Advanced Algebra I

Homework 10

due on Dec. 5, 2003

(1) Consider the field $\mathbb{Q}[u] := \mathbb{Q}[x]/(x^3 + x + 1)$, where u denote the coset of x.
  (a) Find u^{-2} in $\mathbb{Q}[u]$.
  (b) Find minimal polynomial of u^2.

(2) Let F/K be a field extension. Let $\mathcal{A} \subset F$ be those elements in F which is algebraic over K. Show that $\mathcal{A}$ is a field.

(3) In the group $\mathrm{GL}(2, \mathbb{F}_q)$, there is the Borel group B of upper triangular matrices and diagonal subgroup D. There is a group homomorphism $B \to D$ by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Fix $\alpha, \beta : \mathbb{F}_q^* \to \mathbb{C}^*$, one has

$$D \to \mathbb{F}_q^* \times \mathbb{F}_q^* \xrightarrow{(\alpha,\beta)} \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{mult} \mathbb{C}^*$$

It follows that the composition $B \to \mathbb{C}^*$ is a representation. Let $W_{\alpha,\beta}$ be the induced representation on $\mathrm{GL}(2, \mathbb{F}_q)$. Compute the character of $W_{\alpha,\beta}$ and determine the irreducibility.

Recall that conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$ are represented by elements of the following four types $a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, $b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$, $c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$

(4) Let F/K be a field extension of degree 2 and $char(K) \neq 2$. Show that there is an element $u \in F - K$ such that $u^2 \in K$.

(5) Prove or disprove: Let F/K be a field extension of degree n and $d|n$. Then there is an intermediate field E, $K \subset E \subset F$, such that $[E : K] = d$.