Basic Algebra (Solutions)

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Exercises (§1.11, p.69)

1. Let S be a subset of a group G such that $g^{-1}Sg \subset S$ for any $g \in G$. Show that the subgroup $\langle S\rangle$ generated by S is normal. Let T be any subset of G and let $S = \bigcup_{g\in G} g^{-1}Tg$. Show that $\langle S\rangle$ is the normal subgroup generated by T.

Proof. Omitted. □

The following three exercises are taken from Burnside's The Theory of Groups of Finite Order, 2nd ed., 1911. (Dover reprint, pp.464–465.)

2. Using the generators $(12), (13), \ldots, (1n)$ (See exercise 5, §1.6) for S_n, show that S_n is defined by the following relations on $x_1, x_2, \ldots, x_{n-1}$ in $FG^{(n-1)}$:

$$x_i^2,\ (x_ix_j)^3,\ (x_ix_jx_ix_k)^2,\ i, j, k \neq .$$

Proof. (I) Step 1. In S_n, let $(1i) = x_{i-1}$, then they satisfy $x_i^2 = 1$, $(x_ix_j)^3 = (1ji)^3 = 1$, $(x_ix_jx_ix_k)^2 = (ij)^2(1k)^2 = 1$, for i, j, k, distinct. Hence S_n is a homomorphic image of $FG^{(n-1)}/K$ where K is the normal subgroup generated by the above relations.

Step 2. We shall now show that $|FG^{(n-1)}/K| \leq n!$ which will imply that $S_n \simeq FG^{(n-1)}/K$. Let H be the subgroup generated by $x_1, x_2, \ldots, x_{n-2}$. It is enough to prove that $FG^{(n-1)}/K : H] \leq n$ and then get our result by induction.

Step 3. We prove that for any g, g can be written in the form of h, hx_{n-1} or $hx_{n-1}x_i$ for some $h \in H$, $1 \leq i \leq n-2$.

Note that the relation $(x_ix_j)^3 = 1$ will imply

(α) $x_{n-1}x_ix_{n-1} = x_ix_{n-1}x_i$, $1 \leq i \leq n-2$. The relation $(x_ix_jx_ix_k)^2 = 1$ will imply

(β) $x_{n-1}x_ix_j = x_ix_jx_ix_{n-1}x_i$, i, j, $n-1$ are distinct. Given any $g \in FG^{(n-1)}/K$ write g as a word in $x_1, \ldots, x_{n-1}$, say, $g = x_{i_1}\cdots x_{i_l}x_{n-1}x_{j_1}\cdots x_{j_m}$ where $x_{i_1}\cdots x_{i_l} \in H$. We write $x_{i_1}\cdots x_{i_l} = h$ for simplicity. We also assume $x_{j_1} \neq x_{n-1}$ since $x_{n-1}^2 = 1$. We prove the assertion by induction on m.

(1) If $x_{j_2} = x_{n-1}$, then $g = hx_{j_1}x_{n-1}x_{j_1}x_{j_3}\cdots x_{j_m}$ by (α),

(2) If $x_{j_2} \neq x_{n-1}$, then $g = hx_{j_1}x_{j_2}x_{j_1}x_{n-1}x_{j_1}x_{j_3}\cdots x_{j_m}$ by (β). In any case, $g = h'x_{n-1}x_{j_1}x_{j_3}\cdots x_{j_m}$ for $h' \in H$. The number m is reduced. By induction, it is easy to see that g can be transformed into the desired forms. □

Proof (II). Induction on n, $n = 2, 3$. Now let

$$G = \langle x_1, \ldots, x_{n-1} \rangle, \; H = \langle x_1, \ldots, x_{n-2} \rangle.$$

Define a homomorphism ϕ

$$\begin{aligned} \phi : \; & G \to S_n \\ & x_i \mapsto (1, i+1). \end{aligned}$$

ϕ is an epimorphism. To prove that ϕ is an isomorphism, it suffices to show that $|G| \leq n!$. $\square$

By induction hypothesis, $H \simeq S_{n-1}$. Consider $\widetilde{H} \stackrel{\text{def}}{=} H \cup Hx_{n-1} \cup Hx_{n-1}x_1x_{n-1} \cup \cdots \cup Hx_{n-1}x_{n-2}x_{n-1}$. If we can show that $\widetilde{H} = G$, then $|G| \leq n \cdot |H| = n!$.

We claim that $\widetilde{H}x_i \subset \widetilde{H}$ for all $i = 1, 2, \ldots, n-1$.

Assuming the above claim, we find that $x_i \in \widetilde{H}$ (since $1 \in H \subset \widetilde{H}$) and $\widetilde{H}$ is closed under multiplication. It follows that $\widetilde{H} = G$.

Now we shall prove that $\widetilde{H}x_i \subset \widetilde{H}$.

Case 1. $1 \leq i \leq n-2$.

$H \cdot x_i = H \subset \widetilde{H}$

$Hx_{n-1} \cdot x_i = H(x_{n-1}x_i) = H(x_ix_{n-1})^2 = Hx_i \cdot x_{n-1}x_ix_{n-1} = Hx_{n-1}x_ix_{n-1} \subset \widetilde{H}$.

$$Hx_{n-1}x_jx_{n-1} \cdot x_i = \begin{cases} Hx_ix_{n-1} = Hx_{n-1} \subset \widetilde{H} & \text{if } j = i. \\ Hx_ix_{n-1}x_jx_{n-1} = Hx_{n-1}x_jx_{n-1} \subset \widetilde{H} & \text{if } j \neq i. \end{cases}$$

Case 2. $i = n-1$.

$H \cdot x_{n-1} = Hx_{n-1} \subset \widetilde{H}$

$Hx_{n-1} \cdot x_{n-1} = H \subset \widetilde{H}$

$Hx_{n-1}x_jx_{n-1} \cdot x_{n-1} = Hx_{n-1}x_j = Hx_jx_{n-1}x_jx_{n-1} = Hx_{n-1}x_jx_{n-1} \subset \widetilde{H}$.

3. Using the generators $(12), (23), \ldots, (n-1n)$ for S_n show that this group is defined by $x_1, \ldots, x_{n-1}$ subjected to the relations:

$$x_i^2, \; (x_ix_{i+1})^3, \; (x_ix_j)^2, \; j > i+1.$$

Proof. Step 1. Let $x_i = (i, i+1)$, it is easy to see that they satisfy the relations: x_i^2, $(x_ix_{i+1})^3$, $(x_ix_j)^2$, $j > i+1$.

Step 2. By the similar arguments as in exercise 2, it is enough to show that $FG^{(n-1)}/K = H \cup Hx_{n-1} \cup Hx_{n-1}x_{n-2} \cup \cdots \cup Hx_{n-1}x_{n-2} \cdots x_1$ where H is the subgroup generated by $x_1, \ldots, x_{n-2}$:

Let $g = hx_{n-1}x_{j_1} \cdots x_{j_m} \in FG^{(n-1)}/K$ where $h \in H$, we prove this assertion by induction on m. We first note that the relations $(x_ix_{i+1})^3 = 1$ and $(x_ix_j)^2 = 1$, $j > i+1$ imply that

(α) $x_j x_i = x_i x_j$, if $j > i+1$,

(β) $x_{n-1}x_{n-2}x_{n-1} = x_{n-2}x_{n-1}x_{n-2}$.

(i) If $x_{j_1} \neq x_{n-2}$, then $g = hx_{j_1}x_{n-1}x_{j_2}\cdots x_{j_m}$ by (α).

(ii) If $x_{j_1} = x_{n-2}$ and $x_{j_2} \neq x_{n-1}$ ($x_{j_2} \neq x_{n-2}$), then $g = hx_{n-1}x_{n-2}x_{j_2}\cdots x_{j_m} = hx_{n-1}x_{j_2}x_{n-2}x_{j_3}\cdots x_{j_m}$ (by (α)) $= hx_{j_2}x_{n-1}x_{n-2}x_{j_3}\cdots x_{j_m}$ (by (α)).

(iii) If $x_{j_1} = x_{n-2}$, $x_{j_2} = x_{n-1}$, then $g = hx_{n-1}x_{n-2}x_{n-1}x_{j_3}\cdots x_{j_m} = hx_{n-2}x_{n-1}x_{n-2}x_{j_3}\cdots x_{j_m}$ by (β).

In any case, the number m is reduced. And the proof is completed.

Remark. As in proof (II) of the above exercise, we can show that $\widetilde{H}x_i \subset \widetilde{H}$ where $\widetilde{H} \stackrel{\text{def}}{=} H \cup Hx_{n-1} \cup Hx_{n-1}x_{n-2} \cup \cdots \cup Hx_{n-1}x_{n-2}\cdots x_1$.

4. Show that A_n can be defined by the following relations on $x_1, x_2, \ldots, x_{n-2}$

$$x_1^3;\ x_i^2,\ i > 1;\ (x_i x_{i+1})^3;\ (x_i x_j)^2,\ j > i+1.$$

Proof. Step 1. In A_n, set $x_1 = (123)$, $x_i = (12)(i+1, i+2)$ for $2 \leq i \leq n-2$. It is easy to verify that they satisfy the given relations.

Step 2. Similar to the arguments in exercise 2, it is enough to show that $[FG^{(n-2)}/K : H] \leq n$ where H is the subgroup generated by $x_1, x_2, \ldots, x_{n-3}$. For this purpose, we shall show that $FG^{(n-2)}/K = H \cup Hx_{n-2} \cup Hx_{n-2}x_{n-3} \cup Hx_{n-2}x_{n-3}x_{n-4} \cup \cdots \cup Hx_{n-2}x_{n-3}\cdots x_2x_1 \cup Hx_{n-2}\cdots x_2x_1^2$:

For any $g = hx_{n-2}x_{j_1}\cdots x_{j_m} \in G$ where $h \in H$, we shall prove this assertion by induction on m. ¿From the given relations we have (α) $x_j x_i = x_i x_j$, $j > i+1$; and (β) $x_{n-3}x_{n-2}x_{n-3} = x_{n-2}x_{n-3}x_{n-2}$.

The remaining proof are quite similar to exercise 3 and left to the reader. □