Financial Time Series

Topic 9: Modelling Return Distributions

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OUTLINE

1. Empirical finding on return distributions
2. Models for Return Distributions
3. Determining the Tail Shape of a Return Distribution
4. Estimation of μ and σ in Ito's process
5. Revisit the Pricing of Enropean Option

Empirical finding on return distributions

- Characteristics:
  - Fat tails
  - High peakedness (excess kurtosis)
  - Skew
- Three return series: Figure 5.1
  - daily returns of the London FT30 from 1935 to 1994
  - daily returns of the $S\&P500$
  - daily dollar/sterling exchange rate
- Descriptive statistics:
  Splus command summary: It gives min, 1st Qu, median, mean, 3rd Qu, max.
  mean: 0.022, 0.020, -0.008
  sd: 1.004, 1.154, 0.647
  median: 0.000, 0.047, 0.000
  range: 23.1, 33.1, 13.2
  kurt: 14.53, 25.04, 6.51

- Graphical representations:
  - smoothed function of the histogram
  - Q-Q plots: Plot empirical cumulative distribution against normal distribution.

Histogram

- pdf:

$$P(-h/2 \leq X < h/2) = \int_{-h/2}^{h/2} f(x)dx$$

- law of large numbers:

$$P(-h/2 \leq X < h/2) \approx n^{-1}\#\{X_i \in [-h/2, h/2)\}$$

- approximation:

$$P(-h/2 \leq X < h/2) \approx f(\xi)h$$

- density estimate:

$$\hat{f}_h(x) = (nh)^{-1} \sum_{i=1}^{n} \sum_{j} I(X_i \in B_j) I(x \in B_j)$$

Consider the kernel density estimate

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right),$$

where K is a kernel function with the following properties

- Kernel function is symmetric around 0 and integrate to 1.
- Kernel is a density function and the kernel estimate is a density too.
- Kernel estimates do not depend on any choice of origin.

Quantile-Quantile plots

- If X is a continuous random variable with a strictly increasing distribution function, F, the pth quantile ofthe distribution is the value of x such that $F(x) = p$ or $x_p = F^{-1}(p)$.
- In a Q-Q plot, the quantiles of one distribution are plotted against those of another.
- Suppose $G(y) = F(y - h)$. Then

$$y_p = x_p + h$$

  and a Q-Q plot would be a straight line with slope 1 and intercept h.
- Let $r_1, \cdots, r_n$ be the returns of a portfolio in the sample period.
  The order statistics of the sample are these

values arranged in increasing order.
$r_{(1)}$ is the sample minimum and $r_{(n)}$ the sample maximum.

- For $\ell = np$,

$$\sqrt{n}(r_\ell - x_p) \sim N(0, p(1-p)/f^2(x_p)),$$

if $f(x_p) \neq 0$.

Models for Return Distributions

- fat-tailed and highly peaked
- Stable distributions
  It is a generalization of normal in that they are stable under addition.
  But non-normal stable distributions do not have a finite variance.
- Mixture distributions
  Hierarchical model approach: regression
  Facts:
  - $EX = E(E(X|Y))$, $X|Y \sim Bin(Y, p)$, $Y \sim Poisson(\lambda)$
  - $VarX = E(Var(X|Y)) + Var(E(X|Y))$
  - Mixed discrete-continuous distribution
    Slice a data set into different, supposedly more homogeneous, subsets.
  - Scale-Mixture Normals
    As an example, consider
    $$r_t \sim (1-\alpha)N(\mu, \sigma_1^2) + \alpha N(\mu, \sigma_2^2),$$
    where $0 \le \alpha \le 1$, σ_1^2 is small, and σ_2^2 is large.

The large value of σ^2 enables the mixture to put more mass at the tails of its distribution.

Note that

$$\begin{aligned} E(r_t) &= EE(r_t|I) = E\mu = \mu \\ V(r_t) &= EV(r_t|I) + V(E(r_t|I)) \\ &= E\sigma_I^2 + V(\mu) \\ &= (1-\alpha)\sigma_1^2 + \alpha\sigma_2^2 \end{aligned}$$

– Normal and Stable

– Student or double Weibull distributions

- Stable distributions:

  – Characteristic function:

  $$\varphi(t) = \int_{-\infty}^{\infty} \exp(itx) dF(x)$$

  – The symmetric (about zero) stable characteristic function:

  $$\varphi(t) = \exp(-\sigma^\alpha |t|^\alpha)$$

  where $0 < \alpha \leq 2$ is the characteristic exponent and σ is a scale paramter.

  – Probability distribution

  $$F(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\sigma^\alpha |t|^\alpha) \exp(-itX) dt$$

$$= \frac{1}{\pi} \int_0^\infty \exp(-\sigma^\alpha |t|^\alpha) \cos(tX) dt. \quad (1)$$

– Normal distribution: $\alpha = 2$

– For $\alpha < 2$, all moments greater than α are infinite.

– Fat tail: relative to the normal

– regular variation at infinity

$$\lim_{s \to \infty} \frac{1 - F(sX)}{1 - F(s)} = X^{-\alpha}$$

The stable distribution displays a power declining tail, $X^{-\alpha}$, rather than an exponential decline as is the case with the normal.

– α: the tail index

– Why the stable distribution should be an appropriate generating process for financial data?
Mandelbrot (1963): The limiting distribution of an appropriately scaled sum of independent and identically distributed random variables exists then it must be a member of the stable class, even if these random variables have infinite variance.

– If daily returns follow a stable distribution, then weakly, monthly and quarterly returns can be viewed as the sum of daily returns, they too will follow stable distributions having identical characteristic exponents.

- Volatility clustering
  – GARCH class of models: serial correlation of conditional variances
  – Consider $ARCH(1)$ with normal innovations' process for X_t.

$$X_t = U_t \sigma_t \tag{2}$$

where $U_t \sim NID(0, 1)$ and

$$\sigma_t^2 = w + \beta X_{t-1}^2. \tag{3}$$

From the above two equations, we have

$$X_t^2 = wU_t^2 + \beta U_t^2 X_{t-1}^2 = B_t + A_t X_{t-1}^2. \tag{4}$$

  – X_t is serially uncorrelated but is not independent.
  – $ARCH(1)$ process may also exhibit fat tails.

de Haan et al. (1989) show that the X_t of (4) regularly varies at infinity and has a tail index ζ defined explicitly by the equation

$$\Gamma\left(\frac{\zeta+1}{2}\right) = \pi^{1/2}(2\beta)^{-\zeta/2}.$$

Determining the tail shape of a return distribution

Estimation of μ and σ in Ito's process

- Treating price of an asset as a random variable that evolves over time, the prices series forms a stochastic process.
- The observed price series is a realization of the underlying stochastic process.
- Consider two types of stochastic processes.
  - Discrete-time stochastic process: Consider the daily closing price of IBM stock on the NYSE. Here the price change occurs only at the closing of a trading day.
  - Continuous-time stochastic processes: Assume the price changes continuously even when the stock is not traded.
  - A continuous-time continuous stochastic process can be written as $\{x(\eta, t)\}$, where t denotes time and is continuous in $[0, \infty)$. For a given t, $x(\eta, t)$ is a continuous random variable defined on a probability space and η is an element of the space.
  - For a given η, $\{x(\eta, t)\}$ is a time series with value depending on time.

- The counterpart of white noise process to a discrete-time econometric model in a continuous-time model is the *Wiener Process*, which is also known as *Brownian motion*.

A continuous-time stochastic process $\{w_t\}$ is a Wiener Process if it satisfies

- $\triangle w_t = w_{t+\triangle t} - w_t = \epsilon\sqrt{\triangle t}$, where $\epsilon \sim N(0, 1)$.
  - $\triangle w_t \sim N(0, \triangle t)$.
  - Define $N = t/\triangle t$. Then
  $$w_t - w_0 = \sum_{i=1}^{N} \epsilon_i \sqrt{\triangle t} \sim N(0, t).$$
- $\triangle w_t$ is independent of w_j for all $j < t$.
  - This is a Markov property.
  - $w_{t_1+\triangle_1 t} - w_{t_1}$ and $w_{t_2+\triangle_2 t} - w_{t_2}$ are independent for any two non-overlapping time intervals $\triangle_1$ and $\triangle_2$.

This suggests that we can simulate Wiener process on the unit time interval $[0, 1]$ using the following statistical property.

Property:

- Assume that $\{z_t\}_{t=1}^{n}$ is a sequence of independent standard normal random variables.
- For any $t \in [0, 1]$, let $[nt]$ be the integer part of nt.
- Define $w_{n,t} = n^{-1/2} \Sigma_{i=1}^{[nt]} z_i$.
- $w_{n,t}$ converges in distribution to the Wiener process w_t as n goes to infinity.

Let P_t be the price of a security at time t, which is continuous in $[0, \infty)$. In the literature, it is common to assume that P_t follows the following Ito's process

$$dP_t = \mu(P_t, t)dt + \sigma(P_t, t)dW_t. \qquad (5)$$

Here

- μ and σ are referred to as the drift and volatility parameters of the process P_t.
- W_t is a Wiener process.
- Use the notation dy for a small change in the variable y.
- When $\mu(P_t, t) = \mu P_t$ and $\sigma(P_t, t) = \sigma P_t$ where μ and σ are constants, apply the Ito's

lemma to obtain

$$d\ln(P_t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t.$$

Hence, the change in logarithm of price (log return) between current time t and some future time T is normally distributed with mean $(\mu-\sigma^2/2)(T-t)$ and variance $\sigma^2(T-t)$.

- To simulate P_t, we can use the following recursive form

  $$\ln P_{t+\triangle} = \ln P_t + (\mu - \sigma^2/2)\triangle + \sigma\triangle^{1/2}Z,$$

  where $Z \sim N(0,1)$.

How do we estimate these two unknown parameters μ and σ?

- Assume that we have $n+1$ observations of stock price P_i before time t at equally spaced time interval Δ.
- Denote the observed prices as $\{P_0, P_1, \cdots, P_n\}$ and let $r_i = \ln(P_i) - \ln(P_{i-1})$ for $i = 1, \ldots, n$.
- r_i is normally distributed with mean $(\mu - \sigma^2/2)\Delta$ and variance $\sigma^2\Delta$.
  $r_i's$ are not serially correlated.

- Let $\bar{r}$ and s_r be the sample mean and standard deviation of the data.
- Estimate σ by $s_r/\sqrt{\Delta}$ and μ by

$$\frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2}.$$

- Consider the daily log returns of IBM in 1998.
  - It has 252 observations.
  - The sample ACF of the data indicates that the log returns are indeed serially uncorrelated. The Ljung-Box statistic gives $Q(10) = 4.9$, which is highly insignificant compared with a chi-square distribution with 10 degrees of freedom.
  - Assume that the price of IBM in 1998 follows the Ito's process.
  - $\bar{r} = 0.002276$, $s_r = 0.01915$, and $\Delta = 1/252$ year.
  - $\hat{\sigma} = s_r/\sqrt{\Delta} = 0.3040$.
  - $\hat{\mu} = 0.6198$
  - The estimated expected return is 61.98% and the standard deviation is 30.4% per annum for IBM in 1998.

Distributions of stock prices and log returns

- Conditional on the price P_t at time t, the log price at time $T > t$ is

$$\begin{aligned} \ln(P_T) &= \ln(P_t) + X_{t,T} \qquad (6) \\ X_{t,T} &\sim N((\mu - \sigma^2/2)(T-t), \sigma^2(T-t)). \end{aligned}$$

This gives the information on the future price of P_T.

- Lognormal distribution with parameters μ and σ^2:

$$\begin{aligned} E(Y) &= \exp(\mu + \sigma^2/2) \\ V(Y) &= \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]. \end{aligned}$$

- The conditional mean and variance of P_T given P_t are

$$\begin{aligned} E(P_T) &= P_t \exp[\mu(T-t)], \\ V(P_T) &= P_t^2 \exp[2\mu(T-t)]\{\exp[\sigma^2(T-t)] - 1\}. \end{aligned}$$

- Suppose the current price of stock A is \$50, the expected return is 15% per annum, and the volatility is 40% per annum.
  Then the expected price and variance of stock A in 6-month are

$$E(P_T) = 50 \exp(0.15 \times 0.5) = 53.89$$

and

$$
\begin{aligned}
V(P_T) &= 2500 \exp(0.3 \times 0.5)[\exp(0.16 \times 0.5) - 1] \\
&= 241.92.
\end{aligned}
$$

The standard deviation is around 15.55.

- Let r be the continuously compounded rate of return per annum from time t to T.
  Note that
  $$P_T = P_t \exp[r(T-t)]$$
  and
  $$r = \frac{1}{T-t} \ln \frac{P_T}{P_t}.$$
- The distribution of r is
  $$N(\mu - \sigma^2/2, \sigma^2/(T-t)).$$
- Consider a stock with an expected rate of return of 15% per annum and a volatility of 10% per annum.
  The distribution of r is $N(.15-.01/2, (.1/\sqrt{2})^2)$.
  A 95% CI for r is $(0.145 \pm 1.96 \times 0.071) = (0.6\%, 28.4\%)$.

What is the effect on using estimated μ and σ^2?

Refer to next topic for the European call option.

Revisit the Pricing of Enropean Option

Black-Scholes Pricing Formulas

Risk-Neutral World

- The expected return on all securities is the risk-free interest rate r.
- The present value of any cash flow can be obtained by discounting its expected value at the risk-free rate.
- Under the no arbitrage assumption, the portfolio V_t must be riskless during the small time interval.
  Here V_t is the value of the portfolio.
  The portfolio must instantaneously earn the same rate of return as other short-term risk-free securities.
- The expected value of a European call option at maturity in a risk-neutral world is

$$E_*[\max(P_T - K, 0)]$$

  where E_* denotes expected value in a risk-neutral world.

- The price of the call option at time t is

$$c_t = \exp[-r(T-t)]E_*[\max(P_T - K, 0)]. \quad (7)$$

We need to specify the distribution of P_T.

- In a risk-neutral world, we have $\mu = r$ and by (6),

$$\ln(P_T) \sim \ln(P_t) + N((r - \sigma^2/2)(T-t), \sigma^2(T-t)).$$

- Let $g(P_T)$ be the probability density function of P_T. Then the price of the call option in (7) is

$$c_t = \exp[-r(T-t)] \int_K^\infty (P_T - K) g(P_T) dP_T.$$

The above formular holds for general price process.
Under Black-Scholes model, the distribution is log-normal with mean $\ln(P_t) + (r - \sigma^2/2)(T - t)$ and variance $\sigma^2(T-t)$.

- By changing variable in the integration and some algebraic calculations, we have

$$c_t = P_t\Phi(h_1) - K\exp[-r(T-t)]\Phi(h_2) \quad (8)$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable evaluated at x,

$$\begin{aligned} h_1 &= \frac{\ln(P_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ h_2 &= \frac{\ln(P_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ &= h_1 - \sigma\sqrt{T-t}. \end{aligned}$$

- Similarly, the price of a European put option is

$$p_t = K\exp[-r(T-t)]\Phi(h_2) - P_t\Phi(h_1). \qquad (9)$$

- How the price c_t depends on the estimated σ^2?
  differentiate c_t with respect to σ
- Can we solve c_t without using analytic technique?

Example 2

- Suppose the current price of Intel stock is \$80 per share with volatility $\sigma = 20\%$ per annum.
- The risk-free interest is 8% per annum.
- Find the price of a European call option on Intel with as striking price of \$90 that will expire in 3 months?
  - $P_t = 80$, $K = 90$, $T - t = 0.25$, $\sigma = 0.2$ and $r = 0.08$.
  - We have

$$\begin{aligned} h_1 &= \frac{\ln(80/90) + (.08 + .2^2/2)0.25}{0.2\sqrt{0.25}} \\ &= -0.9278 \\ h_2 &= h_1 - 0.2\sqrt{0.25} = -1.0278. \end{aligned}$$

  - $\Phi(-0.9278) = 0.1768$, $\Phi(-1.0278) = 0.1520$.
  - The price of a European call option is

$$\begin{aligned} c_t &= \$80 \times \Phi(-0.9278) \\ &\quad -\$90 \times \Phi(-1.0278)\exp(-0.02) = \$0.46. \end{aligned}$$

  - The stock price has to rise by \$1.05 for the purchaser of the put option to break even

- \$90 may be too high.
  Now we find the price of a European call option on Intel with as striking price of \$85 that will expire in 3 months?
  - $P_t = 80$, $K = 85$, $T - t = 0.25$, $\sigma = 0.2$ and $r = 0.08$.
  - We have

$$\begin{aligned} h_1 &= \frac{\ln(80/85) + (.08 + .2^2/2)0.25}{0.2\sqrt{0.25}} \\ &= -0.356246 \\ h_2 &= h_1 - 0.2\sqrt{0.25} = -0.456246. \end{aligned}$$

  - $\Phi(-0.356246) = 0.3608$, $\Phi(-0.456246) = 0.3241$.
  - The price of a European call option is

$$\begin{aligned} c_t &= \$80 \times \Phi(-0.356246) \\ &\quad -\$85 \times \Phi(-0.456246) \exp(-0.02) = \$1.86. \end{aligned}$$

  - The stock price has to rise by \$6.86 for the purchaser of the put option to break even

- Under the above assumptions, the price of a Europrean put option is

$$p_t = \$85 \exp(-0.02)\Phi(0.456246)$$

$$-\$80 \times \Phi(-0.356246) = \$5.18.$$

The stock has to fall 0.18 for the purchaser of the put option to break even.

Call values when conditional variances change (product process)

- The Black-Scholes formula assumes the price logarithm followed a continuous-time Wiener process:daily returns then have independent and identical distributions.
- Suppose the conditional standard deviations $\sigma_t, \sigma_{t+1}, \cdots, \sigma_T$ are generated by the Gaussian process

$$\ln(\sigma_{t+h}) - \alpha = \phi[\ln(\sigma_{t+h-1}) - \alpha] + \eta_t \quad (10)$$

for $1 \leq h \leq T - t$ with $0 < \phi < 1$.

- The unconditional distribution of $\ln(\sigma_{t+h})$ is $N(\alpha, \beta^2)$, the unconditional variance of σ_{t+h} is $\sigma_1^2 = \exp(2\alpha + 2\beta^2)$ and the η_t are independently distributed as $N(0, \beta^2[1 - \phi^2])$.
- Suppose the return X_t has distribution $N(\mu, \sigma_t^2)$ for some constant μ.
- At time t, we know P_t and σ_t. Then

$$\ln(P_T) = \ln(P_t) + \sum_{i=1}^{T-t} r_{t+i}$$

where $r_{t+i} = \ln P_{t+i} - \ln P_{t+i-1}$.
To determine c_t, we need to find the distribution of $\Sigma_{i=1}^{T-t} r_{t+i}$. Under the above assumption, $\ln(P_T)$ is normally distributed with conditional mean $\ln(P_t) + (T-t)\mu$ and conditional variance

$$\sum_{h=1}^{T-t} E(\sigma_{t+h}^2 | \sigma_t) \tag{11}$$

because the returns r_{t+h} are uncorrelated.

- (11) can be evaluated via (10) to give

  $$\ln(\sigma_{t+h}|\sigma_t) \sim N(\alpha + \phi^h[\ln(\sigma_t) - \alpha], \beta^2[1 - \phi^{2h}])$$

  and hence

  $$E(\sigma_{t+h}^2|\sigma_t) = \exp\{2\alpha + 2\phi^h[\ln(\sigma_t) - \alpha] + 2\beta^2[1 - \phi^{2h}]\}.$$

- Note that we can still use c_t derived under Black-Scholes model but we need to make corresponding change on different variance.

- Replace $\sigma\sqrt{T-t}$ by σ_1 and σ^2 by $\sigma_1^2/(T-t)$.