Chapter 1

Ample and Nef Line Bundles

This chapter contains the basic theory of positivity for line bundles and divisors on a projective algebraic variety.

After some preliminaries in §1.1 on divisors and linear series, we present in §1.2 the classical theory of ample line bundles. The basic conclusion is that positivity can be recognized geometrically, cohomologically or numerically. Section 3 develops the formalism of **Q**- and **R**-divisors, which is applied in §4 to study limits of ample bundles. These so-called nef divisors are central to the modern view of the subject, and §4 contains the core of the general theory. Most of the remaining material is more concrete in flavor. Section 5 is devoted to examples of ample cones and to further information about their structure, while §6 focuses on inequalities of Hodge type. After a brief review of the definitions and basic facts surrounding amplitude for a mapping, we conclude in §8 with an introduction to Castelnuovo-Mumford regularity.

1.1 Preliminaries: Divisors, Line Bundles and Linear Series

In this section we collect some facts and notation that will be used frequently in the sequel.

1.1.A Divisors and line bundles

To fix notation and ideas we start with a quick review of the definitions and facts concerning Cartier divisors, following [194, p. 140 ff], [313, Chapters 9 and 10] and [240]. We take up first the very familiar case of reduced and irreducible varieties, and then pass to more general schemes.

Consider then an irreducible complex variety X, and denote by $\mathcal{K}_X = \mathbf{C}(X)$ the (constant) sheaf of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and so there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{K}_X^*$ of sheaves of multiplicative abelian groups.

Definition 1.1.1. (Cartier divisors). A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. We denote by $\mathrm{Div}(X)$ the group of all such, so that

$$\mathrm{Div}(X) \;=\; \Gamma\big(X, \mathcal{K}_X^*/\mathcal{O}_X^*\big). \quad \square$$

Concretely, then, a divisor $D \in \mathrm{Div}(X)$ is represented by data $\big\{(U_i, f_i)\big\}$ consisting of an open covering $\{U_i\}$ of X together with elements $f_i \in \Gamma\big(U_i, \mathcal{K}_X^*\big)$, having the property that on $U_{i,j} = U_i \cap U_j$ one can write

$$f_i \;=\; g_{ij} f_j \quad \text{for some} \quad g_{ij} \;\in\; \Gamma\big(U_{ij}, \mathcal{O}_X^*\big). \tag{1.1}$$

f_i is called a *local equation* for D at any point $x \in U_i$. Two such collections determine the same Cartier divisor if there is a common refinement $\{V_k\}$ of the open coverings on which they are defined so that they are given by data $\big\{(V_k, f_k)\big\}$ and $\big\{(V_k, f'_k)\big\}$ with

$$f_k \;=\; h_k f'_k \quad \text{on} \quad V_k \quad \text{for some} \quad h_k \;\in\; \Gamma\big(V_k, \mathcal{O}_X^*\big).$$

The group operation on $\mathrm{Div}(X)$ is always written additively: if $D, D' \in \mathrm{Div}(X)$ are represented respectively by data $\big\{(U_i, f_i)\big\}$ and $\big\{(U_i, f'_i)\big\}$, then $D + D'$ is given by $\big\{(U_i, f_i f'_i)\big\}$. The *support* of a divisor $D = \big\{(U_i, f_i)\big\}$ is the set of points $x \in X$ at which a local equation of D at x is not a unit in $\mathcal{O}_x X$. D is *effective* if $f_i \in \Gamma\big(U_i, \mathcal{O}_X\big)$ is regular on U_i: this is written $D \succcurlyeq 0$.

Suppose now that X is a possibly non-reduced algebraic scheme. Then the same definition works except that one has to be more careful about what one means by $\mathcal{K}_X$, which now becomes the sheaf of total quotient rings of $\mathcal{O}_X$.[1] As explained in [313, Chapter 9] there is a unique sheaf $\mathcal{K}_X$ on X characterized by the property that if $U = Spec(A)$ is an affine open subset of X, then

$$\Gamma\big(U, \mathcal{K}_X\big) \;=\; \Gamma\big(U, \mathcal{O}_X\big)_{\mathrm{tot}} \;=\; A_{\mathrm{tot}}$$

is the total ring of fractions of A, i.e. the localization of A at the set of non zero-divisors.[2] Similarly, on the stalk level there is an isomorphism $\mathcal{K}_{X,x} = \big(\mathcal{O}_{X,x}\big)_{\mathrm{tot}}$. As before one has an inclusion $\mathcal{O}_X^* \subseteq \mathcal{K}_X^*$ of multiplicative groups of units, and Definition 1.1.1 — as well as the discussion following it — remains valid without change.

[1] Grothendieck speaks of the sheaf of "meromorphic functions" on X (cf. [175, 20.1]). However since we are working with complex varieties this seems potentially confusing, and we prefer to follow the terminology of [194].

[2] See [240] for a discussion of how one should define $\mathcal{K}_X$ on arbitrary open subsets.

Convention 1.1.2. (Divisors). By a divisor on X, we always understand unless otherwise stated a Cartier divisor. □

One should think of Cartier divisors as "cohomological" objects, but one can also define "homological" analogues:

Definition 1.1.3. (Cycles and Weil divisors). Let X be a variety or scheme of pure dimension n. A *k-cycle* on X is a $\mathbf{Z}$-linear combination of irreducible subvarieties of dimension k. The group of all such is written $Z_k(X)$. A *Weil divisor* on X is an $(n-1)$-cycle, i.e. a formal sum of codimension one subvarieties with integer coefficients. We often use $\mathrm{WDiv}(X)$ in place of $Z_{n-1}(X)$ to denote the group of Weil divisors. □

Remark 1.1.4. (Cycle map for Cartier divisors). There is a *cycle map*

$$\mathrm{Div}(X) \longrightarrow \mathrm{WDiv}(X) \quad , \quad D \mapsto [D] = \sum \mathrm{ord}_V(D) \cdot [V]$$

where $\mathrm{ord}_V(D)$ is the order of D along a codimension one subvariety. In general this homomorphism is neither injective nor surjective, although it is one-to-one when X is a normal variety and an isomorphism when X is non-singular. (See [141, Chapter 2.1] for details and further information.) □

A global section $f \in \Gamma\big(X, \mathcal{K}_X^*\big)$ determines in the evident manner a divisor

$$D = \mathrm{div}(f) \in \mathrm{Div}(X).$$

As usual, a divisor of this form is called *principal* and the subgroup of all such is $\mathrm{Princ}(X) \subseteq \mathrm{Div}(X)$. Two divisors D_1, D_2 are *linearly equivalent*, written $D_1 \equiv_{\mathrm{lin}} D_2$, if $D_1 - D_2$ is principal.

Let D be a divisor on X. Given a morphism $f : Y \longrightarrow X$, one would like to define a divisor f^*D on Y by pulling back the local equations for D. The following condition is sufficient to guarantee that this is meaningful:

> Let $V \subseteq Y$ be any *associated subvariety* of Y, i.e. the subvariety defined by an associated prime of $\mathcal{O}_Y$ in the sense of primary decomposition. Then $f(V)$ should not be contained in the support of D.

If Y is reduced the requirement is just that no component of Y map into the support of D.

A similar condition allows one to define the divisor of a section of a line bundle L on X. Specifically, let $s \in \Gamma\big(X, L\big)$ be a global section of L. Assume that s does not vanish on any associated subvariety of X — for example, if X is reduced, this just means that s shouldn't vanish identically on any component of X. Then a local equation of s

determines in the natural way a divisor $\mathrm{div}(s) \in \mathrm{Div}(X)$. We leave it to the reader to formulate the analogous condition under which a "rational section" $s \in \Gamma\big(X, L \otimes_{\mathcal{O}_X} \mathcal{K}_X\big)$ gives rise a divisor.

A Cartier divisor $D \in \mathrm{Div}(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X, leading to a canonical homomorphism:

$$\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \quad , \quad D \mapsto \mathcal{O}_X(D) \tag{1.2}$$

of abelian groups, where $\mathrm{Pic}(X)$ denotes as usual the Picard group of isomorphism classes of line bundles on X. Concretely, if D is given by data $\big\{(U_i, f_i)\big\}$ as above, then one can build $\mathcal{O}_X(D)$ by using the g_{ij} in (1.1) as transition functions. More abstractly, one can view $\mathcal{O}_X(D)$ as the image of D under the connecting homomorphism

$$\mathrm{Div}(X) \;=\; \Gamma\big(X, \mathcal{K}_X^*/\mathcal{O}_X^*\big) \longrightarrow H^1\big(X, \mathcal{O}_X^*\big) \;=\; \mathrm{Pic}(X)$$

determined by the exact sequence $0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0$ of sheaves on X. Evidently

$$\mathcal{O}_X(D_1) \;\cong\; \mathcal{O}_X(D_2) \quad \Longleftrightarrow \quad D_1 \equiv_{\mathrm{lin}} D_2.$$

If D is effective then $\mathcal{O}_X(D)$ carries a global section $s = s_D \in \Gamma\big(X, \mathcal{O}_X(D)\big)$ with $\mathrm{div}(s) = D$. In general $\mathcal{O}_X(D)$ has a rational section with the analogous property.

The question of whether every line bundle comes from a divisor is more delicate. On the positive side, one has:

Example 1.1.5. (i). If X is reduced and irreducible, or merely reduced, then the homomorphism in (1.2) is surjective.

(ii). If X is projective then the same statement holds even if it is non-reduced.

(If X is reduced and irreducible then any line bundle L has a rational section s, and one can take $D = \mathrm{div}(s)$. For the second statement — which is due to Nagata [**?**] — one can use the theorem of Cartan-Grothendieck-Serre (Theorem 1.2.5) to reduce to the case in which L is globally generated (Definition 1.1.9). But then one can find a section $s \in \Gamma\big(X, L\big)$ that does not vanish on any of the associated subvarieties of X, in which case $D = \mathrm{div}(s)$ gives the required divisor.) □

On the other hand, there is also

Example 1.1.6. (Kleiman's example). Following [241] and [363] we construct a non-projective non-reduced scheme X on which the mapping (2) is not surjective.[3] Start by taking Y to be Hironaka's example of a smooth non-projective threefold containing two disjoint smooth rational curves A and A with $A + B \sim_{\mathrm{alg}} 0$ as described in [194, Appendix

[3]We will use here the basic facts of intersection theory recalled later in this section.

B, Example 3.4.1]. Now fix points $a \in A, b \in B$ and introduce nilpotents at a and b to produce a non-reduced scheme X containing Y. Note that X has depth zero at a and b, so these points must be disjoint from the support of every Cartier divisor on X. Observe also that $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is an isomorphism thanks to the fact that $\ker\big(\mathcal{O}_X^* \longrightarrow \mathcal{O}_Y^*\big)$ is supported on a finite set, and so has vanishing H^1 and H^2.

We claim that there exists a line bundle L on X with the property that

$$\int_A c_1(L) \ > \ 0. \tag{*}$$

In fact, it follows from Hironaka's construction that one can find a line bundle on Y satisfying the analogous inequality, and by what have said above this bundle extends to X. Suppose now that $L = \mathcal{O}_X(D)$ for some divisor D on X. Decompose the corresponding Weil divisor as a sum $[D] = \sum m_i[D_i]$ of prime divisors with $m_i \neq 0$. None of the D_i can pass through a or b, so each D_i is Cartier and

$$D \ = \ \sum m_i D_i$$

as Cartier divisors on X. Now $\big(D_i \cdot A\big) \geq 0$ and $\big(D_i \cdot B\big) \geq 0$ since each of the D_i — avoiding as they do the points a and b — meet A and B properly. On the other hand, it follows from (*) that there is at least one index i such that $\big(m_i D_i \cdot A\big) > 0$: in particular $m_i > 0$ and $\big(D_i \cdot A\big) > 0$. But $\big(D_i \cdot B\big) = -\big(D_i \cdot A\big)$ since $B \sim_{\text{alg}} -A$ and therefore $\big(D_i \cdot B\big) < 0$, a contradiction. (As Schröer observes, the analogous but slightly simpler example appearing in [191, I.1.3] is erroneous.) □

Later on, the canonical bundle of a smooth variety will play a particularly important role:

Notation 1.1.7. (Canonical bundle and divisor). Let X be a non-singular complete variety of dimension n. We denote by $\omega_X = \Omega_X^n$ the canonical line bundle on X, and by K_X any canonical divisor on X. Thus $\mathcal{O}_X(K_X) = \omega_X$. □

Finally, a word about terminology. There is always a certain amount of tension between the additive language of divisors and the multiplicative formalism of line bundles. Our convention is to work additively with divisors and multiplicatively with line bundles. However on many occasions it is natural or customary to stay in additive mode when nonetheless one has line bundles in mind. In these circumstances we will speak of divisors but use notation suggestive of bundles, as for instance in the following statement of the Kodaira vanishing theorem:

> Let L be an ample divisor on a smooth projective variety X of dimension n.
> Then $H^j\big(X, \mathcal{O}_X(-L)\big) = 0$ for every $j < n$.

(The mathematics here appears in Chapter 4.) While for the most part this convention seems to work well, it occasionally leads us to make extraneous projectivity or integrality hypotheses in order to be able to invoke Example 1.1.5. Specifically, we will repeatedly work with the Neron-Severi group $N^1(X)$ of X and the corresponding real vector space $N^1(X)_{\mathbf{R}} = N^1(X) \otimes \mathbf{R}$. Here additive notation seems essential, so we are led to view $N^1(X)$ as the group of divisors modulo numerical equivalence (Definition 1.1.13). On the other hand, functorial properties are most easily established by passing to $\mathrm{Pic}(X)$. For this to work smoothly one wants to know that every line bundle comes from a divisor, and this is typically guaranteed by simply assuming that X is either a variety or a projective scheme. We try to flag this artifice when it occurs.

1.1.B Linear series

We next review some basic facts and definitions concerning linear series.

Let X be a variety (or scheme), L a line bundle on X, and $V \subseteq H^0\big(X, L\big)$ a non-zero subspace of finite dimension. We denote by $|\,V\,| = \mathbf{P}_{\mathrm{sub}}(V)$ the projective space of one-dimensional subspaces of V. When X is a complete variety, $|V|$ is identified with the linear series of divisors of sections of V in the sense of [194, Chapter II, §7], and in general we refer to $|\,V\,|$ as a linear series. Taking $V = H^0\big(X, L\big)$ — assuming this space is finite dimensional, as will be the case for instance if X is complete — yields the *complete linear series* $|\,L\,|$. Given a divisor D, we also write $|\,D\,|$ for the complete linear series associated to $\mathcal{O}_X(D)$.

Evaluation of sections in V gives rise to a morphism of vector bundles on X:

$$\mathrm{eval}_V : V \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow L.$$

Definition 1.1.8. (Base locus and base ideal). The *base ideal* of $|\,V\,|$, written

$$\mathfrak{b}\big(\,|\,V\,|\,\big) \;=\; \mathfrak{b}\big(X\,,\,|\,V\,|\big) \;\subseteq\; \mathcal{O}_X$$

is the image of the map $V \otimes_{\mathbf{C}} L^* \longrightarrow \mathcal{O}_X$ determined by eval_V. The *base locus*

$$\mathrm{Bs}\big(\,|\,V\,|\,\big) \subseteq X$$

of $|\,V\,|$ is the closed subset of X cut out by the base ideal $\mathfrak{b}\big(\,|\,V\,|\,\big)$. When we wish to emphasize the scheme structure on $\mathrm{Bs}(|\,V\,|)$ determined by $\mathfrak{b}(|V|)$ we will refer to $\mathrm{Bs}(|\,V\,|)$ as the *base scheme* of $|\,V\,|$. □

Very concretely, then, $\mathrm{Bs}\big(\,|\,V\,|\,\big)$ is the set of points at which all the sections in V vanish, and $\mathfrak{b}\big(\,|\,V\,|\,\big)$ is the ideal sheaf spanned by these sections.

The easiest linear series with which to deal are those for which the base-locus is empty.

Definition 1.1.9. (Free linear series). One says that $|V|$ is *free*, or *basepoint-free*, if its base-locus is empty, i.e. if $\mathfrak{b}\big(|V|\big) = \mathcal{O}_X$. A divisor D or line bundle L is *free* if the corresponding complete linear series is so. In the case of line bundles one says synonymously that L is *generated by its global sections* or *globally generated.* □

In other words, $|V|$ is free if and only if for each point $x \in X$ one can find a section $s = s_x \in V$ such that $s(x) \neq 0$.

Assume now (in order to avoid trivialities) that $\dim V \geq 2$, and set $B = \mathrm{Bs}(|V|)$. Then $|V|$ determines a morphism

$$\phi \;=\; \phi_{|V|} \;:\; X - B \;\longrightarrow\; \mathbf{P}\big(V\big)$$

from the complement of the base-locus in X to the projective space of one-dimensional quotients of V. Given $x \in X$, $\phi(x)$ is the hyperplane in V consisting of those sections vanishing at x. If one chooses a basis $s_0, \ldots, s_r \in V$, this amounts to saying that ϕ is given in homogeneous coordinates by the (somewhat abusive!) expression

$$\phi(x) \;=\; [s_0(x), \ldots, s_r(x)] \;\in \mathbf{P}^r.$$

When X is an irreducible variety it is sometimes useful to ignore the base-locus, and view $\phi_{|V|}$ as a rational mapping $\phi : X \dashrightarrow \mathbf{P}(V)$. If $|V|$ is free then $\phi_{|V|} : X \longrightarrow \mathbf{P}(V)$ is a globally defined morphism.

At least when $B = \emptyset$ these constructions can be reversed, so that a morphism to projective space gives rise to a linear series. Specifically, suppose given a morphism

$$\phi \,:\, X \;\longrightarrow\; \mathbf{P} = \mathbf{P}(V)$$

from X to the projective space of one dimensional quotients of a vector space V, and assume that $\phi(X)$ does not lie on any hyperplanes. Then pull-back of sections via ϕ realizes $V = H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)\big)$ as subspace of $H^0\big(X, \phi^*\mathcal{O}_{\mathbf{P}}(1)\big)$, and $|V|$ is a free linear series on X. Moreover ϕ is identified with the corresponding morphism $\phi_{|V|}$.

Example 1.1.10. If X is a non-singular variety and $B \subseteq X$ has codimension ≥ 2, then a similar construction works starting with a morphism

$$\phi \;:\; X - B \;\longrightarrow\; \mathbf{P} = \mathbf{P}(V).$$

In fact, $\phi^*\mathcal{O}_{\mathbf{P}}(1)$ extends uniquely to a line bundle L on X — corresponding to the divisor obtained by taking the closure of the pull-back of a hyperplane — and ϕ^* realizes V as a subspace of $H^0\big(X, L\big)$, with $\mathrm{Bs}\big(|V|\big) \subseteq B$. □

Example 1.1.11. (Projection). Suppose that $W \subseteq V$ is a subspace (say of dimension ≥ 2). Then $\mathrm{Bs}\big(|V|\big) \subseteq \mathrm{Bs}\big(|W|\big)$, so that $\phi_{|V|}$ and $\phi_{|W|}$ are both defined on $X - \mathrm{Bs}(|W|)$. Viewed as morphisms on this set one has the relation $\phi_{|W|} = \pi \circ \phi_{|V|}$, where

$$\pi \;:\; \mathbf{P}(V) - \mathbf{P}(V/W) \;\longrightarrow\; \mathbf{P}(W)$$

is linear projection centered along the subspace $\mathbf{P}(V/W) \subseteq \mathbf{P}(V)$. Note that if $|W|$ — and hence also $|V|$ — is free, and if X is complete, then $\pi|X$ is finite (since it is affine and proper). So in this case, the two morphisms

$$\phi_{|V|} : X \longrightarrow \mathbf{P}(V) \quad , \quad \phi_{|W|} : X \longrightarrow \mathbf{P}(W)$$

differ by a finite projection of $\phi_{|V|}(X)$. □

1.1.C Intersection numbers and numerical equivalence

Intersection numbers. Let X be a complete irreducible complex variety. Given Cartier divisors $D_1, \ldots, D_k \in \mathrm{Div}(X)$ together with an irreducible subvariety $V \subseteq X$ of dimension k, the intersection number

$$\big(D_1 \cdot \ldots \cdot D_k \cdot V\big) \ \in \ \mathbf{Z} \tag{1.3}$$

can be defined in various ways. To begin with, of course, the quantity in question arises as a special case of the theory in [141]. However intersection products of divisors against subvarieties do not require the full strength of that technology: a relatively elementary direct approach based on numerical polynomials was developed in the sixties by Snapper [375] and Kleiman [238]. Extensions and modern presentations of the Snapper-Kleiman theory appear in [254, VI.2] and [74, Chapter 1.2]. We prefer to minimize foundational issues by working topologically, referring to [141] for additional properties as needed.

Specifically, in the above situation each of the line bundles $\mathcal{O}_X(D_i)$ has a Chern class

$$c_1\big(\mathcal{O}_X(D_i)\big) \ \in \ H^2\big(X;\mathbf{Z}\big),$$

the cohomology group in question being ordinary singular cohomology of X with its classical topology. The cup product of these classes is then an element

$$c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big) \ \in \ H^{2k}\big(X;\mathbf{Z}\big) :$$

here and elsewhere we write $\alpha \cdot \beta$ or simply $\alpha\beta$ for the cup product of elements $\alpha, \beta \in H^*\big(X;\mathbf{Z}\big)$. Denoting by $[V] \in H_{2k}\big(X;\mathbf{Z}\big)$ the fundamental class of V, cap product leads finally to an integer

$$\Big(c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big)\Big) \cap [V] \ \in \ H_0\big(X;\mathbf{Z}\big) \ = \ \mathbf{Z} \tag{1.4}$$

which of course is nothing but the quantity appearing in (1.3). We generally use one of the notations

$$\Big(D_1 \cdot \ldots \cdot D_k \cdot V\Big) \quad , \quad \int_V D_1 \cdot \ldots \cdot D_k$$