{H}_1 = L^2(B_1)$. By letting $e_j := \sigma_j^{-1}G_{\mathbf{k}}\varphi_j$, from [KRS21, Proposition 2.3] we know that $\{e_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis of $\mathbb{H}_2$ and satisfies $G_\kappa^* e_j = \sigma_j\varphi_j$, therefore we also refer $\{e_j\}_{j\in\mathbb{N}}$ as the conjugate basis to $\{\varphi_j\}_{j\in\mathbb{N}}$. We denote $\tilde{\varphi}_j := \sigma_j^{-1}e_j$ and see that $G_{\mathbf{k}}^*\tilde{\varphi}_j = \sigma_j^{-1}G_{\mathbf{k}}^*e_j = \varphi_j$, i.e. for each $f\in\mathbb{H}_1$ one has

$$\langle f,\varphi_j\rangle_1 = \langle G_{\mathbf{k}}f,\tilde{\varphi}_j\rangle \quad \text{for all } j\in\mathbb{N},$$

which means that we can express the coordinates of f in the $\{\varphi_j\}$ basis in terms of the action of $\{\tilde{\varphi}_j\}$ on $G_{\mathbf{k}}f$. This suggests us to define

$$\tilde{Y}_j := \langle f,\varphi_j\rangle_1 + \frac{1}{\sqrt{N}}\tilde{Z}_j,$$

where $\tilde{Z}_j$ are mean-zero Gaussian random variables with covariance $\mathbb{E}\tilde{Z}_j\tilde{Z}_{j'} = \delta_{jj'}$. Thus the sequence $\{\tilde{Y}_j\}$ provides an unbiased estimator of the coefficients of the true regression function f in the singular basis $\{\varphi_j\}$, and we define a canonical linear estimator of f:

$$f_N = \sum_{j=1}^{k_N(\mathbf{k})} \tilde{Y}_j\varphi_j,$$

where the integer-valued function $k_N(\mathbf{k})$ is the resolution level to be specified. Let P_k be the orthogonal projection operator onto span $\{\varphi_j : 1 \le j \le k\}$. Now we can decompose the estimator $f_{N,\mathbf{k}}$ into its bias and variance parts

$$f_{N,\mathbf{k}} = P_{k_N(\mathbf{k})}(f) + \frac{1}{\sqrt{N}} \sum_{j=1}^{k_N(\mathbf{k})} \tilde{Z}_j \varphi_j .$$

Now following the lines in the proof of the equation before [Ray13, (4.2))], which involves a version of Borell's inequality for the supremum of Gaussian processes in [Ray13, (4.1)] or [Led01, page 134] (see Borell's paper [Bor75]), given any function $L(\mathbf{k}) > 0$ and any positive function $\{\varepsilon_N(\mathbf{k})\}$, one can obtain that

$$\mathbb{P}\left(\|f_{N,\mathbf{k}} - \mathbb{E} f_{N,\mathbf{k}}\|_1 \ge \frac{1}{\sigma_{k_N(\mathbf{k})}} \left(\sqrt{2L(\mathbf{k})}\varepsilon_N(\mathbf{k}) + \sqrt{\frac{k_N(\mathbf{k})}{N}} \right) \right) \le e^{-LN\varepsilon_N^2(\mathbf{k})}.$$

If there exists a function $\ell_1(\mathbf{k}) \ge 0$, which is independent of N, such that

$$k_N(\mathbf{k}) \le \ell_1^2(\mathbf{k}) N \varepsilon_N^2(\mathbf{k}), \tag{5.1}$$

then it follows

$$\mathbb{P}\left(\|f_{N,\mathbf{k}} - \mathbb{E} f_{N,\mathbf{k}}\|_1 \ge \frac{\sqrt{2L(\mathbf{k})} + \ell_1(\mathbf{k})}{\sigma_{k_N(\mathbf{k})}(\mathbf{k})} \varepsilon_N(\mathbf{k}) \right) \le e^{-L(\mathbf{k})N\varepsilon_N^2(\mathbf{k})}. \tag{5.2}$$

Furthermore, suppose that there exists a function $\ell_2(\mathbf{k}) > 0$, which is independent of N, such that

$$\frac{\varepsilon_N(\mathbf{k})}{\sigma_{k_N}(\mathbf{k})} \le \ell_2(\mathbf{k}) \xi_N(\mathbf{k}), \tag{5.3}$$

where $\xi_N(\mathbf{k})$ is a positive function, representing the contraction rate, then from (5.2) we obtain

$$\mathbb{P}\left(\|f_{N,\mathbf{k}} - \mathbb{E} f_{N,\mathbf{k}}\|_1 \ge (\sqrt{2L(\mathbf{k})} + \ell_1(\mathbf{k}))\ell_2(\mathbf{k})\xi_N(\mathbf{k}) \right) \le e^{-L(\mathbf{k})N\varepsilon_N^2(\mathbf{k})}. \tag{5.4}$$

We now assume that there exists a function $\ell_3(\mathbf{k}) \ge 0$, which is independent of N, such that the bias of f_0 satisfies

$$\|P_{k_N(\mathbf{k})}(f_0) - f_0\|_1 \le \ell_3(\mathbf{k}) \xi_N(\mathbf{k}). \tag{5.5}$$

Given any

$$M_0(\mathbf{k}) > \ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}), \tag{5.6}$$

we consider the function[4]

$$\phi_{N,\mathbf{k}} := \mathbb{1}\left\{ \|f_{N,\mathbf{k}} - f_0\|_1 \ge M_0(\mathbf{k})\xi_N(\mathbf{k}) \right\},$$

where $\mathbb{1}A$ denotes the characteristic function of the set A. Then, by (5.3) and (5.5), we can see that the type-I error satisfies

$$\begin{aligned}
\mathbb{E}_{f_0} \phi_{N,\mathbf{k}} &= \mathbb{P}_{f_0}\left(\|f_{N,\mathbf{k}} - f_0\|_1 \ge M_0(\mathbf{k})\xi_N(\mathbf{k}) \right) \\
&\le \mathbb{P}_{f_0}\left(\|f_{N,\mathbf{k}} - \mathbb{E}_{f_0} f_{N,\mathbf{k}}\|_1 \ge M_0(\mathbf{k})\xi_N(\mathbf{k}) - \|\mathbb{E}_{f_0} f_{N,\mathbf{k}} - f_0\|_1 \right) \\
&\le \mathbb{P}_{f_0}\left(\|f_{N,\mathbf{k}} - \mathbb{E}_{f_0} f_{N,\mathbf{k}}\|_1 \ge (M_0(\mathbf{k}) - \ell_3(\mathbf{k}))\xi_N(\mathbf{k}) \right).
\end{aligned}$$

[4]Such ϕ_N is also called a "test".

Choosing

$$L(\mathbf{k}) = \frac{1}{2}\left(\overbrace{\frac{M_0(\mathbf{k}) - \ell_3(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})}^{>0 \text{ since } (5.6)}\right)^2$$

in (5.4) yields

$$\mathbb{E}_{f_0}\phi_{N,\mathbf{k}} \le \exp\left(-\frac{1}{2}\left(\frac{M_0(\mathbf{k}) - \ell_3(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})\right)^2 N\varepsilon_N^2(\mathbf{k})\right). \tag{5.7}$$

We now let

$$\mathcal{S}_N \text{ be a sequence of subsets of } \left\{f \in \mathbb{H}_1 : \|P_{k_N(\mathbf{k})}(f) - f\|_1 \le \ell_4(\mathbf{k})\xi_N(\mathbf{k})\right\} \tag{5.8}$$

for some function $\ell_4(\mathbf{k}) \ge 0$.

Example 5.1. The choice $\mathcal{S}_N := \left\{f \in \mathbb{H}_1 : f = \sum_{j=1}^{k_N(\mathbf{k})} f_j\varphi_j\right\}$ satisfies (5.8) with $\ell_4 \equiv 0$ and is valid for any choice of $\xi_N(\mathbf{k})$.

We further assume that

$$\ell_1(\mathbf{k}) < \frac{M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k})}{\ell_2(\mathbf{k})}. \tag{5.9}$$

For each $f \in \mathcal{S}_N$ with $\|f - f_0\|_1 \ge M(\mathbf{k})\xi_N(\mathbf{k})$, we see that

$$\begin{aligned}
\mathbb{E}_f(1 - \phi_{N,\mathbf{k}}) &= \mathbb{P}_f\left(\|f_{N,\mathbf{k}} - f_0\|_1 \le M_0(\mathbf{k})\xi_N(\mathbf{k})\right)\\
&\le \mathbb{P}_f\left(\|f_0 - f\|_1 - \|f - \mathbb{E}f_{N,\mathbf{k}}\|_1 - \|\mathbb{E}f_{N,\mathbf{k}} - f_{N,\mathbf{k}}\|_1 \le M_0(\mathbf{k})\xi_N(\mathbf{k})\right)\\
&= \mathbb{P}_f\left(\|\mathbb{E}f_{N,\mathbf{k}} - f_{N,\mathbf{k}}\|_1 \ge \|f_0 - f\|_1 - \|f - \mathbb{E}f_{N,\mathbf{k}}\|_1 - M_0(\mathbf{k})\xi_N(\mathbf{k})\right)\\
&\le \mathbb{P}_f\left(\|\mathbb{E}f_{N,\mathbf{k}} - f_{N,\mathbf{k}}\|_1 \ge (M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k}))\xi_N(\mathbf{k})\right)
\end{aligned}$$

It follows by choosing

$$L(\mathbf{k}) = \frac{1}{2}\left(\overbrace{\frac{M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})}^{>0 \text{ since } (5.9)}\right)^2$$

in (5.4) that

$$\mathbb{E}_f(1 - \phi_{N,\mathbf{k}}) \le \exp\left(-\frac{1}{2}\left(\frac{M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})\right)^2 N\varepsilon_N^2(\mathbf{k})\right) \tag{5.10}$$

Note that (5.6) and (5.9) can be guaranteed by the condition

$$\ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}) < M_0(\mathbf{k}) < M(\mathbf{k}) - \ell_4(\mathbf{k}) - \ell_1(\mathbf{k})\ell_2(\mathbf{k}). \tag{5.11}$$

With (5.7) and (5.10) at hand, we are ready to prove the following proposition by modifying the ideas in [Ray15, Theorem 2.5.3], which is based on [GGvdV00, Theorem 2.1].

Proposition 5.2. *Let the white noise model* (1.5) *with* $A = G_{\mathbf{k}}$ *with singular basis* $\{\varphi_j\}_{j\in\mathbb{N}}$*, which forms an orthonormal basis of* $\mathbb{H}_1 = L^2(B_1)$*. Assume that* $\{\varepsilon_N(\mathbf{k})\}$*,* $\{\xi_N(\mathbf{k})\}$ *and* $\{k_N(\mathbf{k})\}$ *are sequence of positive functions such that*

$$k_N(\mathbf{k}) \le \ell_1(\mathbf{k})^2 N\varepsilon_N(\mathbf{k})^2, \quad \frac{\varepsilon_N(\mathbf{k})}{\sigma_{k_N}(\mathbf{k})} \le \ell_2(\mathbf{k})\xi_N(\mathbf{k}), \tag{5.12a}$$

for some $\ell_1(\mathbf{k}) \ge 0$ *and* $\ell_2(\mathbf{k}) > 0$*. Assume that* Y *has law* $\mathbb{P}_{f_0}$ *with the ground truth* $f_0 \in \mathbb{H}_1$ *such that the bias of* f_0 *satisfies the following condition: there exists a function* $\ell_3(\mathbf{k}) \ge 0$ *such that*

$$\|P_{k_N(\mathbf{k})}(f_0) - f_0\|_1 \le \ell_3(\mathbf{k})\xi_N(\mathbf{k}) \quad \text{for all } N \ge 1. \tag{5.12b}$$

Let P_k *be the orthogonal projection operator onto* $\operatorname{span}\{\varphi_j : 1 \le j \le k\}$*, and let* $\mathcal{S}_N$ *be a sequence of subsets of* $\left\{f \in \mathbb{H}_1 : \|P_{k_N(\mathbf{k})}(f) - f\|_1 \le \ell_4(\mathbf{k})\xi_N(\mathbf{k})\right\}$ *for some* $\ell_4(\mathbf{k}) \ge 0$*. Suppose that there exists a constant* $C > 1$*, which is independent of* $\mathbf{k}$*, such that*

$$\Pi_N(\mathcal{S}_N^\complement) \le C\exp\left(-\frac{1}{2}\left(\frac{M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})\right)^2 N\varepsilon_N(\mathbf{k})^2\right) \tag{5.12c}$$

for all $N \ge 1$ *and with* $M_0(\mathbf{k})$ *satisfying* (5.6)*, and further there exists a function* $h(\mathbf{k}) > 0$ *satisfying*

$$\Pi_N\left(f \in \mathbb{H}_1 : \|G_{\mathbf{k}}f - G_{\mathbf{k}}f_0\|_2 \le \sqrt{2}\varepsilon_N(\mathbf{k})\right) \ge C^{-1}e^{-h(\mathbf{k})N\varepsilon_N(\mathbf{k})^2}. \tag{5.12d}$$

Then for each $M(\mathbf{k}) > 0$ *satisfying* (5.9)*, there exists a positive constant* $\tilde{C}$*, which is independent of* $\mathbf{k}$ *and* N*, such that*

$$\begin{aligned}
&\mathbb{P}_{f_0}\left(\Pi_N\left(f \in \mathbb{H}_1 : \|f - f_0\|_1 \ge M(\mathbf{k})\xi_N | Y^{(N)}\right) \ge 2\vartheta\right) \\
&\quad \le \frac{1}{g(\mathbf{k})^2 N\varepsilon_N^2(\mathbf{k})} + \vartheta^{-1}\tilde{C}\exp\left(-\tilde{M}(\mathbf{k})N\varepsilon_N(\mathbf{k})^2\right) \\
&\qquad + \vartheta^{-1}\exp\left(-\frac{1}{2}\left(\frac{M_0(\mathbf{k}) - \ell_3(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})\right)^2 N\varepsilon_N(\mathbf{k})^2\right)
\end{aligned}$$

for all positive parameter $\vartheta > 0$ *and any function*[5] $g(\mathbf{k}) > 0$*, where*

$$\tilde{M}(\mathbf{k}) = \frac{1}{2}\left(\frac{M(\mathbf{k}) - \ell_4(\mathbf{k}) - M_0(\mathbf{k})}{\ell_2(\mathbf{k})} - \ell_1(\mathbf{k})\right)^2 - 1 - g(\mathbf{k}) - h(\mathbf{k}). \tag{5.13}$$

Proof. Here we denote by $\mathbb{P}_v$ the law of the model (1.5) when the operator A equals the identity. Let $w^v = \frac{d\mathbb{P}_v}{d\mathbb{P}_0}$ be the density of $\mathbb{P}_v$ with respect to the law $\mathbb{P}_0$ of the pure white noise process. By the second equation in the proof of [Ray15, Theorem 2.5.3], we have

$$\begin{aligned}
&\Pi_N\left(f \in \mathbb{H}_1 : \|f - f_0\|_1 \ge M(\mathbf{k})\xi_N | Y\right)(1 - \phi_{N,\mathbf{k}}) \\
&\quad = \frac{\int_{f\in\mathbb{H}_1:\|f-f_0\|_1\ge M(\mathbf{k})\xi_N(\mathbf{k})} \frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)(1 - \phi_{N,\mathbf{k}})}{\int_{\mathbb{H}_1} \frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)}.
\end{aligned} \tag{5.14}$$

[5] In [GGvdV00, Ray13, Ray15], they simply choose $g(\mathbf{k}) = 1$.

Recall from [Ray15, (2.5.12)] that

$$-\mathbb{E}_{f_0}\log\left(\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}\right)=\frac{N}{2}\|G_{\mathbf{k}}f-G_{\mathbf{k}}f_0\|_2^2. \tag{5.15}$$

Denote the event

$$\mathcal{B}_N=\left\{f\in\mathbb{H}_1:\|G_{\mathbf{k}}f-G_{\mathbf{k}}f_0\|_2\leq\sqrt{2}\varepsilon_N(\mathbf{k})\right\},$$

and now the small ball condition (5.12d) reads

$$\Pi_N(\mathcal{B}_N))\geq C^{-1}e^{-h(\mathbf{k})N\varepsilon_N(\mathbf{k})^2}. \tag{5.16}$$

It was derived in the proof of [Ray15, Theorem 2.5.3] (which utilized [Ray15, Lemma 2.5.4]) that

$$\mathbb{P}_{f_0}\left(\int_{\mathcal{B}_N}\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\nu(f)\geq e^{-(1+g(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\right)\geq 1-\frac{2}{g^2(\mathbf{k})N\varepsilon_N(\mathbf{k})^2},$$

where $\nu=\frac{\Pi_N|_{\mathcal{B}_N}}{\Pi_N(\mathcal{B}_N)}$, then from the small ball condition (5.16) we have

$$\begin{aligned}1-\frac{2}{g^2(\mathbf{k})N\varepsilon_N(\mathbf{k})^2}&\leq\mathbb{P}_{f_0}\left(\int_{\mathcal{B}_N}\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\geq e^{-(1+g(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\Pi_N(\mathcal{B}_N)\right)\\&\leq\mathbb{P}_{f_0}\left(\int_{\mathcal{B}_N}\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\geq C^{-1}e^{-(1+g(\mathbf{k})+h(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\right).\end{aligned}$$

In view of this, we consider the event

$$\mathcal{A}_N=\left\{\int_{\mathcal{B}_N}\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\geq C^{-1}e^{-(1+g(\mathbf{k})+h(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\right\},$$

which satisfies

$$\mathbb{P}_{f_0}(\mathcal{A}_N^{\complement})\leq\frac{1}{g(\mathbf{k})^2N\varepsilon_N(\mathbf{k})^2}. \tag{5.17}$$

We now fix any $\vartheta>0$, and from (5.14), we can see that

$$\begin{aligned}&\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N|Y\right)(1-\phi_{N,\mathbf{k}})\geq\vartheta\right)\\&\leq\mathbb{P}_{f_0}(\mathcal{A}_N^{\complement})+\mathbb{P}_{f_0}\left(\begin{matrix}(1-\phi_{N,\mathbf{k}})\displaystyle\int_{f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\\\geq\vartheta C^{-1}e^{-(1+g(\mathbf{k})+h(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\end{matrix}\right).\end{aligned} \tag{5.18}$$

Since

$$\mathbb{E}_{f_0}\left(\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)\right)=1\quad\text{and}\quad\mathbb{E}_{f_0}\left(\frac{w^{G_{\mathbf{k}}f}}{w^{G_{\mathbf{k}}f_0}}(Y)(1-\phi_{N,\mathbf{k}})\right)=\mathbb{E}_f(1-\phi_{N,\mathbf{k}}),$$

from (5.10) and (5.12c), it follows that

$$\begin{aligned}
&\mathbb{E}_{f_0}\left((1-\phi_{N,\mathbf{k}})\int_{f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\frac{w^{G_\mathbf{k}f}}{w^{G_\mathbf{k}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\right)\\
&\quad\leq\int_{\mathcal{S}_N^\complement}\mathbb{E}_{f_0}\left(\frac{w^{G_\mathbf{k}f}}{w^{G_\mathbf{k}f_0}}\right)\mathrm{d}\Pi_N(f)+\sup_{f\in\mathcal{S}_N:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\mathbb{E}_{f_0}\left((1-\phi_{N,\mathbf{k}})\frac{w^{G_\mathbf{k}f}}{w^{G_\mathbf{k}f_0}}(Y)\right)\\
&\quad=\Pi_N(\mathcal{S}_N^\complement)+\sup_{f\in\mathcal{S}_N:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\mathbb{E}_f(1-\phi_{N,\mathbf{k}})\\
&\quad\leq(C+1)\exp\left(-\frac{1}{2}\left(\frac{M(\mathbf{k})-\ell_4(\mathbf{k})-M_0(\mathbf{k})}{\ell_2(\mathbf{k})}-\ell_1(\mathbf{k})\right)^2N\varepsilon_N(\mathbf{k})^2\right).
\end{aligned}$$

Now using Markov's inequality yields

$$\begin{aligned}
&\mathbb{P}_{f_0}\begin{pmatrix}(1-\phi_{N,\mathbf{k}})\int_{f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\frac{w^{G_\mathbf{k}f}}{w^{G_\mathbf{k}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\\ \geq\vartheta C^{-1}e^{-(1+g(\mathbf{k})+h(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}\end{pmatrix}\\
&\qquad\leq\frac{\mathbb{E}_{f_0}\left((1-\phi_{N,\mathbf{k}})\int_{f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})}\frac{w^{G_\mathbf{k}f}}{w^{G_\mathbf{k}f_0}}(Y)\,\mathrm{d}\Pi_N(f)\right)}{\vartheta C^{-1}e^{-(1+g(\mathbf{k})+h(\mathbf{k}))N\varepsilon_N(\mathbf{k})^2}}\\
&\qquad\leq\vartheta^{-1}C(C+1)\exp\left(-\tilde{M}(\mathbf{k})N\varepsilon_N(\mathbf{k})^2\right)
\end{aligned}$$

where $\tilde{M}$ is the function defined in (5.13). Combining the above inequality with (5.17) and (5.18), we have

$$\begin{aligned}
&\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N|Y\right)(1-\phi_{N,\mathbf{k}})\geq\vartheta\right)\\
&\qquad\leq\frac{1}{g(\mathbf{k})^2N\varepsilon_N(\mathbf{k})^2}+\vartheta^{-1}C(C+1)\exp\left(-\tilde{M}(\mathbf{k})N\varepsilon_N(\mathbf{k})^2\right).
\end{aligned}\tag{5.19}$$

Again, by Markov's inequality and (5.7), we obtain

$$\begin{aligned}
&\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})|Y\right)\phi_{N,\mathbf{k}}\geq\vartheta\right)\\
&\quad\leq\vartheta^{-1}\mathbb{E}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})|Y\right)\phi_{N,\mathbf{k}}\right)\leq\mathbb{E}_{f_0}\phi_{N,\mathbf{k}}\\
&\quad\leq\vartheta^{-1}\exp\left(-\frac{1}{2}\left(\frac{M_0(\mathbf{k})-\ell_3(\mathbf{k})}{\ell_2(\mathbf{k})}-\ell_1(\mathbf{k})\right)^2N\varepsilon_N(\mathbf{k})^2\right).
\end{aligned}\tag{5.20}$$

Finally, putting together (5.19) and (5.20) yields

$$\begin{aligned}
&\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})|Y\right)\geq 2\vartheta\right)\\
&\quad\leq\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})|Y\right)(1-\phi_{N,\mathbf{k}})\geq\vartheta\right)\\
&\qquad+\mathbb{P}_{f_0}\left(\Pi_N\left(f\in\mathbb{H}_1:\|f-f_0\|_1\geq M(\mathbf{k})\xi_N(\mathbf{k})|Y\right)\phi_{N,\mathbf{k}}\geq\vartheta\right)\\
&\quad\leq\frac{1}{g(\mathbf{k})^2N\varepsilon_N(\mathbf{k})^2}+\vartheta^{-1}C(C+1)\exp\left(-\tilde{M}(\mathbf{k})N\varepsilon_N(\mathbf{k})^2\right)\\
&\qquad+\vartheta^{-1}\exp\left(-\frac{1}{2}\left(\frac{M_0(\mathbf{k})-\ell_3(\mathbf{k})}{\ell_2(\mathbf{k})}-\ell_1(\mathbf{k})\right)^2N\varepsilon_N(\mathbf{k})^2\right),
\end{aligned}$$

which concludes the proposition. □

6. Proofs of theorems in Section 3.1

By imitating some ideas in [Ray13, Proposition 3.1], we prove Theorem 3.1 by verifying conditions in Proposition 5.2.

Proof of Theorem 3.1. First of all, we consider the choice $\mathcal{S}_N$ in Example 5.1, which allows us to choose $\ell_4(\mathbf{k}) = 0$ in Proposition 5.2. We now verify the small ball condition (5.12d). Let f be distributed according to Π, conditioned on $J = j_0$, and $\mathbb{P}$ be the corresponding probability measure. It is easy to see that

$$\begin{aligned}\mathbb{P}\left(\|G_{\mathbf{k}} f - G_{\mathbf{k}} f_0\|_2 \leq \sqrt{2}\varepsilon_N(\mathbf{k})\right) &= \mathbb{P}\left(\sum_{j=1}^{j_0} |f_j - f_{0,j}|^2 \sigma_j^2 \leq 2\varepsilon_N(\mathbf{k})^2\right) \\ &\geq \mathbb{P}\left(|f_j - f_{0,j}|^2 \sigma_j^2 \leq \frac{2\varepsilon_N(\mathbf{k})^2}{j_0}, j = 1, \cdots, j_0\right) = \prod_{j=1}^{j_0} \mathbb{P}\left(|f_j - f_{0,j}| \leq \eta_{N,j}\right)\end{aligned} \tag{6.1}$$

by the independence of f_j's, where

$$\eta_{N,j} = \frac{\sqrt{2}\varepsilon_N}{\sqrt{j_0}\sigma_j}.$$

If $X : \mathbb{C} \to [0, \infty)$ is a complex-valued random variable with density q satisfying Assumption 1, then we obtain that for all $z \in \mathbb{C}$ and $t > 0$

$$\mathbb{P}(|X - z| \leq t) \geq 2\pi D t e^{-d(|z|+t)^\beta} \tag{6.2}$$

from [Ray13, (5.2)]. On the other hand, since $j_0 < (\mathbf{k} - \frac{3}{2})^2 \leq C'\mathbf{k}^3$, by (2.13) and (2.14) we can see that

$$c^{-1}\varepsilon_N \leq \eta_{N,j} \leq c\mathbf{k}^{\frac{4}{3}}\varepsilon_N \quad \text{for } \ 1 \leq j \leq j_0$$

for some $c = c(j_0) > 1$. Using (6.2)[6] and noting that $(a + b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta)$ for all $a, b \geq 0$ and $\beta \geq 1$, the right hand side of (6.1) is bounded below by

$$\begin{aligned}&\prod_{j=1}^{j_0} 2\pi D \eta_{N,j} \tau_j^{-1} \exp(-d\tau_j^{-\beta} 2^{\beta-1}(|f_{0,j}|^\beta + \eta_{N,j}^\beta)) \\ &= 2\pi D \exp\left(\sum_{j=1}^{j_0}\left[\log\left(\frac{\eta_{N,j}}{\tau_j}\right) - d\tau_j^{-\beta} 2^{\beta-1}(|f_{0,j}|^\beta + \eta_{N,j}^\beta)\right]\right) \\ &\geq 2\pi D \exp\left(j_0\left(\log\varepsilon_N - C_1(1 + \mathbf{k}^{\frac{4\beta}{3}}\varepsilon_N^\beta)\right)\right)\end{aligned} \tag{6.3}$$

for some $C_1 = C_1(j_0, d, \beta, f_0, \tau_1, \cdots, \tau_{j_0}) > 1$. We now choose

$$\varepsilon_N = \varepsilon_N(\mathbf{k}) = \mathbf{k}^{-\frac{4}{3}}\left(\frac{\log(\mathbf{k}N)}{N}\right)^{1/2}$$

[6] with choices $X = \tau_j^{-1} f_j$ (to ensure it has density q), $z = \tau_j^{-1} f_{0,j}$ and $t = \tau_j^{-1}\eta_{N,j}$.

and hence $\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2 = \log(\mathbf{k}N)$ and

$$\begin{aligned}
&\varepsilon_N \exp\left(-C_1(1+\mathbf{k}^{\frac{4\beta}{3}}\varepsilon_N^{\beta})\right) \\
&\quad = \mathbf{k}^{-\frac{4}{3}} \left(\frac{\log(\mathbf{k}N)}{N}\right)^{\frac{1}{2}} \exp\left(-C_1\left(1+\left(\frac{\log(\mathbf{k}N)}{N}\right)^{\frac{\beta}{2}}\right)\right) \\
&\quad \geq C_2 \mathbf{k}^{-\frac{4}{3}} \left(\frac{\log(\mathbf{k}N)}{N}\right)^{\frac{1}{2}} \quad \text{for all sufficiently large } N \gtrsim \log \mathbf{k},
\end{aligned}$$

since

$$\frac{\log(\mathbf{k}N)}{N} = \frac{\log N}{N} + \frac{\log \mathbf{k}}{N} \lesssim 1, \quad \forall \;\; N \gtrsim \log \mathbf{k}.$$

This implies

$$\begin{aligned}
&\log\left(\varepsilon_N \exp\left(-C_1(1+\mathbf{k}^{\frac{4\beta}{3}}\varepsilon_N^{\beta})\right)\right) \\
&\quad \geq \log C_2 - \frac{4}{3}\log \mathbf{k} + \frac{1}{2}\log\log N - \frac{1}{2}\log N \\
&\quad \gtrsim -\log(\mathbf{k}N) = -\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2.
\end{aligned}$$

Next, from (6.1) and (6.3), it follows that

$$\mathbb{P}\left(\|G_{\mathbf{k}}f - G_{\mathbf{k}}f_0\|_2 \leq \sqrt{2}\varepsilon_N\right) \geq 2\pi D \exp\left(-C_2\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2\right)$$

for all sufficiently large $N \gtrsim \log \mathbf{k}$. Now since j_0 is fixed and $\mathbf{m}(j_0) > 0$, we can see that

$$\begin{aligned}
&\Pi_N\left(f \in \mathbb{H}_1 : \|G_{\mathbf{k}}f - G_{\mathbf{k}}f_0\|_2 \leq \varepsilon_N\right) \\
&\quad \geq 2\pi D\mathbf{m}(j_0)\exp\left(-C_2\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2\right),
\end{aligned}$$

which verifies (5.12d) with $h(\mathbf{k}) = C_2\mathbf{k}^{\frac{8}{3}}$.

In order to fulfill the first condition in (5.12a), let $L > 1$ be a constant to be determined later and take $\ell_1(\mathbf{k}) = \sqrt{2L}\mathbf{k}^{\frac{4}{3}}$ and

$$k_N = k_N(\mathbf{k}) := \lceil L\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2 \rceil = \lceil L\log(\mathbf{k}N)\rceil \leq 2L\log(\mathbf{k}N) = 2L\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2.$$

Since $\|P_{k_N}(f_0) - f_0\|_1 = 0$ for all $N \geq \exp(j_0)$, it immediately implies (5.12b) with $\ell_3(\mathbf{k}) = 0$ and is valid for arbitrary choices of $M(\mathbf{k})$ and $M_0(\mathbf{k})$. In view of the upper bound on $\mathbf{m}$, we then have

$$\Pi_N(\mathcal{S}_N^{\complement}) \lesssim e^{-bk_N} \leq \exp\left(-bL\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2\right),$$

which guarantees (5.12c) with $M(\mathbf{k}) = \ell_2(\mathbf{k})(1+\sqrt{b})\sqrt{2L}\mathbf{k}^{\frac{4}{3}} + M_0(\mathbf{k})$. Now we simply choose $M_0(\mathbf{k}) = 2\ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k})$ to verify (5.11). From (5.13), we can choose $g(\mathbf{k}) = \mathbf{k}^{\frac{8}{3}}$ and $L = \frac{1}{b}(C_2+2) + \frac{1}{b}$ to see that

$$\tilde{M}(\mathbf{k}) = bL\mathbf{k}^{\frac{8}{3}} - 1 - \mathbf{k}^{\frac{8}{3}} - C_2\mathbf{k}^{\frac{8}{3}} > \mathbf{k}^{\frac{8}{3}}.$$

In order to fulfill the second condition in (5.12a), one shall choose $\ell_2(\mathbf{k}) = 1$ and set $\tilde{\xi}_N = \frac{\varepsilon_N}{\sigma_{k_N}}$. Finally, (2.14) implies

$$\tilde{\xi}_N(\mathbf{k}) \lesssim \left(\frac{\log(\mathbf{k}N)}{N}\right)^{1/2} \cdot \begin{cases} 1 & \text{if } k_N < \left(\mathbf{k} - \frac{3}{2}\right)^2, \\ \exp\left(3k_N^{\frac{1}{2}} \log k_N\right) & \text{if } k_N \geq \left(\mathbf{k} - \frac{3}{2}\right)^2, \end{cases}$$

$$= \left(\frac{\log(\mathbf{k}N)}{N}\right)^{1/2} \cdot \begin{cases} 1 \quad \text{if } \lceil L\log(\mathbf{k}N)\rceil < \left(\mathbf{k} - \frac{3}{2}\right)^2, \\ \exp\left(3\lceil L\log(\mathbf{k}N)\rceil^{\frac{1}{2}} \log(\lceil L\log(\mathbf{k}N)\rceil)\right) \\ \quad \text{if } \lceil L\log(\mathbf{k}N)\rceil \geq \left(\mathbf{k} - \frac{3}{2}\right)^2. \end{cases}$$

We now use Proposition 5.2 to conclude

$$\begin{aligned} &\mathbb{P}_{f_0}\left(\Pi_N\left(f \in \mathbb{H}_1 : \|f - f_0\|_1 \geq M(\mathbf{k})\tilde{\xi}_N(\mathbf{k}) | Y^{(N)}\right) \geq \vartheta\right) \\ &\lesssim (\log(\mathbf{k}N))^{-1} + \exp\left(-\log(\mathbf{k}N)\right) + \exp\left(-\frac{1}{2}bL\log(\mathbf{k}N)\right) \\ &\lesssim (\log(\mathbf{k}N))^{-1}. \end{aligned}$$

Note that we can write

$$M(\mathbf{k})\tilde{\xi}_N(\mathbf{k}) = M\xi_N$$

for some constant $M > 0$, which is independent of $\mathbf{k}$, where ξ_N is given as (3.7). The proof of Theorem 3.1 is now completed. □

The strategy for proving Theorem 3.2 is essentially same as Theorem 3.1.

Proof of Theorem 3.2. First of all, we will verify the small-ball condition (5.12d). By the triangle inequality

$$\|G_{\mathbf{k}}f - G_{\mathbf{k}}f_0\|_2 \leq \|G_{\mathbf{k}}f - G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0))\|_2 + \|G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0)) - G_{\mathbf{k}}f_0\|_2,$$

where $k_N(\mathbf{k}) \to \infty$ will be specified later. By (2.13), we see that there exists an absolute constant $C_* > 1$, which is independent of $\mathbf{k}$ and j, such that $\sigma_j \leq C_*$ for all $j \in \mathbb{N}$. For $f_0 \in H^s_{\exp}$, we can estimate

$$\begin{aligned} &\|G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0)) - G_{\mathbf{k}}f_0\|_2^2 \\ &= \sum_{j=k_N(\mathbf{k})+1}^{\infty} \sigma_j^2 j^{-2s} \exp\left(-6j^{\frac{1}{2}}\log j\right) j^{2s} \exp\left(6j^{\frac{1}{2}}\log j\right) |f_{0,j}|^2 \\ &\leq C_*^2 k_N(\mathbf{k})^{-2s} \exp\left(-6k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right) \|f_0\|^2_{H^s_{\exp}}. \end{aligned}$$

Let $k_N(\mathbf{k})$ be an integer larger than e^2 and define

$$\varepsilon_N(\mathbf{k}) := 2C_* k_N(\mathbf{k})^{-s} \exp\left(-3k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right) \|f_0\|_{H^s_{\exp}}. \tag{6.4}$$

Consider f as a finite series of $\{\varphi_j\}$ of degree $k_N(\mathbf{k})$, i.e.

$$f = \sum_{j=1}^{k_N(\mathbf{k})} f_j \varphi_j.$$

Similar to the derivations of (6.1) and (6.3), using (6.2), we obtain

$$\begin{aligned}
&\mathbb{P}\left(\|G_{\mathbf{k}}f - G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0))\|_2 \le \frac{\varepsilon_N(\mathbf{k})}{2}\right) \\
&\ge \prod_{j=1}^{k_N(\mathbf{k})} \mathbb{P}\left(|f_j - f_{0,j}| \le \frac{\varepsilon_N(\mathbf{k})}{2\sigma_j\sqrt{k_N(\mathbf{k})}}\right) \\
&\ge 2\pi D \exp\left(\sum_{j=1}^{k_N(\mathbf{k})} \left[\log\left(\frac{\tilde{\eta}_{N,j}}{\tau_j}\right) - 2d\tau_j^{-2}(|f_{0,j}|^2 + \tilde{\eta}_{N,j}^2)\right]\right),
\end{aligned} \tag{6.5}$$

where $\tilde{\eta}_{N,j} = \frac{\varepsilon_N(\mathbf{k})}{2\sigma_j\sqrt{k_N(\mathbf{k})}}$. By (3.8) and (6.4), for each $j = 1, \cdots, k_N(\mathbf{k})$, we see that

$$\begin{aligned}
\frac{\tilde{\eta}_{N,j}}{\tau_j} &\ge k_N(\mathbf{k})^{-\frac{1}{2}} k_N(\mathbf{k})^{-s} \exp\left(-3k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right) \|f_0\|_{H^s_{\exp}} \tau_j^{-1} \\
&\gtrsim k_N(\mathbf{k})^{-\frac{1+2s}{2}} \exp\left(-3k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right),
\end{aligned}$$

and hence

$$\sum_{j=1}^{k_N(\mathbf{k})} \log\left(\frac{\tilde{\eta}_{N,j}}{\tau_j}\right) \ge -Ck_N(\mathbf{k})^{\frac{3}{2}} \log k_N(\mathbf{k}) \ge -C\left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^3 \tag{6.6}$$

for some constant $C > 0$, which is independent of $\mathbf{k}$.

Next, from (3.8), it is easy to see that

$$\sum_{j=1}^{k_N(\mathbf{k})} \tau_j^{-2}|f_{0,j}|^2 \lesssim \sum_{j=1}^{k_N(\mathbf{k})} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} |f_{0,j}|^2 \lesssim \sum_{j=1}^{\infty} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} |f_{0,j}|^2 \lesssim \|f_0\|^2_{H^s_{\exp}}. \tag{6.7}$$

On the other hand, it follows from (2.14) that

$$\frac{\tilde{\eta}_{N,j}^2}{\tau_j^2} \lesssim \begin{cases} \mathbf{k}^{\frac{8}{3}} \dfrac{\varepsilon_N(\mathbf{k})^2}{k_N(\mathbf{k})} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} & \text{for all } j < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ \mathbf{k}^{\frac{8}{3}} \dfrac{\varepsilon_N(\mathbf{k})^2}{k_N(\mathbf{k})} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} \exp\left(6j^{\frac{1}{2}} \log j\right) & \text{for all } j \ge \left(\mathbf{k} - \dfrac{3}{2}\right)^2. \end{cases}$$

From above, we then have that

$$
\sum_{j=1}^{k_N(\mathbf{k})} \frac{\tilde{\eta}_{N,j}^2}{\tau_j^2} \lesssim \begin{cases} \sum_{j=1}^{k_N(\mathbf{k})} \mathbf{k}^{\frac{8}{3}} \frac{\varepsilon_N(\mathbf{k})^2}{k_N(\mathbf{k})} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} & \text{if } \ k_N(\mathbf{k}) < \left(\mathbf{k} - \frac{3}{2}\right)^2, \\ \sum_{j=1}^{k_N(\mathbf{k})} \mathbf{k}^{\frac{8}{3}} \frac{\varepsilon_N(\mathbf{k})^2}{k_N(\mathbf{k})} \left(j^{\frac{1}{2}} \log j\right)^{3+\delta} \exp\left(6 j^{\frac{1}{2}} \log j\right) & \\ \qquad \text{if } \ k_N(\mathbf{k}) \geq \left(\mathbf{k} - \frac{3}{2}\right)^2, \end{cases}
\tag{6.8}
$$

$$
\lesssim \begin{cases} \mathbf{k}^{\frac{8}{3}} \varepsilon_N(\mathbf{k})^2 \left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^{3+\delta} & \text{if } \ k_N(\mathbf{k}) < \left(\mathbf{k} - \frac{3}{2}\right)^2, \\ \mathbf{k}^{\frac{8}{3}} \varepsilon_N(\mathbf{k})^2 \left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^{3+\delta} \exp\left(6 k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right) & \\ \qquad \text{if } \ k_N(\mathbf{k}) \geq \left(\mathbf{k} - \frac{3}{2}\right)^2, \end{cases}
$$

where the implied constant is independent of $\mathbf{k}$, $\varepsilon_N(\mathbf{k})$ and $k_N(\mathbf{k})$.

Now substituting (6.6), (6.7) and (6.8) into (6.5) and noting that $k_N(\mathbf{k})$ is the smallest integer satisfying (3.11), we can see that:

Case 1. If $k_N(\mathbf{k}) \geq (\mathbf{k} - \frac{3}{2})^2$, then

$$
\begin{aligned}
&\mathbb{P}\left(\|G_{\mathbf{k}} f - G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0))\|_2 \leq \frac{\varepsilon_N(\mathbf{k})}{2} \right) \\
&\geq 2\pi D \exp\left(\begin{aligned} &- C \left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^3 \times \\ &\times \left(1 + \mathbf{k}^{\frac{8}{3}} \left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^{\delta} \varepsilon_N(\mathbf{k})^2 \exp\left(6 k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)\right) \end{aligned} \right) \\
&\geq 2\pi D \exp\left(-C \left(k_N(\mathbf{k})^{\frac{1}{2}} \log k_N(\mathbf{k})\right)^3 \left(1 + \mathbf{k}^{\frac{8}{3}} k_N(\mathbf{k})^{-2s+\frac{\delta}{2}} \left(\log k_N(\mathbf{k})\right)^{\delta}\right)\right) \\
&\geq 2\pi D \exp\left(C_0 \left(\log \varepsilon_N(\mathbf{k})\right)^3\right),
\end{aligned}
$$

for some $C_0 > 1$, since $\mathbf{k}^{\frac{8}{3}} k_N(\mathbf{k})^{-2s+\frac{\delta}{2}} \left(\log k_N(\mathbf{k})\right)^{\delta} \leq 1$, which is guaranteed by $k_N(\mathbf{k}) \geq \left(\mathbf{k} - \frac{3}{2}\right)^2 > \mathbf{k}^{\frac{16}{3(4s-\delta-\epsilon)}}$ for some suitable chosen $\epsilon > 0$ depends on s and δ due to $s > \frac{2}{3} + \frac{\delta}{4}$.

We now choose $\varepsilon_N = \varepsilon_N(\mathbf{k})$ satisfying

$$
\varepsilon_N(\mathbf{k}) \simeq \mathbf{k}^{-\frac{4}{3}} \left(\frac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}},
\tag{6.9}
$$

i.e.,

$$
\mathbf{k}^{\frac{8}{3}} N \varepsilon_N^2 \simeq (\log(\mathbf{k}N))^3.
$$

Thus, we have

$$
(\log \varepsilon_N)^3 \simeq \left(-\log(\mathbf{k}N) + \log\left((\log(\mathbf{k}N))^3\right)\right)^3 \simeq -(\log(\mathbf{k}N))^3 \simeq -\mathbf{k}^{\frac{8}{3}} N \varepsilon_N^2,
$$

for all N large. This choice yields

$$(\log(\mathbf{k}N))^3 \simeq \mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2 \simeq (-\log\varepsilon_N)^3 \simeq \left(k_N^{\frac{1}{2}}\log k_N\right)^3 \gtrsim k_N, \tag{6.10}$$

which verifies the first condition of (5.12a) with $\ell_1(\mathbf{k}) = \mathbf{k}^{\frac{4}{3}}$, and

$$\mathbb{P}\left(\|G_{\mathbf{k}}f - G_{\mathbf{k}}(P_{k_N}(f_0))\|_2 \le \frac{\varepsilon_N}{2}\right) \gtrsim \exp\left(-C_1\mathbf{k}^{\frac{8}{3}}N\varepsilon_N^2\right). \tag{6.11}$$

Case 2. If $k_N < (\mathbf{k} - \frac{3}{2})^2$, then we again consider the choice (6.9) and use (6.10) to see that

$$\begin{aligned}
&\mathbb{P}\left(\|G_{\mathbf{k}}f - G_{\mathbf{k}}(P_{k_N(\mathbf{k})}(f_0))\|_2 \le \frac{\varepsilon_N}{2}\right)\\
&\ge 2\pi D\exp\left(-C\left(\left(k_N^{\frac{1}{2}}\log k_N\right)^3\left(1 + \mathbf{k}^{\frac{8}{3}}\varepsilon_N^2\left(k_N^{\frac{1}{2}}\log k_N\right)^{\delta}\right)\right)\right)\\
&\ge 2\pi D\exp\left(-C'\left(\left(k_N^{\frac{1}{2}}\log k_N\right)^3\left(1 + \frac{(\log(\mathbf{k}N))^{3+\delta}}{N}\right)\right)\right)\\
&\ge 2\pi D\exp\left(-2C'\left(k_N^{\frac{1}{2}}\log k_N\right)^3\right)
\end{aligned}$$

for all sufficiently large $N \gtrsim (\log\mathbf{k})^{3+\delta}$. This again implies (6.11) for all $N \gtrsim (\log\mathbf{k})^{3+\delta}$.

From (6.4), it is readily seen that

$$\|G_{\mathbf{k}}(P_{k_N}(f_0)) - G_{\mathbf{k}}f_0\|_2 \le \frac{1}{2}\varepsilon_N,$$

and consequently by (6.11), (3.9) and (6.10), we obtain that

$$\begin{aligned}
&\Pi_N\left(f \in \mathbb{H}_1 : \|G_{\mathbf{k}}f - G_{\mathbf{k}}f_0\|_2 \le \varepsilon_N\right)\\
&\ge \Pi_N\left(f \in \mathbb{H}_1 : \|G_{\mathbf{k}}(P_{k_N}(f_0)) - G_{\mathbf{k}}f\|_2 \le \frac{1}{2}\varepsilon_N\right)\\
&\gtrsim \mathbf{m}(k_N)\exp\left(-C_1\mathbf{k}^{\frac{8}{3}}N\varepsilon_N^2\right) \gtrsim \exp\left(-b\left(k_N^{\frac{1}{2}}\log k_N\right)^3\right)\exp\left(-C_1\mathbf{k}^{\frac{8}{3}}N\varepsilon_N^2\right)\\
&\gtrsim \exp(-h_*\mathbf{k}^{\frac{8}{3}}N\varepsilon_N^2)
\end{aligned}$$

and thus the small ball condition (5.12d) is satisfied with $h(\mathbf{k}) = \mathbf{k}^{\frac{8}{3}}h_*$ for some positive constant h_*, which is independent of $\mathbf{k}$.

Noting from (6.4) and (6.10) that

$$\log(\mathbf{k}N) \simeq -\log\varepsilon_N \simeq k_N^{\frac{1}{2}}\log k_N \quad \text{for all large } N \gtrsim (\log\mathbf{k})^{3+\delta}. \tag{6.12}$$

In order to fulfill the second condition in (5.12a), we will choose $\ell_2(\mathbf{k}) = \ell_2$ for some $\mathbf{k}$-independent constant $\ell_2 > 0$, to be determined later, and in view of (2.14) and (6.9), we can see that

$$\frac{1}{\ell_2}\frac{\varepsilon_N}{\sigma_{k_N}} \lesssim \frac{1}{\ell_2}\cdot\begin{cases}\mathbf{k}^{\frac{4}{3}}\varepsilon_N & \text{if } k_N < \left(\mathbf{k}-\frac{3}{2}\right)^2,\\ \mathbf{k}^{\frac{4}{3}}\exp\left(3k_N^{\frac{1}{2}}\log k_N\right)\varepsilon_N & \text{if } k_N \geq \left(\mathbf{k}-\frac{3}{2}\right)^2,\end{cases}$$
$$\leq \frac{1}{\ell_2}\cdot\begin{cases}\mathbf{k}^{\frac{4}{3}}\varepsilon_N & \text{if } k_N^{\frac{1}{2}}\log k_N < \alpha\log\mathbf{k},\\ \mathbf{k}^{\frac{4}{3}}\exp\left(3k_N^{\frac{1}{2}}\log k_N\right)\varepsilon_N & \text{if } k_N^{\frac{1}{2}}\log k_N \geq \alpha\log\mathbf{k},\end{cases}$$
$$\simeq \frac{1}{\ell_2}\cdot\begin{cases}\left(\frac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}} & \text{if } k_N^{\frac{1}{2}}\log k_N < \alpha\log\mathbf{k},\\ \mathbf{k}^{\frac{4}{3}}k_N^{-s} & \text{if } k_N^{\frac{1}{2}}\log k_N \geq \alpha\log\mathbf{k},\end{cases}$$
$$\lesssim \xi_N(\mathbf{k}) := \frac{1}{\ell_2}\cdot\begin{cases}\left(\frac{(\log(\mathbf{k}N))^3}{N}\right)^{\frac{1}{2}} & \text{if } k_N^{\frac{1}{2}}\log k_N < \alpha\log\mathbf{k},\\ \mathbf{k}^{\frac{4}{3}}k_N^{-\frac{\delta}{4}} & \text{if } k_N^{\frac{1}{2}}\log k_N \geq \alpha\log\mathbf{k},\end{cases}$$

for any $\alpha > 1$. Taking $\ell_2 = \frac{2}{C_*}$ yields that

$$\|P_{k_N}(f_0) - f_0\|_1 \leq \frac{1}{2}\varepsilon_N \leq \frac{C_*}{2}\ell_2\xi_N = \xi_N,$$

which verifies (5.12b) with $\ell_3(\mathbf{k}) = 1$ and we see that

$$\ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}) = \mathbf{k}^{\frac{4}{3}}\ell_2 + 1.$$

In this case, we will not consider the choice described in Example 5.1. Argue as in the proof of [Ray13, Proposition 3.4], for $f \in \operatorname{supp}(\Pi_m)$ and by the Karhunen-Loéve expansion, we can express

$$\|P_{k_N}(f) - f\|_1 = \sup_{h\in\mathbb{B}_0}\mathcal{G}_N(h), \tag{6.13}$$

where $\mathbb{B}_0$ is a weak $*$ dense subset of $\{h \in \mathbb{H}_1 : \|h\|_1 \leq 1\}$ and $\mathcal{G}_N$ is the Gaussian process

$$\mathcal{G}_N(h) = \langle h, P_{k_N}(f) - f\rangle_1 = \sum_{j=k_N+1}^{\infty}\tau_j\zeta_j\langle h,\varphi_j\rangle_1,$$

where $\{\zeta_k\}$ are iid standard normal random variables. Applying Jenson's inequality to the bias, using (3.8) and (6.10), as well as the inequality $\sum_{j=k_N+1}^{\infty} j^{-w} \leq \frac{k_N^{1-w}}{w-1}$ for any $w > 1$, we obtain the bias estimate

$$\begin{aligned}\mathbb{E}\|P_{k_N}(f) - f\|_1 &\leq \left(\sum_{j=k_N+1}^{\infty}\tau_j^2\right)^{\frac{1}{2}} \simeq \left(\sum_{j=k_N+1}^{\infty}\left(j^{\frac{1}{2}}\log j\right)^{-(3+\delta)}\right)^{\frac{1}{2}}\\ &\lesssim \left(\sum_{j=k_N+1}^{\infty} j^{-\frac{3+\delta}{2}}\right)^{\frac{1}{2}}(\log k_N)^{-\frac{3+\delta}{2}} \lesssim \left(k_N^{\frac{1}{2}}\log k_N\right)^{-\frac{1+\delta}{2}} \simeq (\log(\mathbf{k}N))^{-\frac{1+\delta}{2}}.\end{aligned} \tag{6.14}$$

For the variance, note that for any $h \in \mathbb{B}_0$,

$$\begin{aligned}\mathbb{E}\mathcal{G}_N(h)^2 &= \sum_{j=k_N+1}^{\infty} \tau_j^2 |\langle h, \varphi_j\rangle_1|^2 \le \tau_{k_N+1}^2 \|h\|_1^2 \le \tau_{k_N}^2 \\ &\lesssim \left(k_N^{\frac{1}{2}} \log k_N\right)^{-(3+\delta)} \simeq \mathbf{k}^{-\frac{8}{3}} \left(N\varepsilon_N^2\right)^{-1} (\log(\mathbf{k}N))^{-\delta}.\end{aligned} \tag{6.15}$$

In view of (6.13), (6.14) and (6.15), applying the version of Borell's inequality for the supremum of Gaussian process in [Led01, page 134] (similar to [Ray13, (4.1)]) gives

$$\begin{aligned}&\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \ge x + C_1(\log(\mathbf{k}N))^{-\frac{1+\delta}{2}}\right) \\ &\le \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 - \mathbb{E}\|P_{k_N}(f) - f\|_1 \ge x\right) \\ &= \mathbb{P}\left(\sup_{h\in\mathbb{B}_0} \mathcal{G}_N(h) - \mathbb{E}\sup_{h\in\mathbb{B}_0} \mathcal{G}_N(h) \ge x\right) \\ &\le \exp\left(-\frac{x^2}{2\sup_{h\in\mathcal{B}_0} \mathbb{E}\mathcal{G}_N(h)^2}\right) \le \exp\left(-\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2 \frac{(\log(\mathbf{k}N))^{\delta}}{2C_2} x^2\right) \quad \text{for all } x > 0.\end{aligned}$$

For any $L > 1$, we choose $x = \sqrt{2C_2L(\log(\mathbf{k}N))^{-\delta}}$ in the inequality above to see that

$$\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \ge \sqrt{2C_2L}(\log(\mathbf{k}N))^{-\frac{\delta}{2}} + C_1(\log(\mathbf{k}N))^{-\frac{1+\delta}{2}}\right) \le \exp\left(-\mathbf{k}^{\frac{8}{3}} L N\varepsilon_N^2\right). \tag{6.16}$$

We now discuss the following two cases.

Case 1. If $k_N^{\frac{1}{2}} \log k_N \ge \alpha \log \mathbf{k}$, then by (6.12) we have

$$\xi_N \gtrsim k_N^{-\frac{\delta}{4}} = \left(k_N^{\frac{1}{2}} \log k_N\right)^{-\frac{\delta}{2}} (\log k_N)^{\frac{\delta}{2}} \simeq (\log(\mathbf{k}N))^{-\frac{\delta}{2}} (\log k_N)^{\frac{\delta}{2}}.$$

For each sufficiently large $N \gtrsim (\log \mathbf{k})^{3+\delta}$ satisfying $(\log k_N)^{\frac{\delta}{2}} \gtrsim \sqrt{L}$, we obtain from (6.16) that

$$\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \ge \xi_N\right) \le \exp\left(-\mathbf{k}^{\frac{8}{3}} L N\varepsilon_N^2\right).$$

By considering $\ell_4(\mathbf{k}) = 1$ and

$$\mathcal{S}_N := \left\{f \in \mathbb{H}_1 : \|P_{k_N}(f) - f\|_1 \le \xi_N\right\},$$

we see that

$$\Pi_N(\mathcal{S}_N^{\complement}) \le \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \ge \xi_N\right) \cdot \sum_{\ell=k_N+1}^{\infty} \mathbf{m}(\ell) \lesssim \exp\left(-(b' + L)\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2\right),$$

which verifies (5.12c) with $M(\mathbf{k}) = (\sqrt{2(b' + L)} + 1)\ell_2\mathbf{k}^{\frac{4}{3}} + 1 + M_0(\mathbf{k}) > 0$. We now take $M_0(\mathbf{k}) = \mathbf{k}^{\frac{4}{3}}\ell_2 + 2$ and choose a sufficiently large L, if necessary, to see that

$$\begin{aligned}M(\mathbf{k}) - \ell_4(\mathbf{k}) - \ell_1(\mathbf{k})\ell_2(\mathbf{k}) &= \ell_2\mathbf{k}^{\frac{4}{3}}\sqrt{2(b' + L)} + M_0(\mathbf{k}) > M_0(\mathbf{k}) = 2 + \mathbf{k}^{\frac{4}{3}}\ell_2 \\ &> 1 + \mathbf{k}^{\frac{4}{3}}\ell_2 = \ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}),\end{aligned}$$

which implies (5.11). Next, setting $g(\mathbf{k}) = \mathbf{k}^{\frac{8}{3}}$, it follows from (5.13) that

$$\tilde{M}(\mathbf{k}) = (b' + L)\mathbf{k}^{\frac{8}{3}} - 1 - \mathbf{k}^{\frac{8}{3}} - h_*\mathbf{k}^{\frac{8}{3}} \ge \mathbf{k}^{\frac{8}{3}},$$

which holds true by (possibly) replacing L with the larger one. Note that the choice of L is independent of $\mathbf{k}$. For each positive parameter $\vartheta > 0$, we can use Proposition 5.2 to conclude that

$$\mathbb{P}_{f_0}\left(\Pi_N(f \in \mathbb{H}_1 : \|f - f_0\|_1 \geq M\xi_N(\mathbf{k})|Y^{(N)}) \geq \vartheta\right) \lesssim \frac{1}{(\log(\mathbf{k}N))^3}. \tag{6.17}$$

Case 2. If $k_N^{\frac{1}{2}} \log k_N < \alpha \log \mathbf{k}$, then (3.11) implies

$$\begin{aligned}\xi_N = \frac{C_*}{2}\left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}} &\simeq \mathbf{k}^{\frac{4}{3}}\varepsilon_N \simeq \mathbf{k}^{\frac{4}{3}} k_N^{-s} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \\ &\geq \mathbf{k}\exp\left(-3\alpha \log \mathbf{k}\right) > \mathbf{k}^{-3\alpha} \exp\left(\frac{1}{\alpha} k_N^{\frac{1}{2}} \log k_N\right).\end{aligned}$$

From (6.16) we also have

$$\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq \sqrt{2C_2L} + C_1\right) \leq \exp\left(-\mathbf{k}^{\frac{8}{3}} LN\varepsilon_N^2\right).$$

In view of (6.12), for any sufficiently large N, one has $\exp\left(\frac{1}{\alpha} k_N^{\frac{1}{2}} \log k_N\right) \geq \sqrt{2C_2L} + C_1$, and so

$$\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq \mathbf{k}^{3\alpha}\xi_N\right) \leq \exp\left(-\mathbf{k}^{\frac{8}{3}} LN\varepsilon_N^2\right).$$

By considering $\ell_4(\mathbf{k}) = \mathbf{k}^{3\alpha}$ and setting

$$\mathcal{S}_N := \left\{f \in \mathbb{H}_1 : \|P_{k_N}(f) - f\|_1 \leq \mathbf{k}^{3\alpha}\xi_N\right\},$$

it yields that

$$\begin{aligned}\Pi_N(\mathcal{S}_N^{\complement}) &\leq \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq \mathbf{k}^{3\alpha}\xi_N\right) \cdot \sum_{\ell = k_N+1}^{\infty} \mathbf{m}(\ell) \\ &\lesssim \exp\left(-(b' + L)\mathbf{k}^{\frac{8}{3}} N\varepsilon_N^2\right),\end{aligned}$$

which verifies (5.12c) with $M(\mathbf{k}) = (\sqrt{2(b'+L)} + 1)\ell_2\mathbf{k}^{\frac{4}{3}} + \mathbf{k}^{3\alpha} + M_0(\mathbf{k}) > 0$. As above, we set $M_0(\mathbf{k}) = 2 + \mathbf{k}^{\frac{4}{3}}\ell_2$ and, if necessary, choose a sufficiently large L (which is independent of $\mathbf{k}$) to see that

$$\begin{aligned}M(\mathbf{k}) - \ell_4(\mathbf{k}) - \ell_1(\mathbf{k})\ell_2(\mathbf{k}) &= \ell_2\sqrt{2(b'+L)} + M_0 > M_0 = 2 + \mathbf{k}^{\frac{4}{3}}\ell_2 \\ &> 1 + \mathbf{k}^{\frac{4}{3}}\ell_2 = \ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}),\end{aligned}$$

which verifies (5.11). Choosing $g(\mathbf{k}) = \mathbf{k}^{\frac{8}{3}}$ and from (5.13) we can check that

$$\tilde{M}(\mathbf{k}) = (b' + L)\mathbf{k}^{\frac{8}{3}} - 1 - \mathbf{k}^{\frac{8}{3}} - h_*\mathbf{k}^{\frac{8}{3}} \geq \mathbf{k}^{\frac{8}{3}},$$

which holds by (possibly) replacing L with the larger one, which is still independent of $\mathbf{k}$. Now for each positive parameter $\vartheta > 0$, we can use Proposition 5.2 to conclude that

$$\mathbb{P}_{f_0}\left(\Pi_N(f \in \mathbb{H}_1 : \|f - f_0\|_1 \geq M\xi_N(\mathbf{k})|Y^{(N)}) \geq \vartheta\right) \lesssim \frac{1}{(\log(\mathbf{k}N))^3}. \tag{6.18}$$

Conclusion. Finally, combining (6.17) and (6.18) ends the proof of Theorem 3.2. □

7. Proofs of theorems in Section 3.2

The central idea of the proof is to the small-ball asymptotics of a Gaussian measure in a separable Hilbert space. Some suitable lower bounds are obtained in [Ray13], which can be proved using either direct methods [HJSD79] or via the metric entropy of the unit ball of the RKHS [KL93] (both of which yield the same result). We also remark that small-ball asymptotics of a Gaussian measure in a Hilbert space also have been exactly characterized by Sytava in [Syt74], and Sytava's result was rediscovered in [Ibr82, Zol86], see also [DMWZ95, Lif97, MWZ93].

As mentioned in Section 3.2, a Gaussian distribution has support equal to the closure of its RKHS $\mathbb{H}$ and so posterior consistency is achievable only when $G_{\mathbf{k}} f_0$ is contained in this set. For $f \sim \mathcal{N}(0, \Lambda)$, by the Karhunen-Loève expansion, we can write $f \overset{\mathrm{d}}{=} \sum_{j\in\mathbb{N}} \tau_j \zeta_j \varphi_j$, where $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}_1$ and $\{\zeta_j\}_{j\in\mathbb{N}}$ are iid standard normal random variables. Recall that $\{\varphi_j\}_{j\in\mathbb{N}}$ is identical to the singular basis of $G_{\mathbf{k}}$ as described in Assumption 2. We can easily characterize its RKHS $\mathbb{H}_f$ in terms of ellipsoids (see [vdVvZ08b] for more details):

$$a = \sum_{j\in\mathbb{N}} a_j \varphi_j \in \mathbb{H}_f \quad \text{if and only if} \quad \|a\|^2_{\mathbb{H}_f} := \sum_{j\in\mathbb{N}} \frac{a_j^2}{\tau_j^2} < +\infty. \tag{7.1}$$

Recall that the concentration function of a zero-mean Gaussian distribution W in $\mathbb{H}_2$ with RKHS $\mathbb{H}_W$ is defined by

$$\phi_{w_0}(\varepsilon) := \inf_{h\in\mathbb{H}_W : \|h-w_0\|_2<\varepsilon} \left(\|h\|^2_{\mathbb{H}_W} - \log \mathbb{P}(\|W\|_2 < \varepsilon) \right). \tag{7.2}$$

Note that the second term in (7.2) is exactly the small deviation function defined in [Lif12, Section 11.4].

From [vdVvZ08a, Theorem 2.1], the following statement holds: for w_0 contained in the closure of $\mathbb{H}_W$

$$\text{if } \phi_{w_0}\left(\frac{1}{\sqrt{2}} \varepsilon_N(\mathbf{k}) \right) \le \frac{1}{2} N \varepsilon_N(\mathbf{k})^2, \text{ then } \mathbb{P}\left(\|W - w_0\| \le \sqrt{2}\varepsilon_N(\mathbf{k}) \right) \ge e^{-\frac{1}{2} N \varepsilon_N(\mathbf{k})^2}. \tag{7.3}$$

This property is helpful to verify the small ball condition (5.12d). First of all, we refine [Ray13, Lemma 5.1] in the following lemma in which the dependence on $\mathbf{k}$ is examined carefully.

Lemma 7.1. *Let $f \in \mathcal{N}(0, \Lambda)$ with Λ satisfying Assumption 2 and let $f_0 \in H^{\gamma}_{\exp}$ with $\gamma \ge \rho$ Then $G_{\mathbf{k}} f$ has RKHS equal to*

$$\mathbb{H}_{G_{\mathbf{k}} f} = \left\{ b = \sum_{j=1}^{\infty} b_j e_j \in \mathbb{H}_2 : \|b\|^2_{\mathbb{H}_{G_{\mathbf{k}} f}} = \sum_{j=1}^{\infty} \frac{|b_j|^2}{\tau_j^2 \sigma_j^2} < +\infty \right\} \tag{7.4}$$

and the concentration function of $G_{\mathbf{k}} f$ satisfies

$$\phi_{G_{\mathbf{k}} f_0}(\varepsilon) \lesssim \left(\log\left(\varepsilon^{-1} \right) \right)^3 \quad \text{for all } 0 < \varepsilon \lesssim 1. \tag{7.5}$$

Here all the implied constants are independent of $\mathbf{k}$.

Proof. First of all, we see that $G_{\mathbf{k}} f$ is a Gaussian random variable in $\mathbb{H}_2$ with $G_{\mathbf{k}} f \sim \mathcal{N}(0, G_{\mathbf{k}} \Lambda G^*_{\mathbf{k}})$. By Assumption 2, $G_{\mathbf{k}} \Lambda G^*_{\mathbf{k}}$ has eigenvectors $\{e_j\}_{j\in\mathbb{N}}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}_2$ consisting of conjugate basis of $G_{\mathbf{k}}$ with corresponding eigenvalues $\{\tau_j^2\sigma_j^2\}_{j\in\mathbb{N}}$, then (7.4) follows directly. The expansion $G_{\mathbf{k}}f_0 = \sum_{j=1}^{\infty}\sigma_j f_{0,j}e_j$ gives

$$\|G_{\mathbf{k}}f_0\|_{\mathbb{H}_{G_{\mathbf{k}}f}}^2 = \sum_{j=1}^{\infty}\tau_j^{-2}|f_{0,j}|^2 \lesssim \sum_{j=1}^{\infty} j^{2\rho}\exp\left(6j^{\frac{1}{2}}\log j\right)|f_{0,j}|^2 \leq \|f_0\|_{H_{\exp}^{\gamma}}^2, \tag{7.6}$$

which implies that $G_{\mathbf{k}}f_0$ is contained in RKHS $\mathbb{H}_{G_{\mathbf{k}}f_0}$. Consequently, (7.6) gives a bound on the first term of $\phi_{G_{\mathbf{k}}f_0}(\varepsilon)$.

Next, to give a bound for the second term of $\phi_{G_{\mathbf{k}}f_0}(\varepsilon)$, we will follow the lines used in [Ray13, Lemma 5.2]. Denote $\mathbb{K}_{G_{\mathbf{k}}f}$ be the unit ball in $\mathbb{H}_{G_{\mathbf{k}}f}$ and let $N(\mathbb{K}_{G_{\mathbf{k}}f}, \|\cdot\|_2, \vartheta)$ be the covering number of $\mathbb{K}_{G_{\mathbf{k}}f}$ with respect to the norm $\|\cdot\|_2$, i.e. the minimal number of sets in a covering of $\mathbb{K}_{G_{\mathbf{k}}f}$ by subsets of $\|\cdot\|_2$-diameter not exceeding ε [Lif12, Section 10.1]. In view of (2.13), we see that there exists an absolute constant $C_* > 1$, which is independent of $\mathbf{k}$ and j, such that $\sigma_j \leq C_*$ for all $j \in \mathbb{N}$. For any $b = \sum_{j=1}^{\infty} b_j e_j \in \mathbb{K}_{G_{\mathbf{k}}f}$, using the characterization of $\mathbb{H}_{G_{\mathbf{k}}f}$, we have that $|b_j| \leq \tau_j\sigma_j \lesssim \exp\left(-3j^{\frac{1}{2}}\log j\right)$, in other words, $\mathbb{K}_{G_{\mathbf{k}}f}$ is contained in the infinite rectangle

$$\prod_{j=1}^{\infty}\left[-C\exp\left(-3j^{\frac{1}{2}}\log j\right), C\exp\left(-3j^{\frac{1}{2}}\log j\right)\right]$$

for some constant $C > 0$ which is independent of $\mathbf{k}$. Given any $0 < \varepsilon \leq 1/e$, we see that

$$C\exp\left(-3j^{\frac{1}{2}}\log j\right) < \frac{1}{2}\varepsilon \quad \text{for all } j > J(\varepsilon),$$

where $J(\varepsilon)$ is the smallest integer satisfying

$$C\exp\left(-3J^{\frac{1}{2}}\log J\right) < \frac{1}{2}\varepsilon.$$

Therefore, it suffices to construct an $\varepsilon/2$ cover for the following $J(\varepsilon)$-dimensional cube

$$X = \prod_{j=1}^{J(\varepsilon)}\left[-C\exp\left(-3j^{\frac{1}{2}}\log j\right), C\exp\left(-3j^{\frac{1}{2}}\log j\right)\right].$$

On the other hand, it is enough to cover this set by considering a regular lattice with distance $\varepsilon/(2\sqrt{J(\varepsilon)})$ between adjacent vertices. Consequently, we obtain that

$$\begin{aligned} N\left(X, \|\cdot\|_{\mathrm{eucl}}, \frac{\varepsilon}{2}\right) &\leq \prod_{j=1}^{J(\varepsilon)}\frac{2C\exp\left(-3j^{\frac{1}{2}}\log j\right)}{\varepsilon/(2\sqrt{J})} = \left(\frac{C'\sqrt{J(\varepsilon)}}{\varepsilon}\right)^{J(\varepsilon)}\exp\left(-\sum_{j=1}^{J} j^{\frac{1}{2}}\log j\right) \\ &\leq \left(\frac{C'\sqrt{J(\varepsilon)}}{\varepsilon}\right)^{J(\varepsilon)} \end{aligned}$$

and thus the metric entropy [Lif12, Section 10.1] of $(X, \|\cdot\|_{\mathrm{eucl}})$ is given by

$$\log N\left(X, \|\cdot\|_{\mathrm{eucl}}, \frac{\varepsilon}{2}\right) \lesssim J\left(\log J + \log\left(\varepsilon^{-1}\right)\right) \lesssim (J^{\frac{1}{2}}\log J)^2 + J\log(\varepsilon^{-1}) \lesssim \left(\log\left(\varepsilon^{-1}\right)\right)^3$$

since

$$\log\left(\varepsilon^{-1}\right) \simeq J^{\frac{1}{2}}\log J \gtrsim 1.$$

Finally, we can use [KL93, Theorem 2] (see also [Lif12, Section 11.7] for similar results) and the bound for the first term in $\phi_{G_{\mathbf{k}} f_0}(\varepsilon)$ obtained above to derive (7.5). □

We now ready to prove Theorem 3.3 by modifying the ideas in [Ray13, Proposition 3.5].

Proof of Theorem 3.3. For $\rho > \frac{3}{2}$, we write $\rho = s + t + u$ for some $s > 0$, $t > 1$, and $u > \frac{1}{2}$. We take

$$\varepsilon_N = \varepsilon_N(\mathbf{k}) := \left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}} \tag{7.7}$$

and choose $k_N = k_N(\mathbf{k})$ be the smallest integer satisfying

$$k_N^{-s} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \simeq \varepsilon_N = \left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}}. \tag{7.8}$$

Then we can see that $N\varepsilon_N^2 = (\log N)^3$ and

$$(\log N)^3 \simeq N\varepsilon_N^2 \simeq (-\log \varepsilon_N)^3 \simeq \left(k_N^{\frac{1}{2}} \log k_N\right)^3 \gtrsim k_N, \tag{7.9}$$

which verifies the first condition of (5.12a) with $\ell_1(\mathbf{k}) = \ell_1$, where ℓ_1 is some general constant, independent of $\mathbf{k}$.

In order to fulfill the second condition of (5.12a), choosing $\ell_2(\mathbf{k}) = 1$ and in view of (2.14) and (7.7), we have

$$\xi_N(\mathbf{k}) := \frac{\varepsilon_N}{\sigma_{k_N}} \lesssim \begin{cases} \mathbf{k}^{\frac{4}{3}} \varepsilon_N & \text{if } \ k_N < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ \mathbf{k}^{\frac{4}{3}} \exp\left(3k_N^{\frac{1}{2}} \log k_N\right) \varepsilon_N & \text{if } \ k_N \geq \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \end{cases}$$
$$\simeq \begin{cases} \mathbf{k}^{\frac{4}{3}} \left(\dfrac{(\log N)^3}{N}\right)^{\frac{1}{2}} & \text{if } \ k_N < \left(\mathbf{k} - \dfrac{3}{2}\right)^2, \\ \mathbf{k}^{\frac{4}{3}} k_N^{-s} & \text{if } \ k_N \geq \left(\mathbf{k} - \dfrac{3}{2}\right)^2. \end{cases}$$

It is readily seen that

$$\exp\left(-k_N^{\frac{11}{20}}\right) \lesssim k_N^{-s} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \simeq \left(\frac{(\log N)^3}{N}\right)^{\frac{1}{2}} \lesssim N^{-\frac{1}{4}},$$

which gives

$$k_N \gtrsim (\log N)^{\frac{20}{11}}. \tag{7.10}$$

Applying Lemma 7.1 with $\varepsilon = \frac{1}{\sqrt{2}} \varepsilon_N$ and by (7.9), we obtain that $\phi_{G_{\mathbf{k}} f_0}(\varepsilon_N) \leq N\varepsilon_N^2$ provided $0 < \varepsilon_N \lesssim 1$. Now we use (7.3) to see that

$$\Pi_N\left(f \in \mathbb{H}_1 : \|G_{\mathbf{k}} f - G_{\mathbf{k}} f_0\| \leq \sqrt{2}\varepsilon_N\right) \geq e^{-\frac{1}{2} N\varepsilon_N^2},$$

which verifies the small-ball condition (5.12d) with $h(\mathbf{k}) = \frac{1}{2}$.

As in the proof of Theorem 3.2, for $f \sim \mathcal{N}(0, \Lambda)$, we write

$$\|P_{k_N}(f) - f\|_1 = \sup_{h \in \mathbb{B}_0} \mathcal{G}_N(h), \tag{7.11}$$

where $\mathbb{B}_0$ is a weak $*$ dense subset of $\{h \in \mathbb{H}_1 : \|h\|_1 \leq 1\}$ and $\mathcal{G}_N$ is the Gaussian process

$$\mathcal{G}_N(h) = \langle h, P_{k_N}(f) - f\rangle_1 = \sum_{j=k_N+1}^{\infty} \tau_j \zeta_j \langle h, \varphi_j\rangle_1,$$

where $\{\zeta_k\}$ are iid standard normal random variables. Applying Jenson's inequality to the bias gives

$$\begin{aligned}\mathbb{E}\|P_{k_N}(f) - f\|_1 &\leq \left(\sum_{j=k_N+1}^{\infty} \tau_j^2\right)^{\frac{1}{2}} \simeq \left(\sum_{j=k_N+1}^{\infty} j^{-2s-2t-2u} \exp\left(-6j^{\frac{1}{2}} \log j\right)\right)^{\frac{1}{2}} \\ &\lesssim k_N^{-s-t} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \left(\sum_{j=1}^{\infty} j^{-2u}\right)^{1/2} \lesssim k_N^{-s-t} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right),\end{aligned} \tag{7.12}$$

where we used the fact $2u > 1$. For the variance estimate, observe that for any $h \in \mathbb{B}_0$,

$$\begin{aligned}\mathbb{E}\mathcal{G}_N(h)^2 &= \sum_{j=k_N(\mathbf{k})+1}^{\infty} \tau_j^2 |\langle h, \varphi_j\rangle_1|^2 \leq \tau_{k_N(\mathbf{k})+1}^2 \|h\|_1^2 \leq \tau_{k_N(\mathbf{k})}^2 \\ &\lesssim k_N^{-2s-2t} \exp\left(-6k_N^{\frac{1}{2}} \log k_N\right).\end{aligned} \tag{7.13}$$

In view of (7.11), (7.12) and (7.13), applying the version of Borell's inequality for the supremum of Gaussian process shown in [Led01, page 134] (also see [Ray13, (4.1)]) yields

$$\begin{aligned}&\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq x + C_1 k_N^{-s-t} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right)\right) \\ &\leq \mathbb{P}\left(\|P_{k_N(\mathbf{k})}(f) - f\|_1 - \mathbb{E}\|P_{k_N(\mathbf{k})}(f) - f\|_1 \geq x\right) \\ &= \mathbb{P}\left(\sup_{h\in\mathbb{B}_0} \mathcal{G}_N(h) - \mathbb{E}\sup_{h\in\mathbb{B}_0} \mathcal{G}_N(h) \geq x\right) \\ &\leq \exp\left(-\frac{x^2}{2\sup_{h\in\mathcal{B}_0} \mathbb{E}\mathcal{G}_N(h)^2}\right) \leq \exp\left(-\frac{k_N^{2s+2t}\exp\left(6k_N^{\frac{1}{2}}\log k_N\right)}{2C_2} x^2\right) \quad \text{for all } \ x > 0.\end{aligned}$$

We now substitute $x = k_N^{-s-t} \exp\left(-3k_N^{\frac{1}{2}} \log k_N\right) \sqrt{2C_2 L}\sqrt{N\varepsilon_N^2}$ into the inequality above and using (7.10) to derive that

$$\begin{aligned}&\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq C_3\left(\sqrt{2C_2 L} + C_1\right)\left(\frac{(\log N)^{6-\frac{40}{11}t}}{N}\right)^{\frac{1}{2}}\right) \\ &\leq \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq C_3\left(\sqrt{2C_2 L} + C_1\right) k_N^{-t}\sqrt{N}\varepsilon_N^2\right) \\ &\leq \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq C_3\left(\sqrt{2C_2 L}\sqrt{N\varepsilon_N^2} + C_1\right) k_N^{-t}\varepsilon_N\right) \\ &\leq \mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq \left(\sqrt{2C_2 L}\sqrt{N\varepsilon_N^2} + C_1\right) k_N^{-t}\,\overbrace{k_N^{-s}\exp\left(-3k_N^{\frac{1}{2}}\log k_N\right)}^{\simeq\varepsilon_N}\right) \\ &\leq \exp\left(-LN\varepsilon_N^2\right).\end{aligned} \tag{7.14}$$

Recall from above, (2.13) implies that there exists an absolute constant $C_* > 1$, which is independent of $\mathbf{k}$ and j, such that $\sigma_j \leq C_*$ for all $j \in \mathbb{N}$. We then see that for $t > 1$

$$\xi_N(\mathbf{k}) \geq C_*^{-1} \left(\frac{(\log N)^3}{N} \right)^{\frac{1}{2}} \geq C_3 \left(\sqrt{2C_2 L} + C_1 \right) \left(\frac{(\log N)^{6-\frac{40}{11}t}}{N} \right)^{\frac{1}{2}}$$

and thus

$$\mathbb{P}\left(\|P_{k_N}(f) - f\|_1 \geq \xi_N(\mathbf{k}) \right) \leq e^{-LN\varepsilon_N(\mathbf{k})^2}. \tag{7.15}$$

Choosing $\ell_4(\mathbf{k}) = 1$ and considering

$$\mathcal{S}_N := \left\{ f \in \mathbb{H}_1 : \|P_{k_N(\mathbf{k})}(f) - f\|_1 \leq \xi_N(\mathbf{k}) \right\},$$

then, from (7.15), (5.12c) is satisfied with

$$M(\mathbf{k}) = \sqrt{2L} + 1 + \ell_1 + M_0(\mathbf{k}).$$

On the other hand, note that

$$\begin{aligned} \|P_{k_N}(f_0) - f_0\|_1^2 &= \sum_{j=k_N+1}^{\infty} j^{-2s} \exp\left(-6j^{\frac{1}{2}} \log j \right) j^{2s} \exp\left(6j^{\frac{1}{2}} \log j \right) |f_{0,j}|^2 \\ &\leq k_N^{-2s} \exp\left(-6k_N^{\frac{1}{2}} \log k_N \right) \|f_0\|_{H^s_{\exp}}^2 \lesssim \varepsilon_N^2 = \sigma_{k_N(\mathbf{k})}^2 \xi_N(\mathbf{k})^2 \leq C_*^2 \xi_N(\mathbf{k})^2, \end{aligned} \tag{7.16}$$

which gives (5.12b) with some $\ell_3(\mathbf{k}) = \ell_3 = C^*$. We now take $M_0 = \ell_1 + \ell_3 + 1$ and compute

$$M_0 = \ell_1 + \ell_3 + 1 > \ell_1 + \ell_3 = \ell_1(\mathbf{k})\ell_2(\mathbf{k}) + \ell_3(\mathbf{k}),$$

which satisfies (5.11).

Choosing $g(\mathbf{k}) = 1$ and picking a large number L ($L > \frac{5}{2}$ is sufficient), which is independent of $\mathbf{k}$, Proposition 5.2 implies that

$$\mathbb{P}_{f_0}\left(\Pi_N \left(f \in \mathbb{H}_1 : \|f - f_0\|_1 \geq M\xi_N(\mathbf{k}) | Y^{(N)} \right) \geq \vartheta \right) \lesssim \frac{1}{N\varepsilon_N^2}$$

for all large N and for any positive parameter $\vartheta > 0$. The proof of Theorem 3.3 is now completed. ∎

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