Abitrage Approach to Pricing Derivatives

Jin-Chuan Duan

Hong Kong University of Science and Technology

Correspondence to:
Prof. Jin-Chuan Duan
Department of Finance
Hong Kong University of Science & Technology
Clear Water Bay, Kowloon, Hong Kong
Tel: (852) 2358 7671; Fax: (852) 2358 1749
E-mail: jcduan@ust.hk
Web: http://www.bm.ust.hk/~jcduan

<u>Static vs. dynamic spanning</u>

- Assume that there are *N* possible states at time 1. Every security entitles its holder an *N*-dimensional payoff vector. There are *K* securities with the $N\times K$ payoff matrix *A* and current $K\times 1$ price vetor *P*.

- If *A* has a rank *N*, then the market is <u>complete</u> in the sense that any possbile payoff structure can be spanned (static) by some portfolio of *K* securites.

- If *A*'s rank is less than *N*, the market is <u>incomplete</u>. A payoff structure can still be priced by arbitrage as long as it falls inside the static spanning.

- Arrow-Debreau equilibrium refers to a complete market competitive equilibrium in which allocations are efficient. Note that no arbitrage is a necessary condition of market equilibrium.

- The price vector *P* cannot be arbitrary. To say the least, it cannot permit arbitrage in the sense that any two portfolios with an identical payoff vector must has the same current value.

- If one is allowed to trade between time 0 and 1, the spanning set can be enlarged even though the number of securities remains fixed. In other words, one is more likely to be able to price a payoff structure by arbitrage.

Black-Schole dynamic spanning approach to option valuation

Asset price dynamic

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Its derivative security with payoff function at time T equal to $f(S_T;\theta)$ has a time-t value expressed as $C(S_t, t; \sigma, \mu, T, r, \theta)$ or C_t for short.

Consider a dynamically rebalanced portfolio shorting $\Delta_{t\_}$ units of the underlying asset to hedge the derivative security. The hedged porfolio's value at time t is

$$V_t = C_t - \Delta_{t\_} S_t .$$

Applying Ito's lemma gives rise to

$$\begin{aligned} dV_t &= dC_t - \Delta_{t\_} dS_t \\ &= \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S_t} dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} dt - \Delta_{t\_} dS_t \\ &= \left(\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right) dt + \left(\frac{\partial C_t}{\partial S_t} - \Delta_{t\_} \right) dS_t . \end{aligned}$$

Setting $\Delta_{t\_} = \frac{\partial C_t}{\partial S_t}$ yields a locally risk-free hedged portfolio. Excluding arbitrage, it must be true that

$$\left(\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right) dt = rV_t dt$$

$$= r\left(C_t - S_t \frac{\partial C_t}{\partial S_t} \right) dt$$

or (the Black-Scholes PDE)

$$\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t = 0$$

The solution to this PDE depends on the terminal condition: $f(S_T;\theta)$. It can be solved using separation of variables, Green's function or Fourier/Laplace transformation technique.

A probabilistic way of solving the generalized Black-Scholes PDE

When both μ_t and σ_t are functions of S_t, the Black-Scholes PDE applies.

$$\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t = 0$$

The solution to the generalized PDE can be obtained by directly applying the backward equation for the Kac functional; that is the following conditional expectation satisfying the Black-Scholes PDE:

$$C_t = E\left\{e^{-r(T-t)} f(S_T;\theta) \mid S_t\right\}$$

with respect to the following <u>artificial</u> diffusion system:

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t^*$$

This probablistic solution suggests a new perspective of risk-neutral valuation

<u>Martingale pricing theory</u>

The Kac functional result suggests that $e^{-rt}S_t$ is a martingale with respect to the law Q which W_t^* is a standard Brownian motion. Note that $C_T = f(S_T;\theta)$. The same martingale result is thus true for derivatives as well.

Alternatively, one can show this by the Kunita-Watanabe martingale representation theorem (see Harrison and Kreps (1979), *Journal of Economic Theory*).