CONSISTENCY OF BAYESIAN INFERENCE FOR A SUBDIFFUSION EQUATION

PU-ZHAO KOW AND JENN-NAN WANG

Abstract. In this work, we consider the inverse problem of determining an unknown potential in a subdiffusion equation from its solution using a nonparametric Bayesian approach. Our aim is to establish the consistency of the posterior distribution with Gaussian priors. To do so, we need some key estimates of the forward problem. For the forward problem, we have to overcome the fact that the solution of the subdiffusion equation is less regular than that of the classical heat equation. The main ingredient is the maximum principle for the subdiffusion equation. We show that the posterior contracts to the ground truth at a polynomial rate.

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1. Introduction

In this paper, we study the inverse problem of recovering an unknown potential function of a subdiffusion equation from observations of a solution using Bayesian approach. Here the subdiffusion phenomenon is modeled by the fractional time derivative. The fractional time derivative can be used to describe particle sticking and trapping phenomena [BBMW01, BTM+04, SBMB03]. There are two kinds of fractional time derivatives commonly studied in literature, Riemann-Liouville derivative and Caputo derivative. They are not equal in general, but are equivalent up to an inhomogeneous term containing the initial condition of the function, see (2.4) or [MS19, (2.33)]. In this work, we will consider the subdiffusion equation governed by the Riemann-Liouville derivative. For such subdiffusion equation, the main interest of the proposed inverse problem is to establish the consistency property of the posterior distribution by proving that the posterior contracts to the true parameter. In particular, we derive a polynomial contraction rate as the sample size increases. This work can be viewed as an extension of [Kek22] for the parabolic equation to the subdiffusion equation, see also [GN20] for related results for the elliptic equation. To this end, we need some key estimates of the forward problem. Here, unlike the probability method based on the Feynman-Kac representations used in [GN20, Kek22], the main ingredient is the maximum principle for the subdiffusion equation due to the insufficient regularity of the solution.

1.1. **Mathematical setup.** Let $d \in \mathbb{N}$ and let $\mathcal{O}$ be a bounded open domain in $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. Fix any $T > 0$, we write $\mathcal{O}_T := (0,T) \times \mathcal{O}$ and $(\partial\mathcal{O})_T := (0,T) \times \partial\mathcal{O}$. We fix parameters $M_0 > 0$ and $0 < \theta < 1$. For each $f = f(x) \in C^1(\overline{\mathcal{O}})$ with $0 < f < M_0$ in $\overline{\mathcal{O}}$, under some appropriate regularity assumptions on the source $h = h(t,x)$, the boundary $g = g(t,x)$ and the initial conditions $u_0 = u_0(x)$ (here u_0 and g satisfy some compatibility conditions), let $u = u_f$ be the solution to the following initial-boundary value problem (IBVP) for the subdiffusion equation with the fractional time derivative of order θ:

$$\begin{cases} \partial_t^\theta (u(t,x) - u_0(x)) - \Delta u(t,x) + f(x)u(t,x) = h(t,x) & \text{in } \mathcal{O}_T, \\ u = g & \text{on } (\partial\mathcal{O})_T, \\ u(0,\cdot) = u_0 & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where ∂_t^θ denotes the Riemann-Liouville derivative of order θ (see Section 2.1 below). The use of $u(x,t) - u_0$ in the first term of (1.1) allows us to formulate the IBVP by a Hilbert-space-setting. We will explain the adoption of this term in detail later. It should be emphasized that the IBVP (1.1) is only understood formally here. We will explain the precise definition of ∂_t^θ and the IBVP (1.1), as well as the well-posedness results, in Section 3 below. In this work, we are interested in the recovery of f from the knowledge of the unique solution $u = u_f$ of (1.1).

If we additionally assume that $\min_{\overline{\mathcal{O}}} u_0 > 0$, $\min_{(\partial\mathcal{O})_T} g > 0$ and $h(t,x)$ is sufficiently large, then the maximum principle guarantees that $u_f > 0$ in $\mathcal{O}_T$, see (3.12), and hence from (1.1) one can reconstruct f from the formula

$$f(x) = \frac{h(t,x) - \partial_t^\theta (u_f(t,x) - u_0(x)) + \Delta u_f(t,x)}{u_f(t,x)} \quad \text{for } x \in \mathcal{O} \text{ a.e.}, \tag{1.2}$$

for any fixed $0 < t < T$. It is helpful to remark that, for the heat equation considered in [Kek22], the corresponding classical solution can be explicitly written in terms of the Feynman-Kac formula. The positivity of the solution with positive potential f then follows from the solution formula. For the case of subdiffusion equation, due to the insufficient regularity of the solution u_f to (1.1), $u_f > 0$ is ensured by the maximum principle. We also want to point out that the maximum principles in different formulations were proved and used in [BKT18, Luc09a, Luc09b, Luc10, Luc11, LY17, LY18, LY19, Zac08]. For the subdiffusion equation (1.1), but with the Robin boundary conditions, the maximum principles were derived in [LY23]. On the other hand, a maximum principle for the time-fractional transport equations was proved in [LSY22]. Furthermore, a maximum principle for more general space- and time-space-fractional PDE has been derived in [KT21].

We would like to study the inverse problem of determining the potential function f by measuring u_f in $\mathcal{O}_T$. In practice, however, it is not feasible to measure u_f at all points $(t,x) \in \mathcal{O}_T$. Instead, we measure u_f at (t,x) (referred to as sample) randomly and derive a "good" estimate of f by increasing the number of samples, and the inverse problem is then addressed through a Bayesian procedure. For the general approach of the Bayesian method in infinite-dimensional models, we refer readers to the monographs [GN21, Nic23]. On the other hand, the coefficient determination problem in the subdiffusion equation by suitable measurements of its corresponding solution from a PDE perspective can be found in [KR23, Chapter 10]. To streamline the presentation, we will explain the Bayesian method to inverse problems in detail in later sections. Before describing the statistical model, we first introduce some notations.

1.2. **Notations.** We recall some notations and function spaces used in our previous work [FKW24a]. Throughout this paper, we shall use the symbol $\lesssim$ and $\gtrsim$ for inequalities holding up to a universal constant. For two real sequences (a_N) and (b_N), we say that $\simeq$ if both $a_N \lesssim b_N$ and $b_N \lesssim a_N$ for all sufficiently large N. For a sequence of random variables Z_N and a real sequence (a_N), we write $Z_N = O_{\rm Pr}(a_N)$ if for all $\epsilon > 0$ there exists $M_\epsilon < \infty$ such that for all N large enough, $\Pr(|Z_N| \ge M_\epsilon a_N) < \epsilon$. Denote $\mathcal{L}(Z)$ the law of a random variable Z. We also denote $a \vee b = \max\{a,b\}$ for all $a,b \in \mathbb{R}$.

Organization of the paper. We first introduce measurement model and state main theorems (Theorem 2.6, Theorem 2.8, Theorem 2.11, Theorem 2.13 and Theorem 2.14) in Section 2. Then we prove the well-posedness of the IBVP (2.12) in Section 3 and study the properties of the solution to the subdiffusion equation, especially its regularity in Section 4. We prove main theorems in Section 5. Finally, we make some conclusions in Section 6.

2. Measurement model and theorems

2.1. **Riemann-Liouville fractional derivative.** In order to make the paper self-contained, we first recall some results on the fractional time derivatives proved in [LY17, LY23], see also the monographs [Jin21, KRY20]. We begin with the classical Caputo fractional derivative $\mathrm{d}_t^\theta w(t)$ for $0 < \theta < 1$ defined by

$$\mathrm{d}_t^\theta w(t) := \frac{1}{\theta(1-\theta)} \int_0^t (t-s)^{-\theta} \frac{\mathrm{d}w}{\mathrm{d}s}(s)\,\mathrm{d}s \quad \text{for all } w \in {}_0C^1([0,T]), \tag{2.1}$$

where ${}_0C^1([0,T]) := \{w \in C^1([0,T]) : w(0) = 0\}$. It is possible to extend the Caputo fractional derivative in weak sense.

Let us first define some fractional Sobolev spaces:

$$H_\theta(0,T) := \begin{cases} H^\theta(0,T), & 0<\theta<\frac{1}{2}, \\ \left\{ w \in H^{\frac{1}{2}}(0,T) : \int_0^T \frac{|w(t)|^2}{t}\,\mathrm{d}t < \infty \right\}, & \theta = \frac{1}{2}, \\ \left\{ w \in H^\theta(0,T) : w(0) = 0 \right\}, & \frac{1}{2} < \theta < 1, \end{cases}$$

with corresponding norms given by

$$\|w\|_{H_\theta(0,T)} := \begin{cases} \|w\|_{H^\theta(0,T)}, & \theta \neq \frac{1}{2}, \\ \left(\|w\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|w(t)|^2}{t}\,\mathrm{d}t \right)^{\frac{1}{2}}, & \theta = \frac{1}{2}, \end{cases}$$

where

$$\|w\|_{H^\theta(0,T)} := \left(\|w\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|w(t)-w(s)|^2}{|t-s|^{1+2\theta}}\,\mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{2}}.$$

Remark 2.1. *As pointed out in the correction of* [LY23]*, the space $H_\theta(0,T)$ is the closure of ${}_0C^1([0,T])$ with respect to the norm $\|\cdot\|_{H_\theta(0,T)}$, not with respect to the norm $\|\cdot\|_{H^\theta(0,T)}$, see also* [KRY20, Lemma 2.2]*.*

Next, for each $\theta > 0$, we define the Riemann-Liouville fractional integral operator

$$(J^\theta w)(t) := \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} w(s)\,\mathrm{d}s, \quad 0 < t < T.$$

In view of [KRY20, Theorems 2.1 and 2.2], one can see that

$$J^\theta : L^2(0,T) \to H_\theta(0,T) \text{ is a bijection,} \tag{2.2}$$

and its inverse

$$\partial_t^\theta := (J^\theta)^{-1} : H_\theta(0,T) \to L^2(0,T)$$

induces an equivalent norm for $H_\theta(0,T)$ in the sense that

$$C^{-1}\|\partial_t^\theta w\|_{L^2(0,T)} \le \|w\|_{H_\theta(0,T)} \le C\|\partial_t^\theta w\|_{L^2(0,T)} \quad \text{for all } w \in H_\theta(0,T), \tag{2.3}$$

for some constant $C > 0$, which is independent of w. Moreover, by [KRY20, Theorem 2.4], we have

$$\partial_t^\theta w = \mathrm{d}_t^\theta w \quad \text{for all } w \in {}_0C^1([0,T]). \tag{2.4}$$

In fact, ∂_t^θ is nothing but the Riemann-Liouville fractional derivative:

$$\partial_t^\theta w = \frac{\mathrm{d}}{\mathrm{d}t}(J^{1-\theta} w) \quad \text{for all } w \in H_\theta(0,T).$$

We remark that ∂_t^θ is the smallest closed extension of d_t^θ in $H_\theta(0,T)$ [KRY20, Theorem 2.5]. Later we will also need the following version of coercivity inequality:

Lemma 2.2 ([KRY20, Theorem 3.3(ii)])**.** *For each $T > 0$, one has*

$$\int_0^T \langle \partial_t^\theta u(t,\cdot), u(t,\cdot)\rangle\,\mathrm{d}t \ge \frac{T^{-\theta}}{2\theta(1-\theta)}\|u\|^2_{L^2(0,T;L^2(\mathcal{O}))}$$

for all $u \in H_\theta(0,T;H^{-1}(\mathcal{O})) \cap L^2(0,T;H_0^1(\mathcal{O}))$*. Here* $\langle\cdot,\cdot\rangle$ *denotes the duality pair between* $H^{-1}(\mathcal{O})$ *and* $H_0^1(\mathcal{O})$*.*

2.2. **Precise formulation of the IBVP.** Here we would like to state conditions on g and u_0 before discussing the well-posedness of the IBVP (1.1). To begin, we define some parabolic Hölder norms. For each parameter $\vartheta \in (0,1]$ and set $A \subset \mathbb{R}\times\mathbb{R}^d$, we say that $u \in C^{1+\vartheta/2,2+\vartheta}(A)$ if u satisfies

$$\|u\|_{C^{1+\vartheta/2,2+\vartheta}(A)} := \|\partial_t u\|_{L^\infty(A)} + [\partial_t u]_{\vartheta/2,\vartheta} + \sum_{|\alpha|\le 2}\left(\|\partial^\alpha u\|_{L^\infty(\mathcal{O}_T)} + [\partial^\alpha u]_{\vartheta/2,\vartheta}\right) < \infty,$$

where

$$[v]_{\vartheta/2,\vartheta} = \sup_{(t_1,x_1)\neq(t_2,x_2)\in A}\frac{|v(t_1,x_1)-v(t_2,x_2)|}{(|t_1-t_2|+|x_1-x_2|^2)^{\vartheta/2}}.$$

Assume for simplicity that

$$g \in C^\infty(\overline{(\partial\mathcal{O})_T}) \;\text{ and }\; u_0 \in C^\infty(\overline{\mathcal{O}}), \tag{2.5}$$

where $\overline{(\partial\mathcal{O})_T} = [0,T]\times\partial\mathcal{O}$. In addition, g and u_0 satisfy the following compatibility conditions:

$$g(0,x) = u_0(x) \quad\text{and}\quad \partial_t g(0,x) - \Delta u_0(x) = 0, \quad \forall\, x\in\partial\mathcal{O}. \tag{2.6}$$

For given g and u_0 satisfying regularity and compatibility conditions stated above, there exists a unique solution $w_g \in C^{1+\vartheta/2,2+\vartheta}(\overline{\mathcal{O}_T})$ solving

$$\partial_t w_g - \Delta w_g = 0 \;\text{ in }\; \mathcal{O}_T,\;\; w_g = g \;\text{ on }\; (\partial\mathcal{O})_T,\;\; w_g(0,\cdot) = u_0(\cdot) \;\text{ in }\; \mathcal{O} \tag{2.7}$$

and for each $\vartheta\in(0,1]$

$$\|w_g\|_{C^{1+\vartheta/2,2+\vartheta}(\mathcal{O}_T)} \le C(\|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)} + \|u_0\|_{C^{2+\vartheta}(\mathcal{O})}), \tag{2.8}$$

where C depends on $T,\mathcal{O},d,\vartheta$, see, for example, [Kry96, Theorem 10.4.1] or [Lun95, Theorem 5.1.15]. To utilize the maximum principle, in addition to the regularity and compatibility conditions (2.5), (2.6), we further assume that there exists $c>0$ such that

$$g(t,x)\ge c,\; \forall\,(t,x)\in(\partial\mathcal{O})_T \;\text{ and }\; u_0(x)\ge c,\; \forall\, x\in\overline{\mathcal{O}}. \tag{2.9}$$

Then it follows from the classical maximum principle that w_g satisfies

$$w_g(t,x)\ge c,\; \forall\,(t,x)\in\mathcal{O}_T.$$

To handle the inverse problem, we assume that the source function $h \in L^2(0,T;L^2(\mathcal{O}))$ satisfies

$$h(t,x) \ge \partial_t^\theta(w_g(t,x)-w_g(0,x)) - \Delta w_g(t,x) + M_0 w_g(t,x),\; \forall\,(t,x)\in\mathcal{O}_T \text{ a.e.} \tag{2.10}$$

Condition (2.10) is used to guarantee the unique solution u of (1.1) satisfying $u>0$ in $\mathcal{O}_T$, in order to make sense of the reconstruction formula (1.2).

To discuss the well-posedness of IBVP, we need to refine the estimate (2.8) in terms of Sobolev norms. In view of (2.3), let us denote $H_\theta(0,T;L^2(\mathcal{O}))$ the Hilbert space equipped with the norm

$$\|v\|_{H_\theta(0,T;L^2(\mathcal{O}))} := \|\partial_t^\theta v\|_{L^2(\mathcal{O}_T)}.$$

From (2.7), we see that

$$\partial_t(w_g-u_0) = \partial_t w_g = \Delta w_g \quad\text{in}\;\; \mathcal{O}_T.$$

Thus, for each $0 \le r < 1/2$, we have

$$
\begin{aligned}
\|w_g - u_0\|_{H_{1+r}(0,T;L^2(\mathcal{O}))} &\le C\|\partial_t(w_g - u_0)\|_{H_r(0,T;L^2(\mathcal{O}))} = C\|\Delta w_g\|_{H_r(0,T;L^2(\mathcal{O}))} \\
&= C\left(\|\Delta w_g\|^2_{L^2(\mathcal{O}_T)} + \int_0^T\int_0^T \frac{\|\Delta w_g(t,\cdot) - \Delta w_g(s,\cdot)\|^2_{L^2(\mathcal{O})}}{|t-s|^{1+2r}}\,\mathrm{d}t\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C_{T,\mathcal{O}}\left(\|w_g\|^2_{C^{1,2}(\mathcal{O}_T)} + \sup_{0<t,s<T, x\in\Omega}\frac{|\Delta w_g(t,x) - \Delta w_g(s,x)|^2}{|t-s|}\overbrace{\int_0^T\int_0^T \frac{1}{|t-s|^{2r}}\,\mathrm{d}t\,\mathrm{d}s}^{<\infty \text{ since } 0\le r<1/2}\right)^{\frac{1}{2}} \\
&\le C_{T,\mathcal{O}}\|w_g\|_{C^{\frac{3}{2},3}(\mathcal{O}_T)} \le C,
\end{aligned}
$$

with $C = C(T, \mathcal{O}, g, u_0)$, where we used $\vartheta = 1$ in (2.8). In other words, we obtain that, for $0 \le r < 1/2$,

$$
\|w_g - u_0\|_{H_{1+r}(0,T;L^2(\mathcal{O}))} + \|w_g\|_{L^2(0,T;H^{2r+2}(\mathcal{O}))} \le C \tag{2.11}
$$

for some positive constant C depending on $T, \mathcal{O}, g$, and u_0.

Given g, u_0 satisfy (2.5) and (2.6). Let $f \in L^\infty(\mathcal{O})$ with $f \ge 0$ a.e., and $h \in L^2(0,T;L^2(\mathcal{O}))$. We now formulate the IBVP (1.1) as follows (see [KRY20, Chapter 4]):

$$
\partial_t^\theta(u(t,x) - u_0(x)) - \Delta u(t,x) + f(x)u(t,x) = h(x,t), \ \text{ in } \ H^{-1}(\mathcal{O}),\ t\in(0,T), \tag{2.12a}
$$

$$
u(\cdot,t) - w_g(\cdot,t) \in H_0^1(\mathcal{O}), \ \ t \in (0,T), \tag{2.12b}
$$

$$
u - u_0 \in H_\theta(0,T;L^2(\mathcal{O})). \tag{2.12c}
$$

In the case of $\frac{1}{2} < \theta < 1$, by the Sobolev embedding $H_\theta(0,T) \subset H^\theta(0,T) \subset C[0,T]$, (2.12c) implies $u - u_0 \in C([0,T], L^2(\mathcal{O}))$ and thus $u(0,x) = u_0(x)$ is satisfied in the sense of $\lim_{t\to 0}\|u(\cdot,t) - u_0(\cdot)\|_{L^2(\mathcal{O})} = 0$.

2.3. **The statistical model.** Here we assume that given certain potential function f, u_f is the unique solution to (2.12). We will consider the value of $u_f(t,x)$ for $(t,x) \in \mathcal{O}_T$ and we write the *forward opeartor* $G : L^\infty(\mathcal{O}) \to L^2(0,T;H^1(\mathcal{O}))$ given by

$$
G(f) := u_f. \tag{2.13}
$$

It is more convenient that we randomly choose $N \in \mathbb{N}$ points $Z_i := (t_i, x_i)$ (at which the solution u_f is measured) from the uniform distribution on $\mathcal{O}_T$, that is, for $N \in \mathbb{N}$,

$$
\{Z_i\}_{i=1}^N \overset{\text{iid}}{\sim} \mu, \quad \mu = \frac{\mathrm{d}t \times \mathrm{d}x}{T|\mathcal{O}|}, \tag{2.14}
$$

where $\mathrm{d}x$, $\mathrm{d}t$ are the Lebesgue measures in $\mathbb{R}^d$, $\mathbb{R}^1$, respectively, and $|\mathcal{O}|$ is the volume of $\mathcal{O}$. We will prove in Lemma 3.3 below that u_f is continuous on $\mathcal{O}_T$ under some suitable conditions (see also (4.3b) below), and one can evaluate the forward operator (2.13) pointwise, i.e.,

$$
G(f)(Z_i) := u_f(t_i, x_i)
$$

where (t_i, x_i) is a realization of Z_i. Now we consider the measurement model with a fixed noise level $\sigma > 0$

$$
Y_i = G(f)(Z_i) + \sigma W_i, \quad \{W_i\}_{i=1}^N \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \tag{2.15}
$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution. We also assume that $W^{(N)} := \{W_i\}_{i=1}^N$ and $Z^{(N)} := \{Z_i\}_{i=1}^N$ are independent. We are interested in the inference of f from the observational data $(Y^{(N)}, Z^{(N)})$ with $Y^{(N)} := \{Y_i\}_{i=1}^N$.

We now introduce the space of parameters. For integer $\beta \geq 2$ and $M_0 > 1$, let

$$\mathcal{F}_{M_0}^{\beta} = \left\{ f \in H^{\beta}(\mathcal{O}) : 0 < f < M_0,\ f|_{\partial\mathcal{O}} = 1,\ \partial_{\nu}^j f\big|_{\partial\mathcal{O}} = 0,\ 1 \leq j \leq \beta - 1 \right\}, \tag{2.16}$$

where ∂_ν is the normal derivative in the sense of [LM72a, Theorem 9.4, Chapter 1]. To have more flexibility of choosing priors, we will re-parametrize the potential function f. Although we follow the general approach of [AN19, GK20, GN20, Kek22, NvdGW20], the limited regularity of the solution necessitates adopting our method based on the maximum principle (rather than the Feynman-Kac representations), which in turn requires the use of a link function as in [FKW24a], differing from that in [Kek22].

Assumption 2.3 (link function)**.** Assume that Φ satisfies

(i) $\Phi : (-\infty, \infty) \to (0, M_0)$, $\Phi(0) = 1$, $\Phi'(z) > 0$ for all z;

(ii) for any $k \in \mathbb{N}$

$$\sup_{-\infty < z < \infty} |\Phi^{(k)}(z)| < \infty.$$

Given any link function Φ as in Assumption 2.3, following the argument as in [NvdGW20], the parameter space can be realized as (this only requires assumptions (i) and (ii))

$$\mathcal{F}_{M_0}^{\beta} := \{\Phi \circ F : F \in H_0^{\beta}(\mathcal{O})\}. \tag{2.17}$$

Accordingly, we can define the *reparametrized forward map* by

$$\mathcal{G}(F) := G(\Phi \circ F) \quad \text{for all } F \in H_0^{\beta}(\mathcal{O}), \tag{2.18}$$

where G is the map given in (2.13). Therefore, the model (2.15) can be regarded as a special case of

$$Y_i = \mathcal{G}(F)(Z_i) + \sigma W_i \quad \text{for } i = 1, \cdots, N. \tag{2.19}$$

The random vectors (Y_i, Z_i) on $\mathbb{R} \times \mathcal{O}_T$ are then iid with laws denoted by $\mathbb{P}_F^i$ with Radon-Nikodym density

$$p_F(y,z) := \frac{\mathrm{d}\mathbb{P}_F^i}{\mathrm{d}y \times \mathrm{d}\mu}(y,z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - \mathcal{G}(F)(z))^2}{2\sigma^2} \right) \tag{2.20}$$

for all $y \in \mathbb{R}, z \in \mathcal{O}_T$, where $\mathrm{d}y$ denotes the Lebesgue measure on $\mathbb{R}$. By slightly abusing the notation, we write $\mathbb{P}_F^N = \otimes_{i=1}^N \mathbb{P}_F^i$ for the joint law of $(Y_i, Z_i)_{i=1}^N$ on $\mathbb{R}^N \times (\mathcal{O}_T)^N$, and $\mathbb{E}_F^i, \mathbb{E}_F^N$ denote the corresponding expectation operators of $\mathbb{P}_F^i, \mathbb{P}_F^N$, respectively. We assign a prior on the parameter space F by a Borel probability measure Π supported on the Banach space $C(\mathcal{O})$. Since the map $(F, (y,z)) \mapsto p_F(y,z)$ can be shown to be jointly measurable, the posterior distribution $\Pi(\cdot|Y^{(N)}, Z^{(N)})$ of $F|Y^{(N)}, Z^{(N)}$ arising from the model (2.19) equals to

$$\Pi(B|Y^{(N)}, Z^{(N)}) = \frac{\int_B e^{\ell^N(F)}\, \mathrm{d}\Pi(F)}{\int_{C(\mathcal{O})} e^{\ell^N(F)}\, \mathrm{d}\Pi(F)} \tag{2.21}$$

for any Borel set $B \subset C(\mathcal{O})$, where

$$\ell^N(F) = -\frac{1}{2\sigma^2} \sum_{i=1}^N (Y_i - \mathcal{G}(F)(Z_i))^2$$

is the joint log-likelihood function (up to a constant).

2.4. **Main results.** In this work, we are interested in the frequentist property of the posterior distribution (2.21) in the sense that the observation data $(Y^{(N)}, Z^{(N)})$ are generated through the model (2.17)–(2.19) of law $\mathbb{P}^N_{f_0}$ corresponding to "ground truth" f_0. From now on, we consider $d = 2, 3$ in the rest of paper.

2.4.1. *Rescaled Gaussian priors.* The aim here is to show that the posterior distribution arising from rescaled Gaussian priors concentrates near f_0 and to derive a bound on the rate of contraction as in [FKW24a, GN20, Kek22]. We now describe explicitly Gaussian priors introduced in [FKW24a, GN20, Kek22].

Assumption 2.4. Let $\alpha > \beta + d/2$, $\beta > 1 + d/2$, and $\mathcal{H}$ be a Hilbert space continuously embedded into $H^\alpha_0(\mathcal{O})$. Assume that Π' is a centered Gaussian Borel probability measure on the Banach space $C(\mathcal{O})$ that is supported on a separable measurable linear space of $H^\beta(\mathcal{O})$. Furthermore, let the reproducing-kernel Hilbert space of Π' be equal to $\mathcal{H}$.

An example that satisfies Assumption 2.4 is constructed from the Whittle-Matérn process. The following example is given in [GN20, Example 25].

Example 2.5. Let $\mathcal{O}$ be an open smooth bounded domain in $\mathbb{R}^d$ with $d \geq 2$. For $\alpha > d/2$, let the Whittle-Matérn process with index set $\mathcal{O}$ and smoothness parameter $\alpha - d/2 > 0$ be $\mathcal{M} = \{\mathcal{M}(x) : x \in \mathcal{O}\}$. By [GvdV17, Chapter 11], the RKHS of $\mathcal{M}$ is $H^\alpha(\mathcal{O})$. Furthermore, we can check that $\mathcal{M}$ has a version with paths belonging almost surely to $H^\beta(\mathcal{O})$ with all $\beta < \alpha - d/2$. If $\beta > 1 + d/2$, then by the Sobolev embedding theorem, one can consider $\mathcal{M}$ a C^1-smooth version Whittle-Matérn process with RKHS $H^\alpha(\mathcal{O})$. In what follows, we will assume that $F_0 \in H^\alpha(\mathcal{O})$ has compact support with $\mathrm{supp}(F_0) = \mathcal{K} \subset \mathcal{O}$. Choose a smooth cut-off function $\chi \in C^\infty_0(\mathcal{O})$ with $\chi = 1$ on $\mathcal{K}$ and define $\mathcal{M}' = \chi\mathcal{M}$. Then $\Pi' = \mathcal{L}(\mathcal{M}')$ is a centered Gaussian Borel probability measure supported on $C^1_0(\mathcal{O})$ with $1+d/2 < \beta < \alpha - d/2$, whose RKHS is given by

$$\mathcal{H} = \{\chi F : F \in H^\alpha(\mathcal{O})\}$$

and $\mathcal{H}$ is continuously embedded into $H^\alpha_0(\mathcal{O})$ [GN21, Exercise 2.6.5].

For those Π' given in Assumption 2.4, we consider the rescaled prior

$$\Pi_N = \mathcal{L}(F_N), \quad F_N = \frac{1}{N^{d/(4\alpha+4+2d)}} F', \quad F' \sim \Pi', \tag{2.22}$$

which again defines a centered Gaussian prior on $C(\mathcal{O})$, and its reproducing-kernel Hilbert space is still given by $\mathcal{H}$ but with norm

$$\|F\|_{\mathcal{H}_N} = N^{d/(4\alpha+4+2d)} \|F\|_{\mathcal{H}} \quad \text{for all} \quad F \in \mathcal{H}.$$

In light of the link function, we define the push-forward posterior on the potential f by

$$\tilde{\Pi}_N \left(\cdot|Y^{(N)}, Z^{(N)}\right) := \mathcal{L}(f) \quad \text{with} \quad f = \Phi \circ F : F \sim \Pi_N \left(\cdot|Y^{(N)}, Z^{(N)}\right), \tag{2.23}$$

where $\Pi_N \left(\cdot|Y^{(N)}, Z^{(N)}\right)$ is the posterior arising from observations $(Y^{(N)}, Z^{(N)})$ with prior Π_N as in (2.21). Our first result shows that the posterior contracts toward the ground truth f_0 in L^2-prediction risk. Here f_0 is given by $f_0 = \Phi \circ F_0$, where $F_0 \in H^\alpha(\mathcal{O})$ with compact support such that $\mathrm{supp}(F_0) = \mathcal{K} \subset \mathcal{O}$ as described above.

Theorem 2.6. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. Assume that g and u_0 satisfy (2.5), (2.6), and (2.9). Let $\theta \in (\frac{4}{5}, 1)$ and $h \in H_{\frac{\theta}{2}}(0,T;L^2(\mathcal{O}))$ satisfy (2.10). We consider a rescaled Gaussian prior Π_N given in (2.22) and the base prior $F' \sim \Pi'$ satisfying Assumption 2.4 with corresponding parameters α, β described there having reproducing kernel Hilbert space $\mathcal{H}$. Let f_0 be the given ground truth described above and the observations $(Y^{(N)}, Z^{(N)})$ be generated through the model (2.13)–(2.15) with $f = f_0$. Let $\Pi_N\left(\cdot|Y^{(N)}, Z^{(N)}\right)$ be the resulting posterior arising from observations $(Y^{(N)}, Z^{(N)})$. Then for each $D > 0$, one can find a positive constant L' (depending on D) such that*

$$\begin{aligned}\tilde{\Pi}_N &\left(f : \|f - f_0\|_{L^2(\mathcal{O})} > L'\delta_N^{1/3} | Y^{(N)}, Z^{(N)}\right)\\ &= O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2}) \quad \text{as} \quad N \to +\infty,\end{aligned}$$

where $\delta_N = N^{-(\alpha+1)/(2\alpha+2+d)}$.

Remark 2.7. *The range of $\theta \in (\frac{4}{5}, 1)$ implies that $\frac{\theta}{2} \in (\frac{1}{2} - \frac{\theta}{8}, \frac{1}{2})$, as required for the estimates on forward and inverse problems in Section 4. This assumption is crucial for ensuring the continuity of the solution u to (2.12), which in turn provides pointwise evaluation of the forward operator (2.13).*

To obtain an estimator of the unknown coefficient f, in view of the link function Φ in Assumption 2.3, it is often convenient to derive an estimator of F. The posterior mean $\overline{F}_N := \mathbb{E}^{\Pi_N}\left(F|Y^{(N)}, Z^{(N)}\right)$ of $\Pi_N\left(\cdot|Y^{(N)}, Z^{(N)}\right)$, which can be approximated numerically by an MCMC algorithm, is the most natural choice of estimator. From Theorem 2.6 we also prove a contraction rate for the convergence $\overline{F}_N$ to F_0.

Theorem 2.8. *Assume that the assumptions of Theorem 2.6 hold. Then there exists a constant $C > 0$ such that*

$$\mathbb{P}^N_{F_0}\left(\|\overline{F}_N - F_0\|_{L^2(\mathcal{O})} > C\delta_N^{1/3}\right) \to 0 \quad \text{as} \quad N \to +\infty.$$

As an immediate consequence of (2.23), there exists a constant $C' > 0$ such that

$$\mathbb{P}^N_{f_0}\left(\|\Phi \circ \overline{F}_N - f_0\|_{L^2(\mathcal{O})} > C'\delta_N^{1/3}\right) \to 0 \quad \text{as} \quad N \to +\infty. \tag{2.24}$$

2.4.2. *High-dimensional Gaussian sieve priors.* We now describe Gaussian sieve priors introduced in [FKW24a, GN20, Kek22]. Let $\{\Psi_{\ell r} : \ell \geq -1, r \in \mathbb{Z}^d\}$ be the d-dimensional compactly supported Daubechies wavelets, which forms an orthonormal basis of $L^2(\mathbb{R}^d)$. Let $\mathcal{K}$ be a compact subset in $\mathcal{O}$ and let $\mathcal{R}_\ell = \{r \in \mathbb{Z}^d : \operatorname{supp}(\Psi_{\ell r}) \cap \mathcal{K} \neq \emptyset\}$. Let $\mathcal{K}'$ be another compact subset in $\mathcal{O}$ such that $\mathcal{K} \subsetneq \mathcal{K}'$ and let $\chi \in C_c^\infty(\mathcal{O})$ be a cut-off function with $\chi = 1$ on $\mathcal{K}'$. For any truncation level $j \in \mathbb{N}$ and $\alpha > 1 + d/2$, let the prior Π'_j be given as the law of the Gaussian random sum

$$\Pi'_j \equiv \Pi'_j[\alpha] = \mathcal{L}(\chi F_j), \quad F_j = \sum_{-1 \leq \ell \leq j, r \in \mathcal{R}_\ell} 2^{-\ell\alpha} F_{\ell r} \Psi_{\ell r} \text{ with } F_{\ell r} \overset{\text{iid}}{\sim} \mathcal{N}(0,1). \tag{2.25}$$

Then Π'_j defines a centered Gaussian prior that is supported on the finite-dimensional space

$$\mathcal{H}_j = \operatorname{span}\{\chi \Psi_{\ell r} : -1 \leq \ell \leq j, r \in \mathcal{R}_\ell\} \subset C(\mathcal{O}). \tag{2.26}$$

Theorem 2.9. *Assume that the rescaled prior $\Pi_N \equiv \Pi_N[\alpha]$ is defined as (2.22) with priors $F' \sim \Pi'_{j(N)}$, where the truncation level $j(N) \in \mathbb{N}$ satisfies $2^{j(N)} \simeq N^{1/(2\alpha+2+d)}$. Then the conclusions of Theorem 2.6 and Theorem 2.8 remain valid.*

The proof of Theorem 2.9 only requires minor modification from the proofs of Theorem 2.6 and Theorem 2.8, and all necessary modifications are listed in [GN20, Section 3.2]. We thus refrain from repeating the argument here, see also [GN20, Proposition 7].

2.4.3. *Randomly truncated Gaussian series priors.* Now let J be a random truncation level, which is independent of the random coefficients $F_{\ell r}$ mentioned in (2.25), satisfying the inequalities:

$$\Pr(J > j) = e^{-2^{jd}\log 2^{jd}} \text{ for all } j \in \mathbb{N}, \quad \Pr(J = j) \gtrsim e^{-2^{jd}\log 2^{jd}} \text{ as } j \to \infty.$$

When $d = 1$, log-Poisson random variable satisfies these tail conditions, and for $d > 1$ an example is constructed in [GN20, Example 28]. We now consider the random (conditionally Gaussian) sum

$$\Pi = \Pi'_J, \tag{2.27}$$

where the random sum $\Pi'_j = \Pi'_{J=j}$ is defined as in (2.25). Here we impose a slightly stronger assumption on the link function.

Assumption 2.10. In addition to Assumption 2.3, the link function Φ further satisfies that there exists $a > 1$ such that $\Phi'(t) \gtrsim |t|^{-a}$ when $|t|$ is sufficiently large.

An example of such a link function satisfying the requirements above is demonstrated in [FKW24b]. Similarly, we consider the push-forward posterior $\tilde{\Pi}(\cdot|Y^{(N)}, X^{(N)})$ mentioned in (2.23). We are now ready to prove the following theorem.

Theorem 2.11. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. Assume that g and u_0 satisfy (2.5), (2.6), and (2.9). Let $\theta \in (\frac{4}{5}, 1)$ and $h \in H_{\frac{\theta}{2}}(0, T; L^2(\mathcal{O}))$ satisfy (2.10). For integer $\alpha > 1 + d/2$, we consider the random series prior given in (2.27). There exists a sufficiently large $\alpha_0 = \alpha_0(d, \alpha) > 0$ (see (5.9) below) such that the following statement holds true: Let f_0 be the given ground truth satisfying $f_0 = \Phi(F_0)$, $F_0 \in H^{\alpha_0}(\mathcal{O})$ with $\mathrm{supp}\,(F_0) \subseteq \mathcal{K} \subset \mathcal{O}$ and the observations $(Y^{(N)}, Z^{(N)})$ be generated through the model (2.19) with $F = F_0$. Denote $\Pi_N\left(\cdot|Y^{(N)}, Z^{(N)}\right)$ the resulting posterior arising from observations $(Y^{(N)}, Z^{(N)})$. Then for each $D > 0$, one can find a positive constant L such that*

$$\tilde{\Pi}\left(f : \|f - f_0\|_{L^2(\mathcal{O})} > L\xi_N^{1/3}|Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}^N_{f_0}}(e^{-DN\xi_N^2}) \quad \text{as} \quad N \to \infty,$$

where $\xi_N = N^{-(\alpha_0+1)/(2\alpha_0+2+d)} \log N$.

Similar as in Theorem 2.8, the last contraction theorem also translates into a convergence result for the posterior mean of F.

Theorem 2.12. *Under the hypotheses of Theorem 2.11, let $\overline{F}_N := \mathbb{E}^{\Pi}(F|Y^{(N)}, X^{(N)})$ be the mean of $\Pi(\cdot|Y^{(N)}, X^{(N)})$. Then there exists a constant $C > 0$ such that*

$$\mathbb{P}^N_{F_0}\left(\|\overline{F}_N - F_0\|_{L^2(\mathcal{O})} > C\xi_N^{1/3}\right) \to 0 \quad \text{as} \quad N \to +\infty.$$

As an immediate consequence of (2.23), there exists a constant $C' > 0$ such that

$$\mathbb{P}^N_{f_0}\left(\|\Phi \circ \overline{F}_N - f_0\|_{L^2(\mathcal{O})} > C'\xi_N^{1/3}\right) \to 0 \quad \text{as} \quad N \to +\infty.$$

Likewise, Theorem 2.12 can be proved by following the same line as in the proof of Theorem 2.8. Note that for the random series prior Π given in (2.27), it also holds that $\mathbb{E}^{\Pi}\|F\|^2_{L^2(\mathcal{O})} < \infty$. For brevity, we omit the detailed proof here.

2.4.4. *A lower bound on the contraction rate.* We now give a lower bound on the contraction rate for any parameter (ground truth) in $\tilde{\mathcal{F}}_\alpha$, given below, in the statistical minimax sense. We define the function space

$$\tilde{\mathcal{F}}_\alpha = \left\{ f \in C^\alpha(\mathcal{O}) : \inf_{x\in\mathcal{O}} f(x) \geq c_1, \|f\|_{C^\alpha(\mathcal{O})} \leq c_2 \right\}. \tag{2.28}$$

for any fixed positive constants $0 < c_1 < c_2$. For simplicity, we consider the specific choice $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. The following theorem gives a lower bound that holds for *any estimator* of f, not necessarily the posterior mean mentioned in Theorem 2.8 and Theorem 2.12.

Theorem 2.13 (see also (5.12) below)**.** *Assume that g, u_0 satisfy* (2.5), (2.6), *and $h \in L^2(0,T;L^2(\mathcal{O}))$. Let $(Y^{(N)}, Z^{(N)})$ be observations generated through the model* (2.13)–(2.15) *with $f \in \tilde{\mathcal{F}}_\alpha$. For each $\alpha > d/2$, one has*

$$\inf_{\hat{f}_N} \sup_{f\in\tilde{\mathcal{F}}_\alpha} \mathbb{E}_f^N \|\hat{f}_N - f\|_{L^2(\mathcal{O})} \geq \frac{1}{2} N^{-\frac{\alpha}{2\alpha+2+d}} \quad \textit{for all sufficiently large } N,$$

where the infimum is taken over all measurable functions $\hat{f}_N = \hat{f}_N(Y^{(N)}, Z^{(N)})$.

By further refining the proof of Theorem 2.13, we can also obtain the following result.

Theorem 2.14. *If all assumptions in* Theorem 2.13 *hold, then there exists $c > 0$ such that for each sufficiently small constant $\epsilon > 0$ one has*

$$\liminf_{\hat{f}_N} \sup_{f\in\tilde{\mathcal{F}}_\alpha} \mathbb{P}_f^N \left(\|\hat{f}_N - f\|_{L^2(\mathcal{O})} > cN^{-\frac{\alpha}{2\alpha+2+d}} \right) \geq 1 - \epsilon \quad \textit{as } N \to \infty$$

where the infimum is taken over all measurable functions $\hat{f}_N = \hat{f}_N(Y^{(N)}, Z^{(N)})$.

One may naturally ask that it is possible to find an estimator in Theorem 2.6, Theorem 2.8 and Theorem 2.11 with a faster contraction rate. Here, Theorem 2.14 gives a negative answer: it is not possible to find an estimator $\hat{f}_N$ of f which contracts with a contraction rate faster than $N^{-\frac{\alpha}{2\alpha+2+d}}$. There remains a gap between the contraction rate $\delta_N^{1/3} = \frac{\alpha+1}{3(2\alpha+2+d)}$ from Theorem 2.6 and the rate $\frac{\alpha}{2\alpha+2+d}$ from Theorem 2.13, likely due to methodological limitation.

2.5. **Related results and remarks.** Our study is inspired by the results in [GN20] (for elliptic equations) and [Kek22] (for parabolic equations), both of which focus on local problems. In the local setting, employing smoother Gaussian priors yields smoother forward maps. Consequently, the Hölder exponent in the inverse stability estimate approaches 1 as the solution's smoothness increases. This, in turn, leads to faster posterior contraction toward the ground truth when smoother priors are used. However, when the true parameter lies within the reproducing kernel Hilbert space (RKHS) of the prior, overly smooth priors may lead to under-fitting in the inference process.

In contrast, time-fractional and spatial-fractional equations such as those involving the fractional Laplacian $(-\Delta)^s$ with $0 < s < 1$, exhibit intrinsically limited regularity of solutions. As a result, the stability estimates for the corresponding inverse problems cannot be improved by increasing the smoothness of the prior, leading to inherently slower posterior contraction rates. The key challenge in analyzing nonlocal equations lies in addressing this limited regularity. Moreover, while [GN20, Kek22] rely on Feynman-Kac representations to study solutions, our approach is based instead on the maximum principle. We believe this alternative strategy is more adaptable for handling the regularity challenges posed by nonlocal models.

For the purpose of uncertainty quantification, consistency alone is not sufficient. The ultimate objective is to establish a Bernstein-von Mises (BvM) theorem, which describes the asymptotic shape of the posterior distribution. However, it is well known that the BvM theorem does not generally hold in infinite-dimensional settings. Recent progress has been made in the semiparametric context for local PDEs, e.g., [Nic20, Nic24]. Extending these results to the nonlocal setting remains an open and compelling direction for future research.

3. Well-posedness of the IBVP

We now want to prove the well-posedness of the IBVP (2.12). Recall that w_g is the solution of (2.7) with given g and u_0 satisfying (2.5) and (2.6). First, we observe that

$$w_g(t,x) - w_g(0,x) = w_g(t,x) - u_0(x) \in H_\theta(0,T;H^2(\mathcal{O})). \tag{3.1}$$

By writing $v(t,x) = u_f(t,x) - w_g(t,x)$, we see that

$$\begin{cases} \partial_t^\theta v(t,x) - \Delta v(t,x) + f(x)v(t,x) & \\ \quad = h(t,x) - \partial_t^\theta(w_g(t,x) - w_g(0,x)) + \Delta w_g(t,x) - f(x)w_g(t,x) & \text{in } \mathcal{O}_T, \\ v \in H_0^1(\mathcal{O}), & t \in (0,T), \\ v \in H_\theta(0,T;L^2(\mathcal{O})). & \end{cases} \tag{3.2}$$

Note that, thanks to (3.1), it is easy to see that $v \in H_\theta(0,T;L^2(\mathcal{O}))$ if and only if $u - u_0 \in H_\theta(0,T;L^2(\mathcal{O}))$. Based on this observation, we now able to proof the following lemma.

Lemma 3.1. *Let $0 < \theta < 1$ and $T > 0$. Assume that g and u_0 satisfy (2.5) and (2.6). Let $f \in L^\infty(\mathcal{O})$, $f \geq 0$ a.e., and $h \in L^2(0,T;L^2(\mathcal{O}))$. Then, for each auxiliary parameter $\vartheta \in (0,1]$, there exists a unique solution u_f solving (2.12) and the following estimate holds*

$$\|u_f\|_{L^2(0,T;H^1(\mathcal{O}))} \leq C(1+\|f\|_{L^\infty(\mathcal{O})})(\|h\|_{L^2(0,T;L^2(\mathcal{O}))}+\|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)}+\|u_0\|_{C^{2+\vartheta}(\mathcal{O})}), \tag{3.3}$$

where $C > 0$ depends on $\theta, \vartheta, T, \mathcal{O}$, but are independent *of f.*

Proof. In view of [KRY20, Theorem 4.2], there exists a unique solution v satisfying (3.2) and by Lemma 2.2, (2.8), we have

$$\begin{aligned} &\frac{T^{-\theta}}{2\theta(1-\theta)}\|v\|^2_{L^2(0,T;L^2(\mathcal{O}))} + \|v\|^2_{L^2(0,T;H^1(\mathcal{O}))} \\ &\leq \int_0^T \langle \partial_t^\theta v(t,\cdot), v(t,\cdot)\rangle \,\mathrm{d}t + \int_0^T\int_{\mathcal{O}} |\nabla v|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathcal{O}} f|v|^2\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_{\mathcal{O}} (h - \partial_t^\theta(w_g - w_g(0,\cdot)) + \Delta w_g - fw_g)v\mathrm{d}x\mathrm{d}t \\ &\leq C(1+\|f\|^2_{L^\infty(\mathcal{O})})(\|h\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)} + \|u_0\|_{C^{2+\vartheta}(\mathcal{O})})^2 \\ &\quad + \frac{T^{-\theta}}{4\theta(1-\theta)}\|v\|^2_{L^2(0,T;L^2(\mathcal{O}))}, \end{aligned}$$

which implies

$$\|v\|_{L^2(0,T;H^1(\mathcal{O}))} \leq C(1+\|f\|_{L^\infty(\mathcal{O})})(\|h\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)} + \|u_0\|_{C^{2+\vartheta}(\mathcal{O})}),$$

and thus $u_f := v + w_g$ is the solution of (2.12) satisfying

$$\|u_f\|_{L^2(0,T;H^1(\mathcal{O}))} \leq C(1 + \|f\|_{L^\infty(\mathcal{O})})(\|h\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)} + \|u_0\|_{C^{2+\vartheta}(\mathcal{O})}).$$

It is important to point out that all constants C above depend on $\theta, T, \mathcal{O}$, but are *independent* of f. $\square$

Remark 3.2. *It should be noted that it follows directly from* [KRY20, Theorem 4.2] *that if $f \in C^1(\overline{\mathcal{O}})$, then there exists a unique $v \in L^2(0,T;H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})) \cap H_\theta(0,T;L^2(\mathcal{O}))$ satisfying* (3.2) *and there exists a constant $C > 0$ such that*

$$\begin{aligned}&\|v\|_{H_\theta(0,T;L^2(\mathcal{O}))} + \|v\|_{L^2(0,T;H^2(\mathcal{O}))}\\ &\leq C(1 + \|f\|_{L^\infty(\mathcal{O})})(\|h\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g\|_{C^{1+\vartheta/2,2+\vartheta}((\partial\mathcal{O})_T)} + \|u_0\|_{C^{2+\vartheta}(\mathcal{O})}).\end{aligned}$$

Hence, the solution u_f will satisfy a similar estimate. However, here the constant C depends on f implicitly. In our inverse problem, we need a regularity estimate of u_f with explicit dependence of $\|f\|_{L^\infty(\mathcal{O})}$. This is why we derive the weaker estimate (3.3).

Next, we would like to discuss the improvement of the time regularity of the solution u_f.

Lemma 3.3. *Let $T > 0$, $0 < \theta < 1$. Assume that g and u_0 satisfy* (2.5), (2.6). *Also, suppose $f \in C^1(\overline{\mathcal{O}})$ with $f \geq 0$. Let $\theta' \in (\frac{1}{2} - \frac{\theta}{8}, \frac{1}{2})$ and $h \in H_{\theta'}(0,T;L^2(\mathcal{O}))$. Then the solution $u_f \in C([0,T];C(\overline{\mathcal{O}}))$ of* (3.2) *satisfies*

$$\|u_f\|_{L^\infty(\mathcal{O}_T)} \leq C(1 + \|f\|^2_{L^\infty(\mathcal{O})})(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}) \tag{3.4}$$

with C independent of f, but depends on g and u_0.

Proof. From (2.11) and $(w_g(t,\cdot) - u_0(\cdot))|_{t=0} = 0$, we can see that $w_g(t,x) - u_0(x) \in H_\alpha(0,T;L^2(\mathcal{O}))$ whenever $0 < \alpha < 3/2$. Note that $\theta + \theta' < 1 + \theta' < 3/2$, thus $w_g(t,x) - w_g(0,x) = w_g(t,x) - u_0(x) \in H_{1+\theta'}(0,T;L^2(\mathcal{O})) \subset H_{\theta+\theta'}(0,T;L^2(\mathcal{O}))$. From (2.7) we see that $\partial_t^{\theta'}\Delta w_g = \partial_t^{1+\theta'} w_g \in L^2(\mathcal{O}_T)$, where we have used the norm equivalence (2.3).

As above, let $v(t,x) = u(t,x) - w_g(t,x)$, recall (3.2)

$$\begin{cases}\partial_t^\theta v(t,x) - \Delta v(t,x) + f(x)v(t,x) & \\ \quad = h(t,x) - \partial_t^\theta(w_g(t,x) - w_g(0,x)) + \Delta w_g(t,x) - f(x)w_g(t,x) & \text{in } \mathcal{O}_T, \\ \quad =: \tilde{h}(t,x), & \\ v \in H_0^1(\mathcal{O}), & t \in (0,T), \\ v \in H_\theta(0,T;L^2(\mathcal{O})). & \end{cases} \tag{3.5}$$

Now observe that $\partial_t^{\theta'}\tilde{h} \in L^2(0,T;L^2(\mathcal{O}))$. Next, by the similar idea in the proof of [Yam22, Theorem 12], we consider the equation

$$\begin{cases}\partial_t^\theta w(t,x) - \Delta w(t,x) + f(x)w(t,x) = \partial_t^{\theta'}\tilde{h}(t,x) & \text{in } \mathcal{O}_T, \\ w \in H_0^1(\mathcal{O}), & t \in (0,T), \\ w \in H_\theta(0,T;L^2(\mathcal{O})). & \end{cases} \tag{3.6}$$

By Lemma 3.1 (with $\vartheta = 1$), there exists a unique solution w solving (3.6) and satisfying

$$\|w\|_{L^2(0,T;H^1(\mathcal{O}))} \leq C(1 + \|f\|_{L^\infty(\mathcal{O})})\|\partial_t^{\theta'}\tilde{h}\|_{L^2(0,T;L^2(\mathcal{O}))}, \tag{3.7}$$

where C is independent of f (from (3.3)). Next, we can check that $J^{\theta'}w$ is the unique solution to (3.5). Indeed, let us denote $\tilde{v} = J^{\theta'}w$. Then, from (2.2) and the last condition of (3.6), one has $\tilde{v} \in H_{\theta+\theta'}(0,T;L^2(\mathcal{O}))$. Moreover, in view of Remark 3.2, $w \in L^2(0,T;H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))$. Using [Yam22, Proposition 5 (ii)], we obtain that

$$J^{-\theta}\tilde{v} = J^{\theta'}(J^{-\theta}w).$$

Also, we can see that

$$-\Delta\tilde{v} + f\tilde{v} = J^{\theta'}(-\Delta w + fw).$$

Consequently, we obtain from the first equation of (3.6) that

$$J^{-\theta}\tilde{v} - \Delta\tilde{v} + f\tilde{v} = J^{\theta'}(J^{-\theta'}w - \Delta w + fw) = J^{\theta'}(\partial_t^{\theta'}\tilde{h}) = \tilde{h}.$$

In other words, $v := \tilde{v}$ solves (3.5) and, by (3.7), v satisfies

$$\begin{aligned} \|v\|_{H_{\theta'}(0,T;H^1(\mathcal{O}))} &\le C(1 + \|f\|_{L^\infty(\mathcal{O})})\|\partial_t^{\theta'}\tilde{h}\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\le C(1 + \|f\|_{L^\infty(\mathcal{O})})(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}), \end{aligned} \tag{3.8}$$

where C is independent of f, but depends on g and u_0.

Estimate (3.8) immediately implies

$$\|u_f\|_{H_{\theta'}(0,T;H^1(\mathcal{O}))} \le C(1 + \|f\|_{L^\infty(\mathcal{O})})(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}), \tag{3.9}$$

where $C > 0$ is independent of f. This estimate will be useful later on.

We would now show that the regularity of v in (3.5) can be improved. We rewrite (3.5) as

$$\begin{cases} \partial_t^\theta v(t,x) - \Delta v(t,x) & \\ = h(t,x) - \partial_t^\theta(w_g(t,x) - w_g(0,x)) & \text{in } \mathcal{O}_T, \\ \quad + \Delta w_g(t,x) - f(x)w_g(t,x) - f(x)v(t,x), & \\ v \in H_0^1(\mathcal{O}), & t \in (0,T), \\ v \in H_\theta(0,T;L^2(\mathcal{O})). & \end{cases} \tag{3.10}$$

Applying [Yam22, Theorem 12] to (3.10) and using (3.8) implies that $v \in H_{\theta'}(0,T;H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})) \cap H_{\theta+\theta'}(0,T;L^2(\mathcal{O}))$ and

$$\begin{aligned} &\|v\|_{H_{\theta+\theta'}(0,T;L^2(\mathcal{O}))} + \|v\|_{H_{\theta'}(0,T;H^2(\mathcal{O}))} \\ &\le C\|h - \partial_t^\theta(w_g - w_g(0,\cdot)) + \Delta w_g - fw_g - fv\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))} \\ &\le C(1 + \|f\|_{L^\infty(\mathcal{O})}^2)(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}), \end{aligned} \tag{3.11}$$

where the constant $C > 0$ is independent of f.

For any parameter $\theta'' \in (0,1)$, from (3.11), we can use the interpolation theorem in [KRY20, Section 4.2.1] (applying to $\partial_t^{\theta'}v$) to derive that

$$\|v\|_{H_{\theta\theta''+\theta'}(0,T;H^{2-2\theta''}(\mathcal{O}))} \le C(1 + \|f\|_{L^\infty(\mathcal{O})}^2)(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}).$$

Choosing $\theta' \in (\frac{1}{2} - \frac{\theta}{8}, \frac{1}{2})$ and $\theta'' = \frac{1}{4} - \epsilon$ for small $\epsilon > 0$ gives

$$\theta\theta'' + \theta' > \frac{\theta}{4} - \theta\epsilon + \frac{1}{2} - \frac{\theta}{8} = \frac{1}{2} + \frac{\theta}{8} - \theta\epsilon > \frac{1}{2}, \quad 2 - 2\theta'' > \frac{3}{2}.$$

When $d = 1,2,3$, the Sobolev embedding theorem implies that $v \in C([0,T];C(\overline{\mathcal{O}}))$ and thus

$$\|v\|_{L^\infty(\mathcal{O}_T)} \le C(1 + \|f\|_{L^\infty(\mathcal{O})}^2)(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))}),$$

which, in turn, gives $u_f = u \in C([0,T]; C(\overline{\mathcal{O}}))$ and

$$\|u_f\|_{L^\infty(\mathcal{O}_T)} \le C(1 + \|f\|^2_{L^\infty(\mathcal{O})})(1 + \|h\|_{H_{\theta'}(0,T;L^2(\mathcal{O}))})$$

with C independent of f, but depends on g and u_0. □

We close this section by proving a useful maximum principle for u_f when the source function h satisfies an appropriate lower bound. In addition to the regularity and compatibility conditions (2.5), (2.6), we further assume that there exists $c > 0$ such that

$$g(t,x) \ge c,\ \forall\,(t,x) \in (\partial\mathcal{O})_T \ \text{ and }\ u_0(x) \ge c,\ \forall\, x \in \overline{\mathcal{O}}.$$

Then the classical maximum principle for the heat equation yields

$$w_g(t,x) \ge c \quad \text{for all } (t,x) \in \mathcal{O}_T.$$

Assume further that the source function $h \in L^2(0,T; L^2(\mathcal{O}))$ satisfies

$$h(t,x) \ge \partial_t^\theta(w_g(t,x) - w_g(0,x)) - \Delta w_g(t,x) + M_0 w_g(t,x),\ \forall\,(t,x) \in \mathcal{O}_T \text{ a.e.}$$

Then, by the (weak) maximum principle in [LY17, Theorem 2.1], v of (3.5) satisfies

$$v(t,x) \ge 0 \quad \text{for all } (t,x) \in \mathcal{O}_T \text{ a.e.}$$

which implies

$$u_f(t,x) \ge c \quad \text{for all } (t,x) \in \mathcal{O}_T, \tag{3.12}$$

for all $f \in C(\overline{\mathcal{O}})$ with $\|f\|_{L^\infty(\mathcal{O})} \le M_0$.

4. Estimates on forward and inverse problems

Given $h \in L^2(0,T; L^2(\mathcal{O}))$, g and u_0 satisfying (2.5) and (2.6). Let $f_j \in L^\infty(\mathcal{O})$ with $0 \le f_j$ a.e. in $\mathcal{O}$ for $j = 1, 2$. Denote u_{f_j} the solution of (2.12) corresponding to f_j. Then we can see that

$$\begin{cases} \partial_t^\theta(u_{f_1} - u_{f_2}) - \Delta(u_{f_1} - u_{f_2}) + f_1(u_{f_1} - u_{f_2}) \\ \qquad = -(f_1 - f_2)u_{f_2} & \text{in } \ \mathcal{O}_T, \\ (u_{f_1} - u_{f_2}) \in H_0^1(\mathcal{O}), & t \in (0,T), \\ (u_{f_1} - u_{f_2}) \in H_\theta(0,T; L^2(\mathcal{O})). \end{cases} \tag{4.1}$$

Applying Lemma 3.1 and estimate (3.3) implies

$$\begin{aligned} &\|u_{f_1} - u_{f_2}\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\le \|u_{f_1} - u_{f_2}\|_{L^2(0,T;H^1(\mathcal{O}))} \\ &\le C(1 + \|f_1\|_{L^\infty(\mathcal{O})})\|(f_1 - f_2)u_{f_2}\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\le C(1 + \|f_1\|_{L^\infty(\mathcal{O})})\|u_{f_2}\|_{L^2(0,T;H^1(\mathcal{O}))}\|f_1 - f_2\|_{(H^1(\mathcal{O}))^*} \\ &\le C(1 + \|f_1\|^2_{L^\infty(\mathcal{O})} \vee \|f_2\|^2_{L^\infty(\mathcal{O})})\|f_1 - f_2\|_{(H^1(\mathcal{O}))^*}, \end{aligned} \tag{4.2}$$

where C depends on T, d, $\mathcal{O}$, $\|h\|_{L^2(0,T;L^2(\mathcal{O}))}$, g, u_0, but is independent of f.

In order to apply the method in [GN20] (or [FKW24a]), we will need to verify the following conditions.

Proposition 4.1. *Let $T > 0$, $0 < \theta < 1$ and assume that g and u_0 satisfy (2.5), (2.6). Let $h \in H_{\theta'}(0, T; L^2(\mathcal{O}))$ for some $\theta' \in (\frac{1}{2} - \frac{\theta}{8}, \frac{1}{2})$. For any integer $\beta > 1 + d/2$, consider $f \in \mathcal{F}^{\beta}_{M_0}$ and define $\mathcal{G}(F)$ by (2.18). Then one has*

$$\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathcal{O}_T)} \lesssim (1 + \|F_1\|^2_{C^1(\mathcal{O})} \vee \|F_2\|^2_{C^1(\mathcal{O})})\|F_1 - F_2\|_{(H^1(\mathcal{O}))^*} \tag{4.3a}$$

for all $F_1, F_2 \in H^{\beta}_0(\mathcal{O})$. In addition, one has

$$\|\mathcal{G}\|_{H^{\beta}_0(\mathcal{O}) \to L^\infty(\mathcal{O}_T)} \equiv \sup_{F \in H^{\beta}_0(\mathcal{O})} \|\mathcal{G}(F)\|_{L^\infty(\mathcal{O}_T)} < \infty. \tag{4.3b}$$

Proof of Proposition 4.1. Let $f_1, f_2 \in \mathcal{F}^{\beta}_{M_0}$ and $f_1 = \Phi(F_1)$, $f_2 = \Phi(F_2)$ for some $F_1, F_2 \in H^{\beta}_0(\mathcal{O})$. Combining (4.2) and [NvdGW20, Lemma 29 (6.4)] immediately yields that

$$\begin{aligned}\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathcal{O}_T)} &\leq C(1 + \|\Phi(F_1)\|^2_{L^\infty(\mathcal{O})} \vee \|\Phi(F_2)\|^2_{L^\infty(\mathcal{O})})\|\Phi(F_1) - \Phi(F_2)\|_{(H^1(\mathcal{O}))^*} \\ &\leq C(1 + \|F_1\|^2_{C^1(\mathcal{O})} \vee \|F_2\|^2_{C^1(\mathcal{O})})\|F_1 - F_2\|_{(H^1(\mathcal{O}))^*}.\end{aligned}$$

On the other hand, when $d = 2, 3$, (4.3b) is an easy consequence of (3.4) in Lemma 3.3. □

Next, we will derive a Hölder-type stability estimate for the inverse problem similar to those in [Kek22, Proposition 10] and [NvdGW20, Lemma 28]. The main idea is based on the maximum principle given in (3.12). We begin with the following simple lemma.

Lemma 4.2. *Suppose that g and u_0 satisfy (2.5), (2.6), and (2.9). Let $h \in L^2(0, T; L^2(\mathcal{O}))$ satisfy (2.10), and $0 \leq f_1 \leq M_0, 0 \leq f_2 \leq M_0$ a.e. in $\mathcal{O}$. Denote u_{f_1}, u_{f_2} the solutions of (2.12) corresponding to $f = f_1, f_2$, respectively. Then we have*

$$\begin{aligned}&\|f_1 - f_2\|_{L^2(\mathcal{O})} \\ &\quad \leq C\left(1 + \|f_2\|_{L^\infty(\mathcal{O})}\right)\left(\|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;L^2(\mathcal{O}))} + \|u_{f_1} - u_{f_2}\|_{L^2(0,T;H^2(\mathcal{O}))}\right),\end{aligned} \tag{4.4}$$

for some constant $C > 0$, which is independent *of f_1 and f_2.*

Proof. From the reconstruction formula (1.2), we can write

$$\begin{aligned}T^{1/2}\|f_1 - f_2\|_{L^2(\mathcal{O})} &= \|f_1 - f_2\|_{L^2(\mathcal{O}_T)} \\ &= \left\| \frac{\partial_t^\theta(u_{f_1} - u_0) - \Delta u_{f_1}}{u_{f_1}} - \frac{\partial_t^\theta(u_{f_2} - u_0) - \Delta u_{f_2}}{u_{f_2}} \right\|_{L^2(\mathcal{O}_T)} \\ &\leq \left\| \frac{\partial_t^\theta(u_{f_1} - u_{f_2}) - \Delta(u_{f_1} - u_{f_2})}{u_{f_1}} \right\|_{L^2(\mathcal{O}_T)} \\ &\quad + \|(u_{f_1}^{-1} - u_{f_2}^{-1})(\partial_t^\theta(u_{f_2} - u_0) - \Delta u_{f_2})\|_{L^2(\mathcal{O}_T)} \\ &\leq c^{-1}\left(\|\partial_t^\theta(u_{f_1} - u_{f_2})\|_{L^2(\mathcal{O}_T)} + \|\Delta(u_{f_1} - u_{f_2})\|_{L^2(\mathcal{O}_T)}\right) + \|(u_{f_1}^{-1} - u_{f_2}^{-1})f_2 u_{f_2}\|_{L^2(\mathcal{O}_T)} \\ &\quad + \|(u_{f_1}^{-1} - u_{f_2}^{-1})h\|_{L^2(\mathcal{O}_T)},\end{aligned} \tag{4.5}$$

where we used the maximum principle (3.12) in the first term of the last inequality. Again, by (3.12), we can see that

$$\begin{aligned}\|(u_{f_1}^{-1} - u_{f_2}^{-1})f_2 u_{f_2}\|_{L^2(\mathcal{O}_T)} &= \|u_{f_1}^{-1}(u_{f_2} - u_{f_1})f_2\|_{L^2(\mathcal{O}_T)} \\ &\leq c^{-1}\|f_2\|_{L^\infty(\mathcal{O})}\|u_{f_1} - u_{f_2}\|_{L^2(\mathcal{O}_T)}\end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
\|(u_{f_1}^{-1}-u_{f_2}^{-1})h\|_{L^2(\mathcal{O}_T)} &= \|u_{f_1}^{-1}u_{f_2}^{-1}(u_{f_2}-u_{f_1})h\|_{L^2(\mathcal{O}_T)} \\
&\le c^{-2}\|(u_{f_2}-u_{f_1})h\|_{L^2(\mathcal{O}_T)} = c^{-2}\left(\int_0^T \|(u_{f_2}-u_{f_1})(t,\cdot)h\|_{L^2(\mathcal{O})}^2\,\mathrm{d}t\right)^{1/2} \\
&\le c^{-2}\|h\|_{L^2(\mathcal{O})}\left(\int_0^T \|u_{f_2}-u_{f_1}(t,\cdot)\|_{L^\infty(\mathcal{O})}^2\,\mathrm{d}t\right)^{1/2} \\
&\le Cc^{-2}\|h\|_{L^2(\mathcal{O})}\left(\int_0^T \|u_{f_2}-u_{f_1}(t,\cdot)\|_{H^2(\mathcal{O})}^2\,\mathrm{d}t\right)^{1/2} \\
&= Cc^{-2}\|h\|_{L^2(\mathcal{O})}\|u_{f_1}-u_{f_2}\|_{L^2(0,T;H^2(\mathcal{O}))}
\end{aligned}
\tag{4.7}
$$

where we have used the continuous embedding $H^2(\mathcal{O}) \subset L^\infty(\mathcal{O})$ since $d = 2, 3$. Combining (4.5), (4.6), (4.7), (2.3) yields (4.4). $\square$

Proposition 4.3. *Assume that g and u_0 satisfy (2.5), (2.6), and (2.9). Let $h \in H_{\frac{\theta}{2}}(0,T;L^2(\mathcal{O}))$ satisfy (2.10), and $f_1, f_2 \in C^1(\overline{\mathcal{O}})$ with $0 \le f_1 \le M_0, 0 \le f_2 \le M_0$ in $\mathcal{O}$. Then we have*

$$
\|f_1-f_2\|_{L^2(\mathcal{O})} \le C\,(1+M_0)\,(1+\|f_1\|_{C^1(\mathcal{O})}^{4/3}\vee\|f_2\|_{C^1(\mathcal{O})}^{4/3})\|u_{f_1}-u_{f_2}\|_{L^2(\mathcal{O}_T)}^{1/3},
\tag{4.8}
$$

where C depends on T, d, $\mathcal{O}$, $\|h\|_{H_{\frac{\theta}{2}}(0,T;L^2(\mathcal{O}))}$, g, u_0, but is independent of both f_1 and f_2.

Remark 4.4. *Using the link function Φ, (4.8) can be re-phrased as*

$$
\|\Phi\circ F_1-\Phi\circ F_2\|_{L^2(\mathcal{O})} \le C(1+\|F_1\|_{C^1(\mathcal{O})}^{4/3}\vee\|F_2\|_{C^1(\mathcal{O})}^{4/3})\|\mathcal{G}(F_1)-\mathcal{G}(F_2)\|_{L^2(\mathcal{O}_T)}^{1/3},
$$

where C depends on the quantities described above and M_0.

Proof of Proposition 4.3. The proof mainly utilize the linear estimate in Lemma 4.2 and the interpolation relation in [LM72b, (2.6)–(2.7)].

Step 1: Interpolation with respect to time variable. On the other hand, we write (4.1) as

$$
\begin{cases}
\partial_t^\theta(u_{f_1}-u_{f_2}) - \Delta(u_{f_1}-u_{f_2}) = f_2u_{f_2}-f_1u_{f_1} & \text{in } \mathcal{O}_T, \\
(u_{f_1}-u_{f_2}) \in H_0^1(\mathcal{O}), & t\in(0,T), \\
(u_{f_1}-u_{f_2}) \in H_\theta(0,T;L^2(\mathcal{O})).
\end{cases}
\tag{4.9}
$$

From [Yam22, Theorem 12] and estimate (3.9), it follows that, for each $0 < \theta_0 < 1/2$, we can derive

$$
\begin{aligned}
&\|u_{f_1}-u_{f_2}\|_{H_{\theta+\theta_0}(0,T;L^2(\mathcal{O}))} + \|u_{f_1}-u_{f_2}\|_{H_{\theta_0}(0,T;H^2(\mathcal{O}))} \\
&\quad\le C\left(\|f_2\|_{L^\infty(\mathcal{O})}\|u_{f_2}\|_{H_{\theta_0}(0,T;L^2(\mathcal{O}))} + \|f_1\|_{L^\infty(\mathcal{O})}\|u_{f_1}\|_{H_{\theta_0}(0,T;L^2(\mathcal{O}))}\right) \\
&\quad\le C(1+\|f_1\|_{L^\infty(\mathcal{O})}^2\vee\|f_2\|_{L^\infty(\mathcal{O})}^2)
\end{aligned}
\tag{4.10}
$$

for some constant $C > 0$, depending on T, d, $\mathcal{O}$, $\|h\|_{H_{\theta_0}(0,T;L^2(\mathcal{O}))}$, g, u_0, but is *independent of f_1 and f_2*. Now applying the interpolation relation in [LM72b, (2.7)] implies

$$\begin{aligned}\|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;L^2(\mathcal{O}))} &\le \|u_{f_1} - u_{f_2}\|^{\frac{\theta}{\theta+\theta_0}}_{H_{\theta+\theta_0}(0,T;L^2(\mathcal{O}))} \|u_{f_1} - u_{f_2}\|^{\frac{\theta_0}{\theta+\theta_0}}_{L^2(0,T;L^2(\mathcal{O}))} \\ &\le C\left(1 + \|f_1\|^{\frac{2\theta}{\theta+\theta_0}}_{L^\infty(\mathcal{O})} \vee \|f_2\|^{\frac{2\theta}{\theta+\theta_0}}_{L^\infty(\mathcal{O})}\right) \|u_{f_1} - u_{f_2}\|^{\frac{\theta_0}{\theta+\theta_0}}_{L^2(0,T;L^2(\mathcal{O}))}.\end{aligned}$$

In particular, taking $\theta_0 = \frac{\theta}{2}$, we obtain

$$\|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;L^2(\mathcal{O}))} \le C\left(1 + \|f_1\|^{4/3}_{L^\infty(\mathcal{O})} \vee \|f_2\|^{4/3}_{L^\infty(\mathcal{O})}\right) \|u_{f_1} - u_{f_2}\|^{1/3}_{L^2(0,T;L^2(\mathcal{O}))}. \tag{4.11}$$

Using (4.11) and (3.3), one can derive that

$$\|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;L^2(\mathcal{O}))} \le C\left(1 + \|f_1\|^{5/3}_{L^\infty(\mathcal{O})} \vee \|f_2\|^{5/3}_{L^\infty(\mathcal{O})}\right) \tag{4.12}$$

with C depending on T, d, $\mathcal{O}$, $\|h\|_{H_{\frac{\theta}{2}}(0,T;L^2(\mathcal{O}))}$, g, u_0.

Step 2: Interpolation with respect to spatial variables. Taking the gradient on the first equation in (4.9), and using the interpolation of space-time Sobolev spaces as in [KRY20, Section 4.2.1], one can estimate

$$\begin{aligned}&\|\nabla\Delta(u_{f_1} - u_{f_2})\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\quad\le \|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;H^1(\mathcal{O}))} \\ &\qquad + C(1 + \|f_1\|_{C^1(\mathcal{O})} \vee \|f_2\|_{C^1(\mathcal{O})}) \left(\|u_{f_1}\|_{L^2(0,T;H^1(\mathcal{O}))} + \|u_{f_2}\|_{L^2(0,T;H^1(\mathcal{O}))}\right) \\ &\quad\le \|u_{f_1} - u_{f_2}\|^{1/2}_{H_{3\theta/2}(0,T;L^2(\mathcal{O}))} \|u_{f_1} - u_{f_2}\|^{1/2}_{H_{\theta/2}(0,T;H^2(\mathcal{O}))} \\ &\qquad + C(1 + \|f_1\|_{C^1(\mathcal{O})} \vee \|f_2\|_{C^1(\mathcal{O})}) \left(\|u_{f_1}\|_{L^2(0,T;H^1(\mathcal{O}))} + \|u_{f_2}\|_{L^2(0,T;H^1(\mathcal{O}))}\right).\end{aligned} \tag{4.13}$$

Furthermore, in the right-hand side of (4.13), we apply (4.10) with $\theta_0 = \frac{\theta}{2}$ to its first term and using (3.3) to its second term to get

$$\|\nabla\Delta(u_{f_1} - u_{f_2})\|_{L^2(0,T;L^2(\mathcal{O}))} \le C(1 + \|f_1\|^2_{C^1(\mathcal{O})} \vee \|f_2\|^2_{C^1(\mathcal{O})}), \tag{4.14}$$

where $C > 0$ is independent of f_1 and f_2. On the other hand, by the first equation of (4.9), we have

$$\begin{aligned}&\|\Delta(u_{f_1} - u_{f_2})\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\quad\le \|u_{f_1} - u_{f_2}\|_{H_\theta(0,T;L^2(\mathcal{O}))} + \|f_1\|_{L^\infty(\mathcal{O})}\|u_{f_1}\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\qquad + \|f_2\|_{L^\infty(\mathcal{O})}\|u_{f_2}\|_{L^2(0,T;L^2(\mathcal{O}))} \\ &\quad\le C(1 + \|f_1\|^2_{L^\infty(\mathcal{O})} \vee \|f_2\|^2_{L^\infty(\mathcal{O})}),\end{aligned} \tag{4.15}$$

where C depends on T, d, $\mathcal{O}$, $\|h\|_{H_{\frac{\theta}{2}}(0,T;L^2(\mathcal{O}))}$, g, u_0. In (4.15), we have used (4.12) and (3.3). Putting together (4.14) and (4.15) yields

$$\|\Delta(u_{f_1} - u_{f_2})\|_{L^2(0,T;H^1(\mathcal{O}))} \le C(1 + \|f_1\|^2_{C^1(\mathcal{O})} \vee \|f_2\|^2_{C^1(\mathcal{O})}). \tag{4.16}$$

Observe that $u_{f_1} - u_{f_2} \in H^1_0(\mathcal{O})$. Notice that, for each integer $k \ge 2$, $-\Delta : H^k(\mathcal{O}) \cap H^1_0(\mathcal{O}) \to H^{k-2}(\mathcal{O})$ is an isomorphism and thus $(-\Delta)^{-1} : H^{k-2}(\mathcal{O}) \to H^k(\mathcal{O}) \cap H^1_0(\mathcal{O})$, for example, see [GT01, Theorem 8.13]. Thus, it follows from (4.16) that $u_{f_1} - u_{f_2} \in L^2(0,T;H^3(\mathcal{O}))$ and

$$\|u_{f_1} - u_{f_2}\|_{L^2(0,T;H^3(\mathcal{O}))} \le C(1 + \|f_1\|^2_{C^1(\mathcal{O})} \vee \|f_2\|^2_{C^1(\mathcal{O})}), \tag{4.17}$$

where $C > 0$ is independent of f_1 and f_2. Next, by the interpolation inequality [LM72b, (2.6)], we have

$$\begin{aligned}\|u_{f_1}-u_{f_2}\|_{L^2(0,T;H^2(\mathcal{O}))} &\le C\|u_{f_1}-u_{f_2}\|^{2/3}_{L^2(0,T;H^3(\mathcal{O}))}\|u_{f_1}-u_{f_2}\|^{1/3}_{L^2(0,T;L^2(\mathcal{O}))} \\ &\le C\left(1+\|f_1\|^{4/3}_{C^1(\mathcal{O})}\vee\|f_2\|^{4/3}_{C^1(\mathcal{O})}\right)\|u_{f_1}-u_{f_2}\|^{1/3}_{L^2(0,T;L^2(\mathcal{O}))}.\end{aligned} \tag{4.18}$$

Finally, Proposition 4.3 follows easily from Lemma 4.2, (4.11), and (4.18). □

5. Proof of theorems

We first recall the general contraction rate in [GN20, Theorem 14] without specifying to the operator (2.13). The statement of the theorem is modified to fit into our setting here.

Lemma 5.1. *Let $\mathcal{F}\subset L^2(\mathcal{O})$ be endowed with the trace Borel σ-field of $L^2(\mathcal{O})$, and consider a Borel-measurable forward map $\mathcal{G}:\mathcal{F}\to L^2(\mathcal{D})$, where $\mathcal{D}$ is a bounded measurable subset of $\mathbb{R}^m$ with $m\ge 1$. For $F_0\in\mathcal{F}$, we are given noisy discrete measurement of $\mathcal{G}(F_0)$ over a grid of points drawn uniformly at random on $\mathcal{D}$ as in the model (2.15). We further assume that $\sup_{F\in\mathcal{F}}\|\mathcal{G}(F)\|_{L^\infty(\mathcal{D})}<+\infty$ and there exist $\beta,\gamma,\kappa,\tau\ge 0$ such that*

$$\|\mathcal{G}(F_1)-\mathcal{G}(F_2)\|_{L^2(\mathcal{D})}\lesssim\left(1+\|F_1\|^{\gamma}_{C^\tau(\mathcal{O})}\vee\|F_2\|^{\gamma}_{C^\tau(\mathcal{O})}\right)\|F_1-F_2\|_{(H^\kappa(\mathcal{O}))^*}$$

for all $F_1,F_2\in H_0^\beta(\mathcal{O})\cap\mathcal{F}$. For integer $\alpha>\beta+d/2$, $\beta>\tau+d/2$ with $\tau\ge 1$, we consider a Gaussian prior Π_N constructed in (2.22) with base prior $F'\sim\Pi'$ satisfying Assumption 2.4 with reproducing-kernel Hilbert space $\mathcal{H}$. Let $\Pi_N\left(\cdot|Y^{(N)},Z^{(N)}\right)$ be the resulting posterior arising from observations $(Y^{(N)},Z^{(N)})$ as in (2.21). If $F_0\in\mathcal{H}$, then for each $D>0$ there exists a sufficiently large $L>0$ such that

$$\Pi_N\left(F:\|\mathcal{G}(F)-\mathcal{G}(F_0)\|_{L^2(\mathcal{D})}>L\delta_N|Y^{(N)},Z^{(N)}\right)=O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2})\quad \text{as } N\to+\infty,$$

where $\delta_N=N^{-(\alpha+\kappa)/(2\alpha+2\kappa+d)}$ and there exists a sufficiently large M such that

$$\Pi_N\left(F:\|F\|_{C^\tau(\mathcal{O})}>M|Y^{(N)},X^{(N)}\right)=O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2})\quad \text{as } N\to+\infty. \tag{5.1}$$

We now proof Theorem 2.6 as an application of Lemma 5.1.

Proof of Theorem 2.6. First of all, it follows from Proposition 4.1 that the conditions in Lemma 5.1 are satisfied with $\tau=1$, $\kappa=1$, and $\gamma=2$. Hence, there exists a sufficiently large $L>0$ such that

$$\Pi_N\left(F:\|\mathcal{G}(F)-\mathcal{G}(F_0)\|_{L^2(\mathcal{O})}>L\delta_N|Y^{(N)},Z^{(N)}\right)=O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2})\quad \text{as } N\to+\infty,$$

where $\delta_N=N^{-(\alpha+1)/(2\alpha+2+d)}$ and, furthermore, there exists a sufficiently large M such that

$$\Pi_N\left(F:\|F\|_{C^1(\mathcal{O})}>M|Y^{(N)},X^{(N)}\right)=O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2})\quad \text{as } N\to+\infty.$$

Now, by Remark 4.4, we can get that

$$\begin{aligned}&\Pi_N\left(\begin{matrix}F:\|\Phi\circ F-\Phi\circ F_0\|_{L^2(\mathcal{O})}>L'\delta_N^{1/3},\\ \|F\|_{C^1(\mathcal{O})}\le M\end{matrix}\middle| Y^{(N)},Z^{(N)}\right)\\ &\le\Pi_N\left(F:\|\mathcal{G}(F)-\mathcal{G}(F_0)\|_{L^2(\mathcal{O})}>L\delta_N|Y^{(N)},Z^{(N)}\right)\\ &=O_{\mathbb{P}^N_{F_0}}(e^{-DN\delta_N^2})\quad \text{as } N\to+\infty.\end{aligned}$$

In other words,

$$\begin{aligned}
&\tilde{\Pi}_N \left(f : \|f - f_0\|_{L^2(\mathcal{O}_T)} > L'\delta_N^{1/3} | Y^{(N)}, Z^{(N)}\right) \\
&\quad = \Pi_N \left(F : \|\Phi \circ F - \Phi \circ F_0\|_{L^2(\mathcal{O})} > L'\delta_N^{1/3} | Y^{(N)}, Z^{(N)}\right) \\
&\quad \le \Pi_N \left(F : \begin{array}{l} \|\Phi \circ F - \Phi \circ F_0\|_{L^2(\mathcal{O})} > L'\delta_N^{1/3} \\ \|F\|_{C^1(\mathcal{O})} \le M \end{array} \middle| Y^{(N)}, Z^{(N)}\right) \\
&\qquad + \Pi_N \left(F : \|F\|_{C^1(\mathcal{O})} > M | Y^{(N)}, X^{(N)}\right) \\
&\quad = O_{\mathbb{P}_{F_0}^N}(e^{-DN\delta_N^2}) \quad \text{as } N \to +\infty,
\end{aligned}$$

which implies our theorem. □

We now proceed to prove Theorem 2.8 by modifying the ideas in [GN20, Theorem 6] or [FKW24a, Theorem 2.6].

Proof of Theorem 2.8. In view of Jensen's inequality, it is enough to prove that for $\tilde{C} > 0$,

$$\mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} | Y^{(N)}, Z^{(N)}\right) > \tilde{C}\delta_N^{1/3}\right) \to 0 \quad \text{as } N \to +\infty. \tag{5.2}$$

We split the proof into two parts. Let $D > 0$ be a constant to be determined later, and let M be the constant given in Lemma 5.1.

In the first part, we estimate $\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} > M} | Y^{(N)}, Z^{(N)}\right)$. By the Cauchy-Schwartz inequality, it is clear that

$$\begin{aligned}
&\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} > M} | Y^{(N)}, Z^{(N)}\right) \\
&\quad \le \sqrt{\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})}^2 | Y^{(N)}, Z^{(N)}\right)} \sqrt{\Pi_N \left(F : \|F\|_{C^1(\mathcal{O})} > M | Y^{(N)}, Z^{(N)}\right)}.
\end{aligned}$$

Consequently, from (5.1) (with $\beta = 1$) in Lemma 5.1, we see that

$$\begin{aligned}
&\mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} > M} | Y^{(N)}, Z^{(N)}\right) > \delta_N^{1/3}\right) \\
&\quad \le \mathbb{P}_{F_0}^N \left(\begin{array}{l} \sqrt{\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})}^2 | Y^{(N)}, Z^{(N)}\right)} \\ \qquad \times \sqrt{\Pi_N \left(F : \|F\|_{C^1(\mathcal{O})} > M | Y^{(N)}, Z^{(N)}\right)} > \delta_N^{1/3} \end{array}\right) \\
&\quad \le \mathbb{P}_{F_0}^N \left(\sqrt{\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})}^2 | Y^{(N)}, Z^{(N)}\right)} e^{-\frac{1}{2}DN\delta_N^2} > \delta_N^{1/3}\right) + o(1).
\end{aligned} \tag{5.3}$$

Recall from [GN20, Lemmas 16 and 23] that the set

$$\mathcal{B}_N := \left\{F : \mathbb{E}_{F_0}^1 \left(\log \frac{p_{F_0}(Y_1, Z_1)}{p_F(Y_1, Z_1)}\right) \le \delta_N^2, \mathbb{E}_{F_0}^1 \left(\log \frac{p_{F_0}(Y_1, Z_1)}{p_F(Y_1, Z_1)}\right)^2 \le \delta_N^2\right\}$$

satisfies $\Pi_N(\mathcal{B}_N) \geq ae^{-AN\delta_N^2}$ for some $a, A > 0$. On the other hand, using [GN21, Lemma 7.3.2], we also know that the set

$$\mathcal{C}_N = \left\{ \int_{\mathcal{B}_N} \prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i, Z_i)\,\mathrm{d}\nu(F) \geq e^{-2N\delta_N^2} \right\} \quad \text{with} \quad \nu(\cdot) = \frac{\Pi_N(\cdot \cap \mathcal{B}_N)}{\Pi_N(\mathcal{B}_N)}$$

satisfies $\mathbb{P}_{F_0}^N(\mathcal{C}_N) \to 1$ as $N \to +\infty$. Now from (5.3), it follows that

$$\begin{aligned}
&\mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} > M} | Y^{(N)}, Z^{(N)} \right) > \delta_N^{1/3} \right) \\
&\quad \leq \mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})}^2 | Y^{(N)}, Z^{(N)} \right) e^{DN\delta_N^2} > \delta_N^{1/6}, \mathcal{C}_N \right) + o(1) \\
&\quad = \mathbb{P}_{F_0}^N \left(\begin{array}{l} \dfrac{\int_{C(\mathcal{O})} \|F - F_0\|_{L^2(\mathcal{O})}^2 \prod_{i=1}^N p_F/p_{F_0}(Y_i, Z_i)\,\mathrm{d}\Pi_N(F)}{\Pi(\mathcal{B}_N) \int_{\mathcal{B}_N} \prod_{i=1}^N p_F/p_{F_0}(Y_i, Z_i)\,\mathrm{d}\nu(F)} e^{-DN\delta_N^2} \\ \quad > \delta_N^{1/6} \end{array}, \mathcal{C}_N \right) + o(1) \qquad (5.4) \\
&\quad \leq \mathbb{P}_{F_0}^N \left(\begin{array}{l} \displaystyle\int_{C(\mathcal{O})} \|F - F_0\|_{L^2(\mathcal{O})}^2 \prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i, Z_i)\,\mathrm{d}\Pi_N(F) \\ \quad > \delta_N^{1/6} a e^{(D-A-2)N\delta_N^2} \end{array} \right) + o(1).
\end{aligned}$$

By Markov's inequality and Fubini's theorem, we obtain that (5.4) is bounded above by

$$\begin{aligned}
&\delta_N^{-1/6} a^{-1} e^{-(D-A-2)N\delta_N^2} \int_{C(\mathcal{O})} \|F - F_0\|_{L^2(\mathcal{O})}^2 \mathbb{E}_{F_0}^N \left(\prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i, Z_i) \right) \mathrm{d}\Pi_N(F) \\
&\quad = \delta_N^{-1/6} a^{-1} e^{-(D-A-2)N\delta_N^2} \int_{C(\mathcal{O})} \|F - F_0\|_{L^2(\mathcal{O})}^2 \,\mathrm{d}\Pi_N(F).
\end{aligned}$$

Furthermore, by Fernique's theorem [GN21, Exercises 2.1.1, 2.1.2 and 2.1.5] one has $\mathbb{E}^{\Pi_N} \|F\|_{L^2(\mathcal{O})}^2 < +\infty$. Taking $D > A + 2$, we conclude that

$$\mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} > M} | Y^{(N)}, Z^{(N)} \right) > \delta_N^{1/3} \right) \to 0 \qquad (5.5)$$

as $N \to +\infty$.

For the second part, we estimate $\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} \leq M} | Y^{(N)}, Z^{(N)} \right)$. Since $f = \Phi \circ F$ and $f_0 = \Phi \circ F_0$, by Assumption 2.3(i), mean value theorem and inverse function theorem, there exists η lying between $f_0(x)$ and $f(x)$ such that

$$|F(x) - F_0(x)| = \frac{1}{|\Phi'(\Phi^{-1}(\eta))|} |f(x) - f_0(x)| \quad \text{for all } x \in \mathcal{O}. \qquad (5.6)$$

Since $f, f_0 \in [\Phi(-M), \Phi(M)]$ and $\|F\|_{C^1} \leq M$, we have

$$|F(x) - F_0(x)| \leq \frac{1}{\min_{[-M,M]} \Phi'} |f(x) - f_0(x)| \lesssim |f(x) - f(x_0)| \quad \text{for all } x \in \mathcal{O}.$$

Therefore, we see that

$$\begin{aligned}
&\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} \leq M} | Y^{(N)}, Z^{(N)} \right) \lesssim \mathbb{E}^{\tilde{\Pi}_N} \left(\|f - f_0\|_{L^2(\mathcal{O})} | Y^{(N)}, Z^{(N)} \right) \\
&\quad \leq L'\delta_N^3 + \mathbb{E}^{\tilde{\Pi}_N} \left(\|f - f_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|f - f_0\|_{L^2(\mathcal{O})} > L'\delta_N^3} | Y^{(N)}, Z^{(N)} \right)
\end{aligned}$$

where L' is the constant given in Theorem 2.6. We now repeat the arguments in the first part, except that replacing the event $\left\{F : \|F\|_{C^1(\mathcal{O})} > M\right\}$ by the event $\left\{f : \|f - f_0\|_{L^2(\mathcal{O})} > L'\delta_N^{1/3}\right\}$ and using Theorem 2.6, to show that

$$\mathbb{P}_{F_0}^N \left(\mathbb{E}^{\Pi_N} \left(\|F - F_0\|_{L^2(\mathcal{O})} \mathbb{1}_{\|F\|_{C^1(\mathcal{O})} \le M} | Y^{(N)}, Z^{(N)} \right) > \delta_N^{1/3} \right) \to 0 \tag{5.7}$$

as $N \to +\infty$. Finally, we combine (5.5) and (5.7) to conclude (5.2) with $\tilde{C} = L' + 1$. □

Similar to Lemma 5.1, we now recall the general contraction rate for *random series prior* in [GN20, Theorem 19] without specifying to the operator (2.13).

Lemma 5.2. *Assume that the assumptions of Lemma 5.1 are satisfied. Let Π be the random series prior defined in (2.27), and let $\Pi_N \left(\cdot | Y^{(N)}, Z^{(N)}\right)$ be the resulting posterior arising from observations $(Y^{(N)}, Z^{(N)})$ as in (2.21). Then for each $\alpha_0 \ge \alpha$ and for each $F_0 \in H_0^{\alpha_0}(\mathcal{O})$ with compact support such that $\operatorname{supp}(F_0) \subset \mathcal{O}$ and for each $D > 0$, there exists a sufficiently large $L > 0$ such that*

$$\Pi_N \left(F : \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathcal{D})} > L\xi_N | Y^{(N)}, X^{(N)} \right) = O_{\mathbb{P}_{F_0}^N}(e^{-DN\xi_N^2}),$$

where $\xi_N = N^{-(\alpha_0+\kappa)/(2\alpha_0+2\kappa+d)} \log N$. For each $j \in \mathbb{N}$, let $\mathcal{H}_j$ be the finite-dimensional space given in (2.26). Furthermore, if we choose $j(N) \in \mathbb{N}$ satisfying $2^{j(N)} \simeq N^{1/(2\alpha_0+2\kappa+d)}$, then we obtain that for sufficiently large $M > 0$ such that

$$\Pi_N \left(F \in \mathcal{H}_{j(N)} : \|F\|_{H^\alpha} \ge M 2^{j(N)\alpha} N \xi_N^2 | Y^{(N)}, X^{(N)} \right) = O_{\mathbb{P}_{F_0}^N}(e^{-DN\xi_N^2}).$$

As above, we will prove Theorem 2.11 by adapting the ideas in [GN20, Lemma 12].

Proof of Theorem 2.11. From Lemma 5.2 (with $\kappa = 1$ and $\gamma = 2$), for each $D > 0$ and for each sufficiently large $L, M > 0$, one has

$$\Pi_N(\mathcal{A}_N | Y^{(N)}, X^{(N)}) = 1 - O_{\mathbb{P}_{F_0}^N}(e^{-DN\xi_N^2}),$$

where $\mathcal{A}_N = \{F \in \mathcal{H}_{j(N)} : \|F\|_{H^\alpha} \le M 2^{j(N)\alpha} \sqrt{N} \xi_N, \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2} \le L\xi_N\}$.

We fix any $F \in \mathcal{H}_{j(N)}$. In view of [GN20, (B7)], we have $\|F\|_{H^\alpha} \lesssim 2^{j(N)\alpha} \|F\|_{L^2}$ for all sufficiently large N. Let $P_{\mathcal{H}_j}$ be the projection defined in [GN20, (B4)], then we obtain that

$$\begin{aligned} \|F\|_{H^\alpha(\mathcal{O})} &\le \|F - P_{\mathcal{H}_{j(N)}}(F_0)\|_{H^\alpha(\mathcal{O})} + \|P_{\mathcal{H}_{j(N)}}(F_0)\|_{H^\alpha(\mathcal{O})} \\ &= \|P_{\mathcal{H}_{j(N)}}(F) - P_{\mathcal{H}_{j(N)}}(F_0)\|_{H^\alpha(\mathcal{O})} + \|P_{\mathcal{H}_{j(N)}}(F_0)\|_{H^\alpha(\mathcal{O})} \\ &\le 2^{j(N)\alpha} \|F - F_0\|_{L^2(\mathcal{O})} + \|F_0\|_{H^\alpha(\mathcal{O})}. \end{aligned} \tag{5.8}$$

Furthermore, for $F \in \mathcal{A}_N$, the Sobolev embedding theorem gives $\|F\|_{L^\infty} \le M' 2^{j(N)\alpha} \sqrt{N} \xi_N$ for some $M' > 0$. Now we use (5.6) and Assumption 2.10 to see that

$$\|F - F_0\|_{L^2(\mathcal{O})} \lesssim (2^{j(N)\alpha} \sqrt{N} \xi_N)^a \|f - f_0\|_{L^2(\mathcal{O})}.$$

We now apply the inverse estimate in Proposition 4.3 to see that

$$\|F - F_0\|_{L^2(\mathcal{O})} \lesssim (2^{j(N)\alpha} \sqrt{N} \xi_N)^a (1 + \|f\|_{H^\alpha(\mathcal{O})}^{4/3}) \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathcal{O}_T)}^{1/3}$$

for all $F \in \mathcal{A}_N$ and thus

$$\begin{aligned}2^{j(N)\alpha}\|F-F_0\|_{L^2(\mathcal{O})} &\lesssim 2^{j(N)\alpha}(2^{j(N)\alpha}\sqrt{N}\xi_N)^{a+4/3}\xi_N^{1/3}\\ &\simeq N^{\frac{\alpha}{2\alpha_0+2+d}}(N^{\frac{2\alpha+d}{2(2\alpha_0+2+d)}}\log N)^{\frac{3a+4}{3}}N^{-\frac{\alpha_0+1}{3(2\alpha_0+2+d)}}(\log N)^{1/3}\\ &\simeq N^{\frac{-2(\alpha_0+1)+(2\alpha+d)(3a+4)+6\alpha}{6(2\alpha_0+2+d)}}(\log N)^{\frac{3a+5}{3}}.\end{aligned}$$

Now we choose $a_0 = \alpha_0(\alpha, d)$ sufficiently large such that

$$2(\alpha_0+1) > (2\alpha+d)(3a+4)+6\alpha \tag{5.9}$$

and consequently from (5.8) we have

$$\|F\|_{H^\alpha(\mathcal{O})} \lesssim 1 + \overbrace{N^{\frac{-2(\alpha_0+1)+(2\alpha+d)(3a+4)+6\alpha}{6(2\alpha_0+2+d)}}(\log N)^{\frac{3a+5}{3}}}^{\to 0 \quad \text{as } N\to+\infty} \quad \text{for all } F \in \mathcal{A}_N.$$

Thus, we obtain

$$\begin{aligned}&1 - O_{\mathbb{P}^N_{F_0}}(e^{-DN\xi_N^2}) = \Pi_N(\mathcal{A}_N|Y^{(N)}, X^{(N)})\\ &\quad\le \Pi_N\left(F \in \mathcal{H}_{j(N)} : \|\mathcal{G}(F)-\mathcal{G}(F_0)\|_{L^2(\mathcal{O}_T)} \le L\xi_N, \|F\|_{H^\alpha(\mathcal{O})} \lesssim 1|Y^{(N)}, X^{(N)}\right).\end{aligned}$$

Finally, from Proposition 4.3, it follows that

$$\begin{aligned}&\tilde{\Pi}\left(f : \|f-f_0\|_{L^2(\mathcal{O})} > L\xi_N^{1/3}|Y^{(N)}, X^{(N)}\right)\\ &\quad\le \Pi_N\left(\|\mathcal{G}(F)-\mathcal{G}(F_0)\|_{L^2(\mathcal{O}_T)} \ge L\xi_N, \|F\|_{H^\alpha(\mathcal{O})} \lesssim 1|Y^{(N)}, X^{(N)}\right)\\ &\qquad+ \Pi_N\left(\|F\|_{H^\alpha(\mathcal{O})} \gtrsim 1|Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}^N_{F_0}}(e^{-DN\xi_N^2}) \ \text{ as } \ N\to\infty,\end{aligned}$$

which concludes the theorem. □

We now prove the optimality result in Theorem 2.13 by adopting the proof of [Kek22, Theorem 8]. The idea is to apply [GN21, Theorem 6.3.2] to reduce the problem of estimating the lower bound in the whole parameter space into a test problem in a finite subset of $\tilde{\mathcal{F}}_\alpha$, see also [Nic20, NvdGW20].

Proof of Theorem 2.13. The central idea in the proof is to construct a finite $N^{-\frac{\alpha}{2\alpha+2+d}}$-separated set in $\tilde{\mathcal{F}}_\alpha$ given by (2.28) which is not too small. As described on [Kek22, Page 18], for every $j \in \mathbb{N}$, there exist a small constant $c > 0$ such that $n_j := c2^{jd}$ many Daubechies wavelets $\{\Psi_{jr}\}_{r=1}^{n_j}$ have disjoint compact supports in $\mathcal{O}$. Using the Varshamov-Gilbert bound (see [GN21, Example 3.1.4]), there exists

$$\{b_{m,\cdot} : m = 1, \cdots, M_j\} \in \{-1, +1\}^{n_j} \ \text{ with } \ M_j \ge 3^{n_j/4}$$

such that

$$\sum_{r=1}^{n_j}(b_{m,r}-b_{m',r})^2 \gtrsim n_j.$$

Let $\kappa > 0$ a constant to be determined later and define

$$h_m(x) := \kappa\sum_{r=1}^{n_j} b_{mr}2^{-j(\alpha+d/2)}\Psi_{jr}(x) \quad \text{for all } x \in \mathcal{O},$$

for all $m = 0, 1, \cdots, M_j$. in view of the support condition of Ψ_{jr}, it is clear that supp $(h_m) \subset \mathcal{O}$. In addition, it is proved on [Kek22, Page 18] that one can choose a sufficiently small

$\kappa > 0$, independent of j, such that $\|h_m\|_{C^\alpha(\mathcal{O})} \le 1/2$ for all $m = 1, \cdots, M_j$. Let $f_0 \equiv 1$ and we define

$$f_m := f_0 + h_m, \quad \forall\ m = 1, \cdots, M_j,$$

which satisfies

$$\|f_m\|_{C^\alpha(\mathcal{O})} \le \|f_0\|_{C^\alpha(\mathcal{O})} + \|h_m\|_{C^\alpha(\mathcal{O})} \le 3/2 \ \text{ and } \ \inf_{x\in\mathcal{O}} f_m \ge 1 - \|h_m\|_{L^\infty(\mathcal{O})} \ge 1/2,$$

that is, $\{f_m\}_{m=1}^{M_j} \subset \tilde{\mathcal{F}}_\alpha$.

Observe that

$$\|f_m - f_{m'}\|^2_{L^2(\mathcal{O})} = \kappa^2 2^{-2j(\alpha+d/2)} \sum_{r=1}^{n_j} (b_{mr} - b_{m'r})^2 \gtrsim 2^{-2j(\alpha+d/2)} n_j \simeq 2^{-2j\alpha}.$$

For each $N \in \mathbb{N}$, we now choose $j = j(N) \in \mathbb{N}$ satisfying $2^j \simeq N^{\frac{1}{2\alpha+2+d}}$ and, hence,

$$\|f_m - f_{m'}\|_{L^2(\mathcal{O})} \gtrsim N^{-\frac{\alpha}{2\alpha+2+d}} \tag{5.10}$$

Our next task is to estimate M_j from below to ensure that the set $\{f_m\}_{m=1}^{M_j} \subset \tilde{\mathcal{F}}_\alpha$ is not too small. Based on the Radon-Nikodym density (2.20), the argument carried out in the proof of [Kek22, Theorem 8] shows that

$$\mathrm{KL}\,(\mathbb{P}^N_{f_m}, \mathbb{P}^N_{f_0}) \simeq N\|u_{f_0} - u_{f_m}\|^2_{L^2(\mathcal{O}_T)},$$

where $\mathrm{KL}(\cdot,\cdot)$ is the Kullback-Leibler divergence. Using (4.2), [Kek22, (25)] and the choice $2^j \simeq N^{\frac{1}{2\alpha+2+d}}$, we get that

$$\begin{aligned}\|u_{f_0} - u_{f_m}\|^2_{L^2(\mathcal{O}_T)} &\lesssim \|f_m - f_0\|^2_{(H^1(\mathcal{O}))^*}\\ &\lesssim \|h_m\|^2_{H^{-1}(\mathbb{R}^d)} = \kappa^2 2^{-2j(\alpha+d/2+1)} \sum_{r=1}^{n_j} 1 \simeq \kappa^2 N^{-1} n_j.\end{aligned}$$

By the definition of M_j, we can see that

$$\mathrm{KL}\,(\mathbb{P}^N_{f_m}, \mathbb{P}^N_{f_0}) \le \epsilon \log(M_j) \tag{5.11}$$

for any small $\epsilon > 0$ (by taking κ sufficiently small).

With slightly abuse of the notation, we write $M_N = M_{j(N)}$, where $j = j(N) \in \mathbb{N}$ satisfies $2^j \simeq N^{\frac{1}{2\alpha+2+d}}$. So far, we have proved that $\{f_m\}_{m=1}^{M_N}$ is a $N^{-\frac{\alpha}{2\alpha+2+d}}$-separated set in $\tilde{\mathcal{F}}_\alpha$ in the sense of (5.10) and is not too small in the sense of (5.11). Finally, applying [GN21, Theorem 6.3.2] leads to

$$\inf_{\hat{f}_N} \sup_{f\in\tilde{\mathcal{F}}_\alpha} \mathbb{E}^N_f \|\hat{f}_N - f\|_{L^2(\mathcal{O})} \ge N^{-\frac{\alpha}{2\alpha+2+d}} \frac{\sqrt{M_N}}{1+\sqrt{M_N}} \left(1 - 2\epsilon - \sqrt{\frac{8\epsilon}{\log M_N}}\right), \tag{5.12}$$

which implyies the desired estimate by choosing ϵ small enough and the fact $M_N \to +\infty$ as $N \to +\infty$. □

For each $N \in \mathbb{N}$, let Ψ_N be any $\{0, 1, \cdots, M_N\}$-valued measurable function such that

$$\|\hat{f}_N - f_{\psi_N}\|_{L^2(\mathcal{O})} = \min_{m=0,\cdots,M_N} \|\hat{f}_N - f_m\|_{L^2(\mathcal{O})}.$$

By carefully inspecting the proof of [GN21, Theorem 6.3.2], we can see that

$$\inf_{\Psi_N} \max_{m=0,\cdots,M_N} \mathbb{P}^N_{f_m}(\Psi_N \neq m) \ge \frac{\sqrt{M_N}}{1+\sqrt{M_N}} \left(1 - 2\epsilon - \sqrt{\frac{8\epsilon}{\log M_N}}\right), \tag{5.13}$$

for all sufficiently large N, which is slightly stronger than (5.12). We are now give a proof of Theorem 2.14 based on this observation.

Proof of Theorem 2.14. In view of (5.10), it was proved in [Kek22, Theorem 8] that (with $r_N \simeq N^{-\frac{\alpha}{2\alpha+2+d}}$)

$$\liminf_{\hat{f}_N} \sup_{f \in \tilde{\mathcal{F}}_\alpha} \mathbb{P}_f^N \left(\|\hat{f}_N - f\|_{L^2(\mathcal{O})} > cN^{-\frac{\alpha}{2\alpha+2+d}} \right) \geq \liminf_{\Psi_N} \max_{m=0,\cdots,M_N} \mathbb{P}_{f_m}^N (\Psi_N \neq m).$$

The theorem follows immediately from above and (5.13). $\square$

6. Conclusions

In this paper, we investigate a Bayesian approach to an inverse problem for a time-fractional partial differential equation involving the Riemann-Liouville time derivative. The objective is to recover an unknown potential function from interior measurements of the solution. From a practical standpoint, such measurements are typically available only at discrete points, which naturally raises the question of how accurately the unknown function can be estimated. The Bayesian framework offers a principled way to address this question.

Our first objective is to justify the Bayesian method from a frequentist perspective. Specifically, we show that when the data is generated from a true underlying potential, the posterior distributions which are obtained using various commonly employed priors, contract toward the ground truth at explicit rates as the sample size increases. Because inverse problems based on interior measurements are mildly ill-posed, in the sense that their stability estimates are of Hölder type, we derive polynomial posterior contraction rates, as expected in such settings.

A major challenge in analyzing time-fractional equations is the limited regularity of their solutions. Similar difficulties also arise in spatial-fractional problems, such as those involving the fractional Laplacian. To address this issue, we employ the maximum principle to establish positivity of the solution, rather than relying on the Feynman-Kac representation. We believe this approach is more flexible and may be adapted to a broader class of equations.

For the purpose of uncertainty quantification, posterior consistency alone is not sufficient. The ultimate goal is to establish a Bernstein-von Mises (BvM) theorem, which characterizes the limiting shape of the posterior distribution. However, it is well known that the BvM theorem does not generally hold in infinite-dimensional settings. Some progress has been made in the semiparametric regime for local PDEs, e.g., [Nic20, Nic24]. Whether an analogous result can be established for nonlocal equations remains an open and compelling problem for future research.

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