

Advanced Algebra I

INDUCED REPRESENTATION

In order to introduce the notion of induced representation, we start by looking at some examples.

Example 0.1. Let $H < G$ be a subgroup. Then there is a natural groups action $G \times G/H \rightarrow G/H$ by translation. Hence one has a permutation representation $G \rightarrow \mathbb{C}[G/H]$. This representation can be described as following:

Let $W = \mathbb{C}e_H$ where H denote the trivial coset in G/H . Let $\sigma W = \mathbb{C}e_\sigma$ for $\sigma \in G/H$. Finally, we consider $V = \bigoplus_{\sigma \in G/H} \sigma W$. Then we obtain a representation of G on $V = \mathbb{C}[G/H]$.

One notices that this construction works for any W . So let $\rho : G \rightarrow GL(W)$ be a representation of H . We consider vector spaces $\sigma W := \{w_\sigma | w \in W\}$. And we fix a representative $s_\sigma \in G$ for each coset $\sigma \in G/H$. Now let

$$V := \bigoplus_{\sigma \in G/H} \sigma W.$$

It's easy to verify that there is a representation $\tilde{\rho} : G \rightarrow V$ given as following:

For $s \in G$, $ss_\sigma = s_\tau h$ for some $h \in H$ and $\tau \in G/H$. Then we consider $\tilde{\rho}_s(w_\sigma) = (\rho_h w)_\tau$.

A careful reader might notices that $\tilde{\rho}$ depends on choices of representatives s_σ . However, different choices give isomorphic representations. Thus it's is unique up to isomorphism.

On the other hand, Let $\rho : G \rightarrow GL(V)$ be a representation. It's restriction gives $\rho_H : H \rightarrow GL(V)$ a representation. If $\theta : H \rightarrow GL(W)$ is a subrepresentation of ρ_H , then we can find a G -invariant subspace $\sum_\sigma \sigma W$, where $\sigma W := \rho_{s_\sigma} W$ for any representative s_σ of $\sigma \in G/H$.

Definition 0.2. We say that $\rho : G \rightarrow GL(V)$ is a representation induced by the representation $\theta : H \rightarrow GL(W)$ if $V = \bigoplus_{\sigma \in G/H} \sigma W$.

We remark that on V there are now two representations ρ and $\tilde{\theta}$. We will prove that they are isomorphic by the following Lemma.

Lemma 0.3. Suppose that (V, ρ) is induced by (W, θ) . Let $\rho' : G \rightarrow GL(V')$ be a representation of G and let $f : W \rightarrow V'$ be a linear map such that $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$. Then there exists a unique linear map $F : V \rightarrow V'$ which extends f and satisfying $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.

Proof. We first prove the uniqueness. If $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$, then for $w \in \sigma W$, we pick any $s \in \sigma \in G/H$, one has $\rho_{s^{-1}} w \in W$ and

$$F(w) = F(\rho_s \rho_{s^{-1}} w) = \rho'_s F(\rho_{s^{-1}} w) = \rho'_s f(\rho_{s^{-1}} w).$$

Hence if F is determined by f . It implies that if F' is another extension, then $F' = F$. This proves the uniqueness.

To prove the existence, for $w \in \sigma W$, we define $F(w) := \rho'_s f(\rho_{s^{-1}} w)$. We first check that this is well-defined, i.e. independent of choice of $s \in \sigma$. Suppose we pick $s' = sh$ for some $h \in H$. Then

$$\rho'_{sh} f(\rho_{sh}^{-1} w) = \rho'_s \rho'_h f(\rho_h^{-1} \rho_s^{-1} w) = \rho'_s f(\theta_h \rho_h^{-1} \rho_s^{-1} w) = \rho'_s f(\rho_{s^{-1}} w).$$

Since $V = \oplus \sigma W$, F is thus defined on V . Moreover, it's clear that $F|_W = f$. $\square$

Corollary 0.4. *Suppose that (V, ρ) and (V', ρ') are induced by (W, θ) . Then they are isomorphic.*

Proof. Let $f : W \rightarrow V' = \oplus \sigma W$ be the inclusion. Since $\theta_t = \rho'_t$ for $t \in H$, it's easy to see that $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$. Then f extends to $F : V \rightarrow V'$. Since $F|_W$ is an isomorphism, it follows that $F|_{\sigma W}$ is an isomorphism by the construction of F . Therefore, (V, ρ) and (V', ρ') are isomorphic. $\square$

Corollary 0.5. *Suppose that (V, ρ) is induced by (W, θ) . Then $\rho \cong \tilde{\theta}$.*

Proof. Let $\rho' = \tilde{\theta}$ and $V' = \oplus \sigma W$. Then one sees that

$$\tilde{\theta}_t(w) = \theta_t(w) = \rho_t w$$

for $t \in H$ and $w \in W$. We take $f = id_W$ then we are done. $\square$

We now ready to compute the character.

Theorem 0.6. *Let (V, ρ) be a representation induced by (W, θ) . Let $R = \{s_\sigma\}$ be a system of representatives of G/H . For each $u \in G$, one has*

$$\chi_\rho(u) = \sum_{r \in R, r^{-1}ur \in H} \chi_\theta(r^{-1}ur) = \frac{1}{|H|} \sum_{s \in G, s^{-1}us \in H} \chi_\theta(s^{-1}us).$$

Proof. Since $\rho \cong \tilde{\theta}$, we can compute it by $\tilde{\theta}$ and we may assume that $\rho = \tilde{\theta}$.

We first note that ρ_u maps σW to $u\sigma W$, where $u\sigma$ denote the coset of us_σ . We have non-zero trace only at those subspace σW such that $\sigma W = u\sigma W$. This is equivalent to $\sigma = u\sigma$ and hence equivalently, $s_\sigma^{-1}us_\sigma \in H$.

Now, we compute trace on σW with $s_\sigma^{-1}us_\sigma = h \in H$. By definition, $\tilde{\theta}_s(w_\sigma) = (\theta_h w)_\sigma$. Hence

$$tr(\tilde{\theta}_s|_{\sigma W}) = tr(\theta_h) = \chi_\theta(s_\sigma^{-1}us_\sigma).$$

Add them up, then we get the required equality. $\square$

A convenience way to compute is

Exercise 0.7. If $\mathcal{C}$ is the conjugacy class of u , and $\mathcal{C} \cap H$ decompose into conjugacy classes $D_1, \dots, D_r$ of H . Then

$$\chi_\rho(u) = \frac{|G|}{|H|} \sum_{i=1}^r \frac{|D_i|}{|\mathcal{C}|} \chi_\theta(D_i).$$

Example 0.8. Consider $S_3 < S_4$. There is a irreducible representation (W, θ) of degree 2 on S_3 . It induces a representation $(V, \tilde{\theta})$ of degree 8 on S_4 . It's character has value 8, 0, -1, 0, 0 on 1, (12), (123), (1234), (12)(34) respectively. By orthogonal property, one has $\chi_{\tilde{\theta}} = \chi_3 + \chi_4 + \chi_5$.

We now fix some notations. Let V be a representation of G and $H < G$ a subgroup. Then $\text{Res}V$ denote the restriction of the representation of G on V to H .

On the other hand, if W is a representation of H , then $\text{Ind}W$ denotes the induced representation.

If V, V' are representations of G , then $\text{Hom}_G(V, V')$ denotes the G -invariant linear transformation from V to V' .

Then the extension theorem can be rephrase as

$$\text{Hom}_H(W, \text{Res}V') \cong \text{Hom}_G(\text{Ind}W, V').$$

Corollary 0.9 (Frobenius Reciprocity). If V is a representation of G and W is a representation of $H < G$, then

$$\langle \chi_{\text{Ind}W}, \chi_V \rangle_G = \langle \chi_W, \chi_{\text{Res}V} \rangle_H.$$

Proof. It suffices to prove this when W and V are irreducible. Note that $\langle \chi_{\text{Ind}W}, \chi_V \rangle_G$ is the number of copies of V appears in the decomposition of $\text{Ind}W$, which equals to $\dim(\text{Hom}_G(\text{Ind}W, V))$. And similarly, $\langle \chi_W, \chi_{\text{Res}V} \rangle_H$ is the number of copies of W appears in the decomposition of $\text{Res}V$, which equals to $\dim(\text{Hom}_H(W, \text{Res}V))$. $\square$