Advanced Algebra I

Burnside' Theorem

As an application, we are going to prove Burnside's $p^a q^b$ theorem.

Theorem 0.1. *Let G be a group of order $p^a q^b$. Then G is solvable.*

Proof. If G has a non-trivial normal subgroup $N \lhd G$. Then by induction on (a,b), one sees that both N and G/N are solvable. Hence G is solvable.

We thus assume that G is non-abelian simple. Since G is simple, each representation is either injective or trivial. We remark that if $\rho : G \to GL(V)$ is an irreducible representation of degree 1, then ρ must be trivial cause G can't inject into an abelian group. Thus, any non-trivial irreducible representation is of degree > 1.

Let $\rho_1, ..., \rho_r$ be representative of isomorphic classes of irreducible representations with ρ_1 is the trivial representation. And Let h_i be the number of elements in conjugacy classes c_i (with c_1 being the class of the identity e, hence $h_1 = 1$). One has the following equations:

$$g = p^a q^b = 1 + \sum_{i=2}^{r} h_i,$$

$$0 = \chi_{reg}(s) = 1 + \sum_{i=2}^{r} d_i \chi_i(s),$$

for $s \neq e$.

Modulo the first equation by q, one finds that

$$h_i \not\equiv 0 \pmod{q}$$

for some $i \geq 2$. Recall that for any $s \in c_i$, one has that $h_i |C_G(s)| = |G| = p^a q^b$, it follows that $h_i = p^{a'}$.

Next modulo the second equation by p, one finds that

$$d_i \chi_i(s) \not\equiv 0 \pmod{p}$$

for some $i \geq 2$. In particular, $p \nmid d_i$ and $\chi_i(s) \neq 0$.

Our goal is to prove the following

Claim. Under the condition that (for $i \geq 2$) $(d_i, h_i) = 1$, $\chi_i(s) \neq 0$ and $s \in c_i$. One has $\rho_i(s) = \lambda I$ for some λ.

Grant this for the time being. Then $\rho_i : G \to GL(V_i)$ gives an injection. $\rho_i(s) = \lambda I$ implies that $s \in Z(GL(V_i))$ and hence $s \in Z(\rho_i(G)) \cong Z(G)$. In particular, G has non-trivial center, hence a non-trivial normal subgroup. This completes the proof.

Proof. It remains to prove the claim. Let $e_c = \sum_{t \in c_i} e_t \in \mathbb{C}[G]$, then $e_c \in Z(\mathbb{C}[G])$ and hence $\widetilde{\rho}_i(e_c) = lI$ with $l = \frac{h_i}{d_i}\chi_i(s)$. More precisely, one has $\omega_i : Z(\mathbb{C}[G]) \to \mathbb{C}$ such that $\omega_i(e_c) = l$. Since $e_c \in Z(\mathbb{C}[G])$ is integral over $\mathbb{Z}$. It's clear that l is integral over $\mathbb{Z}$. Moreover, it's clear that $\chi_i(s)$ is integral over $\mathbb{Z}$. Since $(d_i, h_i) = 1$ there exists $x, y \in \mathbb{Z}$ such that $d_i x + h_i y = 1$. Thus one has

$$\frac{1}{d_i}\chi_i(s) = x\chi_i(s) + \frac{h_i y}{d_i}\chi_i(s) \in \mathcal{A}.$$

Now

$$\chi_i(s) = tr(\rho_i(s)) = \lambda_1 + ... + \lambda_{d_i}$$

is sum of root of unity. And $0 \neq \frac{\chi_i(s)}{d_i} \in \mathcal{A}$. We leave it as an exercise to show that

$$\lambda_1 = ... = \lambda_{d_i} = \frac{1}{d_i}\chi_i(s).$$

Therefore, $\rho_i(s) = \lambda_1 I$. $\square$

$\square$