Advanced Algebra I

representation of finite groups,II Characters

Let ρ be a 1-dimensional representation of a group G. Then in this case $\rho = \chi : G \to \mathbb{C}^*$. One sees that $\chi(st) = \chi(s)\chi(t)$ for all $s, t \in G$. Such character is called an abelian character.

Let $\hat{G}$ be the set of all 1-dimensional characters, it forms a group under the multiplication $\chi\chi'(g) := \chi(g)\chi'(g)$.

Exercise 0.1. *Let G be an abelian group. Prove that $G \cong \hat{G}$*

Recall that a representation $\rho : G \to GL(V)$ is the same as a linear action $G \times V \to V$. Suppose now that there are two representation ρ, ρ' on V, V' respectively. A linear transformation $T : V \to V'$ is said to be *G-invariant* if it's compatible with representations. That is,

$$T\rho_s(v) = \rho'_s(Tv),$$

for all $v \in V$.

Thus an isomorphism of representation is nothing but a G-invariant bijective linear transformation.

Exercise 0.2. *It's easy to check that if $T : V \to V'$ is G-invariant, then the $ker(T) \subset V$ and $im(T) \subset V'$ are G-invariant subspaces.*

Theorem 0.3 (Schur's Lemma)**.** *Let ρ, ρ' be two irreducible representation of G on V, V' respectively. And let $T : V \to V'$ be a G-invariant linear transformation. Then*

(1) *Either T is an isomorphism or $T = 0$.*
(2) *If $V = V'$,$\rho = \rho'$, then T is multiplication by a scalar.*

Proof. (1) Since $ker(T)$ is a G-invariant subspace and V is irreducible. One has that either $ker(T) = 0$ or $ker(T) = V$. Hence T is injective or $T = 0$. If T is injective, by looking at $im(T)$, One must have $im(T) = V'$. Therefore T is an isomorphism.

(2) Let λ be an eigenvalue of T. One sees that $T_1 := T - \lambda I$ is also an G-invariant linear transformation. Since $ker(T_1)$ is non-zero, one has that $ker(T_1) = V$. Thus $T_1 = 0$ and hence $T = \lambda I$.

□

Suppose one has $T : V \to V'$ not necessarily G-invariant. One can produce an G-invariant linear transformation by the "averaging process". For $T(v) = s^{-1}T(sv)$, we set

$$\tilde{T}(v) := \frac{1}{g} \sum_{s \in G} s^{-1}T(sv).$$

And it's easy to check that this is G-invariant.

proof of the main theorem. (1) Let ρ, ρ' be two irreducible representation of G on V, V' with character χ, χ' respectively.

Let $T : V \to V'$ be any linear transformation. One can produce a G-invariant transformation $\tilde{T}$.

Suppose first that ρ and ρ' are not isomorphic. Then by Schur's Lemma, $\tilde{T} = 0$ for all T.

We fix bases of V, V' and write everything in terms of matrices.

$$0 = (\tilde{T})_{ij} = \sum_{t,k,l} (R'_{t^{-1}})_{ik}(T)_{kl}(R_t)_{lj}.$$

Take $T = E_{ij}$, then one has

$$0 = \sum_{t,k,l} (R'_{t^{-1}})_{ik}(E_{ij})_{kl}(R_t)_{lj} = \sum_{t} (R'_{t^{-1}})_{ii}(R_t)_{jj}.$$

Hence

$$< \chi', \chi >= \sum_{t,i,j} (R'_{t^{-1}})_{ii}(R_t)_{jj} = 0.$$

Suppose now that $\rho = \rho'$, $\chi = \chi'$. The averaging process and Schur's Lemma gives

$$\lambda I = \tilde{T} = \frac{1}{g} \sum_t R_{t^{-1}} T R_t.$$

One notice that $\lambda d = tr(\tilde{T}) = tr(T)$.

Now we set $T = E_{ii}$, then

$$\frac{1}{d} = (\lambda I)_{ii} = \frac{1}{g} \sum_t (R_{t^{-1}})_{ik}(E_{ii})_{kl}(R_t)_{li} = \sum_t (R_{t^{-1}})_{ii}(R_t)_{ii}.$$

It follows that

$$< \chi, \chi >= \sum_t \sum_i (R_{t^{-1}})_{ii}(R_t)_{ii} = \sum_i \frac{1}{d} = 1.$$

(2) A class function f on a group G is a complex value function such that $f(s) = f(t)$ if s and t are conjugate. The space $\mathcal{C}$ of class function is clearly a vector space of dimension r, where r denotes the number of conjugacy classes of G. We claim that the set of character of irreducible representation form a orthonormal basis of $\mathcal{C}$.

We remark that inner product can be defined on any class function.

Suppose now that ϕ is a class function which is orthogonal to every χ_i. For any character χ of an irreducible representation ρ, we can produce a linear transformation by averaging process $T := \frac{1}{g} \sum_t \overline{\phi(t)} \rho_t$. It's clear that $tr(T) =< \phi, \chi >= 0$. One sees that $T : V \to V$ is G-invariant. By Schur's Lemma, $T = \lambda I$. But $Tr(T) = 0$. Thus $T = 0$ for any character χ.

We apply to the regular representation $\rho : G \to \mathbb{C}[G]$,

$$0 = T(e_1) = \frac{1}{g}\sum_t \overline{\phi(t)}\rho_t(e_1) = \frac{1}{g}\sum_t \overline{\phi(t)}e_t.$$

Since e_t forms a basis for $\mathbb{C}[G]$, it follows that $\phi(t) = 0$ for all $t \in G$ and hence $\phi = 0$.

(3) We may assume that there are r irreducible representation. And suppose that the regular representation ρ is decomposed into $n_1\rho_1 \oplus ... \oplus n_r\rho_r$. One notice that $\rho(1) = g$ and $\rho(t) = 0$ for all $t \neq 1$. By direct computation,

$$d_i =< \chi_\rho, \chi_i >= n_i,$$
$$g =< \chi_\rho, \chi_\rho >= \sum_i d_i^2.$$

To prove that $d_i|g$ need some extra work on the group algebra $\mathbb{C}[G]$ which we will do later.

$\square$