Advanced Algebra I

TRANSCENDENTAL EXTENSION

Before we move on to the transcendental extension. We first complete the proof of the Corollary of last time.

Corollary 0.1. *Let F/K be an algebraic extension with $char(K) = p \neq 0$. We have*

1. *If F/K is separable, then $F = KF^{p^n}$ for each $n \geq 1$.*
2. *If F/K is finite and $F = KF^p$, then F/K is separable.*
3. *In particular, $u \in F$ is separable over K if and only if $K(u^p) = K(u)$.*

Note that F^p is not necessarily an extension over K. So is F^{p^n}. But we can take KF^{p^n}, which is an extension over K.

Proof. We first suppose that F/K is finite, hence finitely generated. Write $F = K(u_1, ..., u_r)$. It's clear that there is $N \geq 1$ such that $u^{p^N} \in S$. Hence $F^{p^N} \subset S$, therefore, $KF^{p^N} \subset S$.

We claim that $S = KF^{p^N}$. To see this, one notices that F is purely inseparable over KF^{p^N}, so is S purely inseparable over KF^{p^N}. And on the other hand, S is separable over K, so is over KF^{p^N}. Hence $S = KF^{p^N}$.

For (1), if F/K is separable and finite, then we have $F = KF^{p^N}$. However, in the proof, one can choose N to be arbitrary large. More precisely, one has $F = KF^{p^N}$ for all $N \geq N_0$. By looking at the inclusion

$$F = KF^{p^N} \subset KF^{p^{N-1}} \subset ... \subset KF^p \subset F.$$

One has $F = KF^{p^n}$ for all $n \geq 1$.

Suppose now that F/K is separable but not necessarily finite. For any $u \in F$, we consider $F_0 := K(u)$ which is separable and finite over K. Thus $u \in F_0 = KF_0^{p^n} \subset KF^{p^n}$ for all $n \geq 1$. This proves (1).

We now prove (2). If $F = KF^p$, then $F = K(KF^p)^p = KF^{p^2}$. Inductively, one has $F = KF^{p^n}$ for all $n \geq 1$. Since we have show that $S = KF^{p^N}$, it follows that $F = S$.

Apply the statement to a single element. We consider $F = K(u)$. $F^p \subset K^p(u^p) \subset K(u^p)$. Indeed, $KF^p = K(u^p)$. By (2), if $K(u) = K(u^p)$, then u is separable. By (1), if u is separable, then $K(u) = K(u^p)$. □

We now start our discussion on transcendental extension. The main purpose is to show that the concept of *transcendental degree*, which is the cardinality of transcendental basis, can be well-defined. Moreover, transcendental degree is a good candidate for defining dimension.

Definition 0.2. *Let F/K be an extension. $S \subset F$ is said to be algebraically dependent (over K) if there is an $n \geq 1$ and an $f \neq 0 \in K[x_1, ..., x_n]$ such that $f(s_1, ..., s_n) = 0$ for some $s_1, ..., s_n$. Roughly speaking, some element of S satisfy a non-zero algebraic relation f over K.*

S is said to be algebraically independent over K if it's not algebraically dependent over K.

Example 0.3. *For any $u \in F$, $\{u\}$ is algebraically dependent over K if and only if u is algebraic over K.*

Example 0.4. *In the extension $K(x_1, ..., x_n)/K$, $S = \{x_1, ..., x_n\}$ is algebraically independent over K.*

The following theorem says that finitely generated purely transcendental extension are just rational function fields.

Theorem 0.5. *If $\{s_1, ..., s_n\} \subset F$ is algebraically independent over K. Then $K(s_1, ..., s_n) \cong K(x_1, ..., x_n)$.*

Proof. We consider the homomorphism $\theta : K[x_1, ..., x_n] \to K[s_1, ..., s_n]$. θ is surjective by definition. It's injective because $\{s_1, ..., s_n\} \subset F$ is algebraically independent. Then θ induces an isomorphism on quotient fields. □

One notices that the notion of being algebraic independent is an analogue of being linearly independent. Therefore, one can try to define the notion of "basis" and "dimension" in a similar way.

Definition 0.6. *$S \subset F$ is said to be a transcendental basis of F/K if S is a maximal algebraically independent set. In other words, for all $u \in F - S$, $S \cup \{u\}$ is algebraically dependent.*

We will then define the *transcendental degree* to be the cardinality of a transcendental basis (in a analogue of dimension). In order to show that this is well-defined. We need to work harder.

Proposition 0.7. *Let $S \subset F$ be an algebraically independent set over K and $u \in F - K(S)$. Then $S \cup \{u\}$ is algebraically independent if and only if u is transcendental over $K(S)$.*

Proof. The proof is straightforward. □

Corollary 0.8. *S is a transcendental basis of F/K if and only if $F/K(S)$ is algebraic.*

Proof. Suppose that S is a transcendental basis of F/K. If $u \in F - K(S)$, then $S \cup \{u\}$ is not algebraically independent. Thus, u is algebraic over $K(S)$ by the Proposition.

On the other hand, suppose that $F/K(S)$ is algebraic. Then for all $u \in F - S$, u is algebraic over $K(S)$. By the Proposition, $S \cup \{u\}$ is algebraically dependent if $u \in F - K(S)$. In fact, it's easy to see directly that $S \cup \{u\}$ is algebraically dependent if $u \in K(S)$. Thus S is a maximal algebraically independent set. □

Corollary 0.9. *Let $S \subset F$ be an subset over such that $F/K(S)$ is algebraic. Then S contains a transcendental basis.*

Proof. By Zorn's Lemma, there exists a maximal algebraically independent subset $S' \subset S$. Then $K(S)$ is algebraic over $K(S')$ and hence F is algebraic over $K(S')$. □

Theorem 0.10. *Let S, T be transcendental bases of F/K. If S is finite, then $|T| = |S|$.*

Proof. Let $S = \{s_1, ..., s_n\}$ and $S' := \{s_2, ..., s_n\}$. We first claim that there is an element $t \in T$, say $t = t_1$ such that $\{t_1, s_2, ..., s_n\}$ is a transcendental basis.

to see this, if every element of T is algebraic over $K(S')$, then F is algebraic over $K(T)$ hence over $K(S')$ which is a contradiction. Thus, there is an element $t \in T$, say $t = t_1$ such that t_1 is transcendental over $K(S')$. And hence $T' := \{t_1, s_2, ..., s_n\}$ is algebraically independent.

By the maximality of S, one sees that s_1 is algebraic over $K(T')$. It follows that F is algebraic over $K(t_1, s_1, ..., s_n)$ and hence algebraic over $K(T')$. Therefore, T' is a transcendental basis.

By induction, one sees that there is a transcendental basis $\{t_1, ..., t_n\} \subset T$. Thus $T = \{t_1, ..., t_n\}$. □

Theorem 0.11. *Let S, T be transcendental bases of F/K. If S is infinite, then $|T| = |S|$.*

Proof. By the previous theorem, we have $|T|$ is infinite.

For each $s \in S$, s is algebraic over $K(T)$. There is a finite subset $T_s \neq \emptyset \subset T$ containing all coefficients of the minimal polynomial of s. And hence s is algebraic over $K(T_s)$. Let $T' := \cup_{s \in S} T_s$. Since every $u \in F$ is algebraic over $K(S)$ and hence algebraic over $K(T')$. It follows that $T' = T$ as $T' \subset T$.

Finally, one shows that

$$|T| = | \cup_{s \in S} | \leq |S||\mathbb{N}| = |S|.$$

Since one can similarly have $|S| \leq |T|$. We are done. □

With the above two theorem, we can have a well-defined notion of *transcendental degree*.

Definition 0.12. *Let F/K be an extension. The transcendental degree of F/K, denoted tr.d.F/K, is the cardinal number $|S|$, where S is a transcendental basis.*

Theorem 0.13. *If F/E and E/K are extensions, then*

$$tr.d.F/K = tr.d.F/E + tr.d.E/K.$$

Proof. Let S be a transcendental basis of E/K and T be a transcendental basis of F/E. It clear that $T \cap E = \emptyset$, thus $T \cap S = \emptyset$. It's enough to show that $S \cup T$ is a transcendental basis of F/K.

Note that E is algebraic over $K(S)$, so E is algebraic over $K(S \cup T)$. It follows that $E(T)$ is algebraic over $K(S \cup T)$. Together with the fact that F is algebraic over $E(T)$. One sees that F is algebraic over $K(S \cup T)$.

It suffices to show that $S \cup T$ is algebraically independent. If f is a non-trivial algebraic relation, i.e. $f(s_1, ..., s_n, t_1, ..., t_m) = 0$ for some $s_i \in S, t_j \in T$. □

Corollary 0.14. *Let F_1/K_1 and F_2/K_2 are extensions and F_1, F_2 are algebraically closed. Then every isomorphism between K_1 and K_2 can be extended to an isomorphism between F_1 and F_2 if $tr.d.F_1/K_1 = tr.d.F_2/K_2$*