Financial Time Series

Topic 11: Value at Risk

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OUTLINE

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Value at Risk

There are several types of risk in financial markets. Credit risk, liquidity risk, and market risk are three examples.

- Value at risk (VaR) is concerned with market risk.
- VaR is a single number measuring the risk of a financial position.
- VaR is used to ensure that the financial institutions can still be in business after a catastrophic event.
- It is the amount that a position could decline in a given period with a specified probability.

A formal definition

- VaR is evaluated under a probabilistic framework.
- time period given: $\Delta t = \ell$
  This might be set by a regulatory committee such as 10 days.
- change in value: $\Delta V(\ell)$

- CDF of the change: $F_\ell(x)$
  It is the focus of econometric modeling.
- Given probability: p
  p can be 0.01 or 0.05.
- a long position:

$$p = Pr(\Delta V(\ell) \leq VaR) = F_\ell(VaR).$$

- a short position: (A loss occurs when the price increases.)

$$p = Pr(\Delta V(\ell) \geq VaR) = 1 - F_\ell(VaR).$$

- VaR is a prediction concerning possible loss of a portfolio in a given time horizon.
  - Use the *predictive distribution* of future returns of the portfolio.
  - For example, the VaR for a one-day horizon of a portfolio using daily returns r_t should be calculated using the predictive distribution of r_{t+1} given information available at time t.
  - Predictive distribution takes into account parameter uncertainty even for a properly specified model.

- Quantile: x_p is the pth quantile of $F_\ell(x)$ if

$$p = F_\ell(x_p)$$

and $F_\ell(\cdot)$ is continuous.
In general,

$$x_p = \inf_x \{x | F_\ell(x) \geq p\}.$$

- Use log returns.

$$\text{VaR} = \text{Value} \times (\text{VaR of log returns}).$$

Data Example

Daily log returns of IBM stock

- Span: July 3rd, 1962 to December 31st, 1998
- Size: 9190 points
- See Figure.
- Position: long on 10 million

RiskMetrics

- Developed by J.P. Morgan
- r_t given $\mathcal{F}_t$: $N(0, \sigma_t^2)$
- σ_t^2 follows the special $IGARCH(1,1)$ model

$$\sigma_t^2 = \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2, \quad 1 > \alpha > 0.$$

- RiskMetrics assumes that $r_t|\mathcal{F}_{t-1} \sim N(\mu_t, \sigma_t^2)$, where $\mu_t = 0$.
  The logarithm of the daily price, $p_t = \ln P_t$, satisfies the difference equation

$$p_t - p_{t-1} = a_t,$$

  where $a_t = \sigma_t \epsilon_t$ is an $IGARCH(1,1)$ process.
- α is often in the interval $(0.9, 1)$.
- Var $= 1.65\sigma_t$ if $p = 0.05$.
- The conditional distribution of a multi-period forecast is easily available.

$$r_{t+h-1}|\mathcal{F}_{t-1} \sim N(0, h\sigma_t^2).$$

- k-horizon:

$$VaR(k) = \sqrt{k}VaR$$

- Example 1:
  - The sample SD of the continuously compounded daily return of the German Marks/US Dollars exchange rate was about 0.53% in June 1997.
  - An investor was long in \$10 millions worth of Mark/Dollar exchange rate contract.
  - The 5% VaR for a one-day horizon of the investor is

    $$10,000,000 \times (1.65 \times 0.0053) = 53,000.$$

    For one-month horizon(30 days) is

    $$\sqrt{30} \times 53,000 = 290,000.$$

- Example 2: IBM data
  - Model: $r_t = a_t$, $a_t = \sigma_t \epsilon_t$,

    $$\sigma_t^2 = 0.9396\sigma_{t-1}^2 + (1 - 0.9396)a_{t-1}^2$$

  - $r_{9190} = -0.0128$, $\hat{\sigma}_{9190}^2 = 0.0003472$, $\hat{\sigma}_{9190}^2(1) = 0.000336$.
  - For $p = 0.05$, VaR of r_t is

    $$-1.65 \times \sqrt{0.000336} = -0.03025.$$

    Or VaR is \$302,500.

A General GARCH Approach to VaR

- Consider general econometric models.
- $r_t = \mu_t + a_t$ given $\mathcal{F}_{t-1}$
  - μ_t: a mean equation
  - σ_t^2: a volatility model
- Write r_t as

$$\mu + \sum_{i=1}^{p} \phi_i r_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j}.$$

- Write σ_t^2 as

$$\alpha_0 + \sum_{i=1}^{u} \alpha_i a_{t-i}^2 + \sum_{j=1}^{v} \beta_j \sigma_{t-j}^2.$$

- What is ϵ_t?
  Assume that ϵ_t is Gaussian or a student-t distribution with h degrees of freedom.

Daily log returns of IBM stock

- Case 1: Assume that ϵ_t is standard normal.
  - Consider $AR(2)-GARCH(1,1)$ model.
  - $r_t = 0.00066 - 0.0247 r_{t-2} + a_t$ where $a_t = \sigma_t \epsilon_t$.
  - $\sigma_t^2 = 0.00000389 + 0.0799 a_{t-1}^2 + 0.9073 \sigma_t^2$.
  - $r_{9189} = -0.00201$, $r_{9190} = -0.0128$, $\sigma_{9190}^2 = 0.00033455$
  - $\hat{r}_{9190}(1) = 0.00071$ and $\hat{\sigma}_{9190}^2(1) = 0.0003211$.
  - If $p = 0.05$, we have

    $0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.$

    Or, VaR is $\$287,700$.
  - If $p = 0.01$, we have

    $0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.$

    Or, VaR is $\$409,738$.
- Case 2: Assume that ϵ_t is Student-t_5.
  - Consider $AR(2)-GARCH(1,1)$ model.
  - $r_t = 0.0003 - 0.0335 r_{t-2} + a_t$ where $a_t = \sigma_t \epsilon_t$.
  - $\sigma_t^2 = 0.000003 + 0.0559 a_{t-1}^2 + 0.9350 \sigma_t^2$.

- $r_{9189} = -0.00201$, $r_{9190} = -0.0128$, $\sigma^2_{9190} = 0.000349$
- $\hat{r}_{9190}(1) = 0.000367$ and $\hat{\sigma}^2_{9190}(1) = 0.0003386$.
- If q is the quantile of a Student-t distribution wih v degrees of freedom for probability p, then $q/\sqrt{v/(v-2)}$ is the quantile of a standardized Student-t distribution with v degrees of freedom and probability p.
- If $p = 0.05$, we have

  $$0.000367 - (2.015/\sqrt{5/3}) \times \sqrt{0.0003386} = -0.028354.$$

  Or, VaR is \$283,520.
- If $p = 0.01$, we have

  $$0.000367 - (3.3659/\sqrt{5/3}) \times \sqrt{0.0003386} = -0.0475953.$$

  Or, VaR is \$475,943.

• Effect of heavy-tails seen with $p = 0.01$.

Empirical Quantile

Assuming that the distribution of the return in the prediction period is the same as that in the sample period, one can use the empirical quantile of the return r_t to calculate VaR.

- Sample of log returns: $\{r_t; t = 1, \ldots, n\}$
- Order statistics:

$$r_{(1)} \le \cdots \le r_{(n)}$$

  $r_{(i)}$ is the ith order statistic of the sample.

- Idea: Use the empirical quantile to estimate the theoretical quantile of r_t.
  - If $np = \ell$ is an integer, it is $r_{(\ell)}$.
  - If np is not an integer, find the two neighboring integers $\ell_1 < np < \ell_2$ and use interpolation.

    The quantile is

$$\hat{x}_p = \frac{p_2 - p}{p_2 - p_1} r_{(\ell_1)} + \frac{p - p_1}{p_2 - p_1} r_{(\ell_2)}.$$

- For IBM data, $np = 459.5$ when $p = 0.05$. Its 5% quantile is

$$(r_{(459)} + r_{(460)})/2 = -0.021603.$$

Or, VaR is $\$216,030$.
If $p = 0.01$, VaR is $\$365,800$.

- Advantage: simplicity and no specific distributional assumption.
- Drawbacks:
  - The distribution of the return r_t remains unchanged from the sample period to the forecast period.
  - VaR is concerned mainly with tail probability. Use empirical quantile implies that predicted loss cannot be greater than that of the historical loss.
  - For extreme quantiles, the empirical quantiles are not efficient estimates of the theoretical quantiles.
  - It fails to take into account the effect of explanatory variables that are relevant to the portfolio under study.

Extreme Value Theory

- A properly normalized $r_{(1)}$ assumes a special distribution:

  $$F_*(x) = \begin{cases} 1 - \exp(-(1+kx)^{1/k}) & \text{if } k \neq 0 \\ 1 - \exp(-\exp(x)) & \text{if } k = 0 \end{cases}$$

  for $x < -1/k$ if $k < 0$ and for $x > -1/k$ if $k > 0$.

  - k: the shape parameter
  - $\alpha = -1/k$: tail index of the distribution
  - The CDF of $r_{(1)}$ is given by

    $$1 - \{1 - F(x)\}^n.$$

  - The extreme value theory is concerned with finding two sequences $\{\beta_n\}$ and $\{\alpha_n\}$, where $\alpha_n > 0$, such that the distribution of $(r_{(1)} - \beta_n)/\alpha_n$ converges to a non-degenerated distribution as n goes to infinity.

- Classification of extreme value distribution:
  - Type I: $k = 0$, the Gumbel family. ($x \in R$)
    The left tail of the distribution declines exponentially.

- Type II: $k < 0$, the Frechet family. $(x < -1/k)$
  The left tail of the distribution declines as a power function.
- Type III: $k > 0$, the Weibull family. $(x > -1/k)$. The left tail of the distribution is finite.

- For risk management, we are interested mainly in the Frechet family that includes stable distributions and Student-t distributions.

- The Gumbel family consists of thin-tailed distributions such as normal and log-normal distributions.

- Estimation: There are three parameters k, β_n, and α_n.
  They are referred to as *shape*, *location*, and *scale* parameters, respectively.

Estimation with EVT distribution

- Divide the sample into non-overlapping subsamples.
- Suppose that there are T data points, we divide the data as

  $\{r_1, \cdots, r_n | r_{n+1}, \cdots, r_{2n} | \cdots | r_{(g-1)n+1}, \cdots, r_{ng}\}$.

  Here n is the size of subgroup.
- Find the minimum of each subgroup.
- Assuming that the subperiod minima $\{r_{n,i}\}$ follow a generalized extreme value distribution.
- Use the method of maximum likelihood to obtain parameter estimates.

Now we consider IBM data again.

- Consider $n = 63$ (quarterly minima). ($g = 145$)
  - We have $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$.
  - If $p = 0.01$, the VaR is

$$-2.583 - \frac{0.945}{-0.335}\left\{1 - [-63 \ln(1 - 0.01)]^{-0.335}\right\}.$$

Or, VaR is $\$304,969$.

- If $p = 0.05$, VaR is $\$166,641$.

- Consider $n = 21$ (monthly minima). ($g = 437$)
  - We have $\hat{\alpha}_n = 0.823$, $\hat{\beta}_n = -1.902$, and $\hat{k}_n = -0.197$.
  - If $p = 0.05$, the VaR is $\$184,127$.
  - If $p = 0.01$, VaR is $\$340,013$.
- Results depend on the choice of n.

Summary of IBM Data:

- If $p = 0.05$, then
  - $\$302,500$ for the RiskMetrics,
  - $\$287,200$ for an $AR(2)-GARCH(1,1)$ model,
  - $\$283,520$ for an $AR(2)-GARCH(1,1)$ model with t_5,
  - $\$216,030$ using the empirical quantile, and
  - $\$184,127$ for EVT with $n = 21$.
- If $p = 0.01$, then
  - $\$426,500$ for the RiskMetrics,
  - $\$409,738$ for an $AR(2)-GARCH(1,1)$ model,
  - $\$475,943$ for an $AR(2)-GARCH(1,1)$ model with t_5,
  - $\$365,800$ using the empirical quantile, and
  - $\$340,013$ for EVT with $n = 21$.
- If $p = 0.001$, then
  - $\$566,443$ for the RiskMetrics,
  - $\$546,641$ for an $AR(2)-GARCH(1,1)$ model,

– \$836,341 for an $AR(2)-GARCH(1,1)$ model with t_5,
– \$780,712 using the empirical quantile, and
– \$666,590 for EVT with $n = 21$.