

Advanced Algebra I

Homework 8

due on Nov.14, 2003

Part A.

- (1) Let $G = S_n$ and $S = \{1, \dots, n\}$. We can have the permutation representation $\rho : G \times \mathbb{C}[S] \rightarrow \mathbb{C}[S]$. ρ factors as $1 \oplus \rho'$. Prove or disprove that ρ' is irreducible.
- (2) Determine the character table of the group $G = \langle x, y \mid x^7 = y^6 = e, yxy^{-1} = x^2 \rangle$

Part B.

- (1) Let V be a n -dimensional vector space with basis $\{e_1, \dots, e_n\}$. One can consider the vector space $V \otimes V$ as the n^2 -dimensional vector space with basis $\{e_i \otimes e_j\}_{i,j=1 \dots n}$.
 - (a) Consider $\theta : V \otimes V \rightarrow V \otimes V$ by $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Show that the space $Sym^2 V := \{z \in V \otimes V \mid \theta(z) = z\}$ has basis $\{e_i \otimes e_j + e_j \otimes e_i\}_{i \leq j}$. And the space $Alt^2 V := \{z \in V \otimes V \mid \theta(z) = -z\}$ has basis $\{e_i \otimes e_j - e_j \otimes e_i\}_{i < j}$. And $V \otimes V = Sym^2 V \oplus Alt^2 V$.
 - (b) If $\rho : G \rightarrow GL(V)$ is a representation with character χ . Show that the induced representation $\rho \otimes \rho : G \rightarrow GL(V \otimes V)$ is a representation with character χ^2 .
 - (c) Show that $Alt^2 V$ and $Sym^2 V$ gives subrepresentations.
 - (d) Show that the induced representation $\rho_{Alt^2} : G \rightarrow GL(Alt^2 V)$ has character $\chi_{Alt^2} = \frac{1}{2}(\chi_\rho(g)^2 - \chi_\rho(g^2))$.
 - (e) Show that the induced representation $\rho_{Sym^2} : G \rightarrow GL(Sym^2 V)$ has character $\chi_{Sym^2} = \frac{1}{2}(\chi_\rho(g)^2 + \chi_\rho(g^2))$.
- (2) Let $\rho_1, \dots, \rho_r$ be representatives of isomorphic classes of irreducible representation of a finite group G . Let $\rho : G \rightarrow GL(V)$ be an arbitrary representation which factors as $\rho \cong n_1 \rho_1 \oplus \dots \oplus n_r \rho_r$. Prove that the vector space $Hom_G(V, V_i)$ of G -invariant linear transformation has dimension n_i . And so is $Hom_G(V_i, V)$.