1.6. Density and Mass Functions

Definition 1.6.1 (Probability Mass Function)

The probability mass function (pmf) of a discrete random variable X is given by

$$f_X(x) = P(X = x) \quad \text{for all } x.$$

Example 1.6.2 (Geometric probabilities)

For the geometric distribution of Example 1.5.4, we have the pmf

$$f_X(x) = P(X = x) = \begin{cases} (1-p)^{x-1}p & \text{for } x = 1, 2, \dots \\ 0 & \textit{otherwise.} \end{cases}$$

From this example, we see that a pmf gives us "point probability". In the discrete case, we can sum over values of the pmf to get the cdf. The analogous procedure in the continuous case is to substitute integrals for sums, and we get

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

Using the Fundamental Theorem of Calculus, if $f_X(x)$ is continuous, we have the further relationship

$$\frac{d}{dx}F_X(x) = f_X(x).$$

Definition 1.6.3 (Probability Density Function)

The probability density function or pdf, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \quad \text{for all } x. \tag{1}$$

A note on notation : The expression "X has a distribution given by $F_X(x)$" is abbreviated symbolically by "$X \sim F_X(x)$", where we read the symbol "$\sim$" as "is distributed as". We can similarly write $X \sim f_X(x)$ or, if X and Y have the same distribution, $X \sim Y$.

In the continuous case we can be somewhat cavalier about the specification of interval probabilities. Since $P(X = x) = 0$ if X is a continuous random variable,

$$P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b).$$

<u>Example 1.6.4 (Logistic probabilities)</u>

For the logistic distribution, we have

$$F_X(x) = \frac{1}{1 + e^{-x}},$$

hence, we have

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1 + e^{-x})^2},$$

and

$$\begin{aligned} P(a < X < b) &= F_X(b) - F_X(a) \\ &= \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx \\ &= \int_{a}^{b} f_X(x)dx. \end{aligned}$$

<u>Theorem 1.6.5</u>

A function $F_X(x)$ is a pdf (or pmf) of a random variable X if and only if

a. $F_X(x) \geq 0$ for all x.

b. $\sum_x f_X(x) = 1$ (pmf) or $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (pdf).

PROOF: If $f_X(x)$ is a pdf (or pmf), then the two properties are immediate from the definitions. In particular, for a pdf, using (1) and Theorem 1.5.3, we have that

$$1 = \lim_{x \to \infty} F_X(x) = \int_{-\infty}^{\infty} f_X(t)dt.$$

The converse implication is equally easy to prove. Once we have $f_X(x)$, we can define $F_X(x)$ and appeal to Theorem 1.5.3. Theorem 1.5.3 is the Theorem in Lecture note 5 on cumulative distribution function. □