Advanced Algebra I

Homework 14

due on Jan. 9, 2004

(1) If F is a finite extension of $\mathbb{Q}$, then F has only a finite number of roots of unity.
(2) Which roots of unity are contained in $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer?
(3) Let F/K be a finite extension over a finite field K. Show that the norm and trace maps are surjective.
(4) Examine that the property of being radical under "extension","lifting", and "compositum".
(5) Find a polynomial which is not solvable by radicals. And explain why.

Part of answer to the problem 4:

Let $K \subset E \subset F$ be field extensions. We are going to give an example that F/K is radical but E/K is not radical.

Let F_0 be the splitting field of $f(x) = x^3 - x - 1$ over $\mathbb{Q}$. It's clear that the Galois group G_f is solvable. Hence F_0 is contained in a radical extension $F/\mathbb{Q}$.

Let $u \in F_0$ be a root of $f(x)$. We claim that $E := \mathbb{Q}(u)/\mathbb{Q}$ is not radical. Suppose on the other hand that $\mathbb{Q}(u)/\mathbb{Q}$ is radical, that is $\mathbb{Q}(u) = \mathbb{Q}(v_1, ..., v_n)$. However, $[\mathbb{Q}(u) : \mathbb{Q}] = 3$, we can therefore assume that $\mathbb{Q}(u) = \mathbb{Q}(v)$ for some $v^3 = d \in \mathbb{Q}$.

Thus $u = av^2 + bv + c$. Since $T(u) = T(v) = T(v^2) = 0$, we have $T(c) = 0$, hence $c = 0$. We can write $u = av^2 + bv$. By direct computation,

$$u^3 - u - 1 = a(1 + 3abd)v^2 + b(1 + 3abd)v + a^3d^2 + b^3d + 1 = 0.$$

One has

$$\begin{cases} a(1 + 3abd) = 0 \\ b(1 + 3abd) = 0 \\ a^3d^2 + b^3d + 1 = 0 \end{cases}$$

If $3abd + 1 = 0$, then $a^2 + 9ab^2 - 3b^4 = 0$ has no non-zero solution in $\mathbb{Q}$. If $3abd + 1 \neq 0$, then $a = b = 0$, which is also impossible. Therefore, $\mathbb{Q}(u)$ can't be a radical extension over $\mathbb{Q}$.