Basic Algebra (Solutions)

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Exercise (§1.7, p.53)

1. Determine the cosets of $\langle\alpha\rangle$ in $S-4$ where $\alpha=(1234)$.

Ans. Let $H=\langle(1234)\rangle$. The right cosets are H, $H(12)=\{(12),(134),(1423),(243)\}$, $H(13)=\{(13),(14)(23),(24),(12)(34)\}$, $H(14)=\{(14),(234),(1243),(132)\}$, $H(23)=\{(23),(241),(1342),(143)\}$, $H(24)=\{(24),(12)(34),(13),(14)(23)\}$. The left coset of H are left for the reader.

2. Show that if G is finite and H and K are subgroups such that $H\supset K$ then $[G:K]=[G:H][H:K]$.

Proof. (I) Since H is a subgroup of G, hence $|G|=[G:H]|H|$. K is a subgroup of H, $|H|=|K|[H:K]$. Thus $|G|=[G:H][H:K]|K|$, K is a subgroup of G, $|G|=[G:K]|K|$. Hence $[G:H][H:K]|K|=[G:H]|K|$ and $[G:H][H:K]=[G:K]$.

(II) Let $G=\bigcup_{i=1}^{n}Hh_i$, $H=\bigcup_{j=1}^{m}Kk_j$ where $n=[G:H]$, $m=[H:K]$. Then $G=\bigcup_{\substack{1\le i\le n\\ 1\le j\le m}}Kk_jh_i$. It is easy to check that $Kk_jh_i\neq Kk_rh_s$ if $(j,i)\neq(r,s)$. Hence $[G:H]=nm=[G:H][H:K]$. □

3. Let H_1 and H_2 be subgroups of G. Show that any right coset relative to $H_1\cap H_2$ is the intersection of a right coset of H_1 with a right coset of H_2. Use this to prove Poincare's Theorem that if H_1 and H_2 have finite index in G then so has $H_1\cap H_2$.

Proof. (1) Let $(H_1\cap H_2)x$ be any coset of $H_1\cap H_2$, we just need to prove that $(H_1\cap H_2)x=H_1x\cap H_2x$:

For $y\in H_1x\cap H_2x$, $y=h_1x$ for $h_1\in H_1$. Since $h_1x\in H_2x$, $h_1=(h_1x)x^{-1}\in H_2$, so $h_1\in H_1\cap H_2$. $y\in(H_1\cap H_2)x$.

(2) Let $\{H_1x_1,\dots,H_1x_n\}$ be cosets of H_1 and $\{H_2y_1,\dots,H_2y_m\}$ cosets of H_2. From (1) any cosets $(H_1\cap H_2)x$ of $H_1\cap H_2$ is the intersection of a right coset H_1x_i of H_1 with a right coset H_2y_j of H_2. Hence $H_1\cap H_2$ has only a finite number ($\le nm$) of cosets. □

4. Let G be a finitely generated group, H a subgroup of finite index. Show that H is finitely generated.

Proof. Let $S = \{g_1, \ldots, g_m\}$ be a finite generating set of G. We may assume that $g_i^{-1} \in S$ for all i. Let $\{Hx_1, Hx_2, \ldots, Hx_n\}$ be the right cosets of H, where $x_1 = 1$. For any i, j, $x_i g_j = u_{ij} x_{i'}$ for some $u_{ij} \in H$ and some coset representative $x_{i'}$. We shall show that H is generated by $\{u_{ij}\}$, hence is finitely generated.

Let $h = g_{i_1} g_{i_2} \cdots g_{i_l} \in H$, where $g_{ij} \in H$. Then

$$\begin{aligned} h &= (x_1 g_{i_1}) g_{i_2} \cdots g_{i_l} = (u_{1_{i_1}} x_{1'}) g_{i_2} \cdots g_{i_l} \\ &= u_{1_{i_1}} (x_{1'} g_{i_2}) g_{i_3} \cdots g_{i_l} = u_{1_{i_1}} (u_{j' i_l} x_{2'}) g_{i_3} \cdots g_{i_l} \\ &= \cdots \\ &= u_{1_{i_1}} u_{1' i_2} \cdots u_{(l-1)' i_l} x_{s'} \in H = Hx_1. \end{aligned}$$

Hence $x_{s'} = x_1 = 1$ and H is generated by $\{u_{ij}\}$.

Remark. If H is any subgroup of a finitely generated group G, it is not necessary that H should be finitely generated. In fact, the commutator subgroup of a free group of rank two is not finitely generated.

5. Let H and K be two subgroups of a group G. Show that the set of maps $x \to hxk$, $h \in H$, $k \in K$ is a group of transformations of the set G. Show that the orbit of x relative to this group is the set $HxK = \{hxk | h \in H, k \in K\}$. This is called the double coset of x relative to the pair (H, K). Show that if G is finite then $|HxK| = |H|[K : x^{-1}Hx \cap K]$.

Proof. We only prove the last statement. We write $M = x^{-1}Hx \cap K$ for simplicity. We shall show that the mapping $Mk \to Hxk$ establishes a one to one correspondence between the cosets of $x^{-1}Hx \cap K$ in K and the cosets of H is HxK. Thus $|HxK|/|K| = [K : x^{-1}Hx \cap K]$, hence the result.

(i) The mapping is well-defined. If $Mk = Mk'$, then $k(k')^{-1} \in M = x^{-1}Hx \cap K$, $k(k')^{-1} \in x^{-1}Hx$, $xkk'^{-1}x^{-1} = xk(xk')^{-1} \in H$. Thus $Hxk = Hxk'$.

(ii) The mapping is one to one. Reversing the implications in (i) will get (ii).

(iii) The mapping is onto obviously. □

Remark. Let H and K be subgroups of G. Then HxK is an orbit under the transformation group stated in the exercise. Hence G has a double coset decomposition $G = \bigcup_{x \in G} HxK$.

6. Let H be a subgroup of finite index in a group G. Show that there exists a set of elements $z_1, z_2, \ldots, z_r \in G$, $r = [G : H]$, which are representatives of both the set of right and the set of left cosets, that is, G is the disjoint union of the Hz_i and also of the z_iH.

Proof. Let H be a subgroup of G. By the remark after exercise 5, G has a double coset decomposition $G = Hg_1H \cup Hg_2H \cup \cdots \cup Hg_lH$. To prove this exercise, it is enough to show that, for each double coset Hg_iH, there exist $z_1, \ldots, z_s$ so that they are representatives of the left and the right cosets of H contained in the double coset Hg_iH.

Let HgH be any double coset. Write $HgH = \bigcup_{i=1}^{s} Hgx_i$ where $x_i \in H$ and $Hgx_i \cap Hgx_j = \emptyset$ if $i \neq j$. We also write $HgH = \bigcup_{i=1}^{s'} y_i gH$ where $y_i \in H$ and $y_i gH \cap y_i gH = \emptyset$ if $i \neq j$. ¿From exercise 5, we have $s = s'$. For any $x_i, y_i \in H$ we have $Hgx_i = Hy_i gx_i$ and $y_i gx_i H = y_i gH$. Hence $\{y_i gx_i\}$ are the representatives which we want. $\square$