Financial Time Series

Topic 8: Additional Nonlinear Models

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Stochastic volatility Models

- $\{x_t\}_1^t$: the product process

$$x_t = \mu + \sigma_t U_t \tag{1}$$

where
$E(U_t) = 0$ and $Var(U_t) = 1$ for all t,
$Var(x_t|\sigma_t) = \sigma_t^2$, and
σ_t is a positive random variable.

- $E(x_t) = \mu$,

$$E(x_t - \mu)^2 = E(\sigma_t^2 U_t^2) = E(\sigma_t^2),$$

and autocovariance

$$E(x_t - \mu)(x_{t-k} - \mu) = E(\sigma_t \sigma_{t-k} U_t U_{t-k})$$
$$= E(\sigma_t \sigma_{t-k} U_t) E(U_{t-k}) = 0.$$

- Typically $U_t = (x_t - \mu)/\sigma_t$ is assumed to be normal and independent of σ_t.

- (1) is motivated by the discrete time approximation to the stochastic differential equation

$$\frac{dP}{P} = d(\log(P)) = \mu dt + \sigma dW$$

where $x_t = \triangle \log(P_t)$ and $W(t)$ is standard Brownian motion.

This is the usual diffusion process used to price financial assets in theoretical models of finance.

- In the world of time series analysis, write the above sdf by setting $dt = 1$.
  We then have

$$\log(P_{t+1}) - \log(P_t) = \mu + \sigma(W_{t+1} - W_t).$$

- Although x_t is a white noise, the squared and absolute deviation, $S_t = (x_t - \mu)^2$ and $M_t = |x_t - \mu|$, can be autocorrelated.

$$\begin{aligned} Cov(S_t, S_{t-k}) &= E(\sigma_t^2 \sigma_{t-k}^2) E(U_t^2 U_{t-k}^2) - (E(\sigma_t^2))^2 \\ &= E(\sigma_t^2 \sigma_{t-k}^2) - (E(\sigma_t^2))^2. \end{aligned}$$

- Fact: Almost all sample paths W of Brownian motion are of unbounded variation.
  They are not differentiable.

How do we model σ_t?

- The distribution of σ_t is skewed to the right.
  To ensure positiveness of the conditional variance, SV models use $\log(\sigma_t^2)$ instead of σ_t^2.
  Consider a log-normal distribution.

- Define
$$h_t = \log(\sigma_t^2) = \gamma_0 + \gamma_1 h_{t-1} + \eta_t \qquad (2)$$
where $\eta_t \sim NID(0, \sigma_\eta^2)$ and is independent of U_t.
h_t represents the random and uneven flow of new information into financial market.

- $x_t = \mu + U_t \exp(h_t/2)$, where U_t is always stationary.
x_t will be stationary if and only if h_t is.
Or, $|\gamma_1| < 1$.

- Moments of x_t or S_t:
For even r,
$$\begin{aligned} E(x_t - \mu)^r &= E(U_t^r) E\left(\exp(r h_t/2)\right) \\ &= \frac{r!}{2^{r/2} r/2!} \exp\left(\frac{r}{2}\mu_h + \frac{r}{2}\frac{\sigma_h^2}{2}\right) \end{aligned}$$
where $\mu_h = E(h_t) = \gamma_0/(1-\gamma_1)$ and $\sigma_h^2 = V(h_t) = \sigma_\eta^2/(1-\gamma_1^2)$.
All odd moments are zero.

- kurtosis:
$$\frac{E(x_t - \mu)^4}{[E(x_t - \mu)^2]^2} = 3\exp(\sigma_h^2) > 3$$

The process has fatter tails than a normal distribution.

- autocorrelation: Refer to page 129.
- Taking logarithms of (1) yields

$$\begin{aligned}\log(S_t) &= h_t + \log(U_t^2) \\ &= \mu_h + \frac{\eta_t}{1-\gamma_1 B} + \log(U_t^2) \\ \log(S_t) &\sim ARMA(1,1)\end{aligned}$$

with non-normal innovations.

- The main difficulty with using SV models is that they are rather difficult to estimate.

The SV model is also extended to allow for long memory in volatility, using the idea of fractional difference.

- A process is said to have long memory if its autocorrelation function decays at a hyperbolic, instead of an exponential, rate as the lag increases.
- In financial time series, it is often found that the autocorrelation function of the squared or absolute-value series of an asset return

often decays slowly, even though the return series itself has no serial correlations.

- A simple long-memory stochastic volatility (LMSV) model can be written as

$$\begin{aligned} a_t &= \sigma_t \epsilon_t, \\ \sigma_t &= \sigma \exp(u_t/2), \\ (1-B)^d u_t &= \eta_t \end{aligned}$$

where $\sigma > 0$, ϵ_t's are iid $N(0,1)$, η_t's are iid $N(0,\sigma_\eta^2)$ and independent of ϵ_t, and $0 < d < 0.5$

Random Coefficient Autoregressive (RCA) Model

- Historically, it is used to obtain a better description of the conditional mean equation of the process.
- The RCA model can be used to account for variability among different subjects under study.
- A time series x_t is said to follow a $RCA(p)$ model if it satisfies

$$x_t = \phi_0 + \sum_{i=1}^{p} (\phi_i + \delta_{it}) x_{t-1} + a_t$$

where $(\delta_{1t}, \ldots, \delta_{pt})^T$ is a sequence of independent random vectors with mean zero and covariance matrix Σ_δ, and $(\delta_{1t}, \ldots, \delta_{pt})^T$ is independent of $\{a_t\}$.

- The conditional mean and variance of the RCA model is

$$\begin{aligned} \mu_t &= \sum_{i=1}^{p} \phi_i a_{t-i}, \\ \sigma_t^2 &= \sigma_a^2 + (x_{t-1}, \ldots, x_{t-p}) \Sigma_\delta \sum_{i=1}^{p} (x_{t-1}, \ldots, x_{t-p})^T. \end{aligned}$$

Threshold Autoregressive (TAR) Model

- It uses piecewise linear models to obtain a better approximation of the conditional mean equation.
- It uses threshold space to improve linear approximation.
- A time series x_t is said to follow a k-regime self-exciting TAR(SETAR) model with threshold variable x_{t-d} if it satisfies

$$(1 - \phi_1^{(j)} B - \cdots - \phi_p^{(j)} B^p) x_t = c^{(j)} + a_t^{(j)},$$

if $\triangle_{j-1} \leq x_{t-d} < \triangle_j$, $j = 1, \ldots, k$ where

  – k and d are positive integers,

  – B is the back-shift operator,

  – $\triangle_i$: Thresholds
  They are real numbers such that $-\infty = \triangle_0 < \triangle_1 < \cdots < \triangle_{k-1} < \triangle_k = \infty$,

  – Regimes are defined by the past values of the time series itself.

  – the supscript (j) is used to denote parameters in the j-th regime,

- $\{a_t^{(j)}\}$ are iid sequences with mean 0 and variance σ_j^2 and are mutually independent for different j.
- It is a generalization of linear autoregressions.
- When d, p, and $\{\triangle_i\}$ were known, the model can be estimated by separating the data into groups by regime and finding the least squares estimates for the parameters in each regime.

• Consier a simple 2-regime $AR(1)$ model

$$x_t = \begin{cases} -1.5x_{t-1} + a_t & \text{if } x_{t-1} < 0 \\ 0.5x_{t-1} + a_t & \text{if } x_{t-1} \geq 0 \end{cases}$$

Here a_j's are iid $N(0,1)$.

- When x_{t-1} is positive, it tends to take multiple time indices for x_t to reduce to negative value.
- If x_{t-1} is negative, then x_t tends to switch to a positive value due to negative and explosive coefficient -1.5.
- This series exhibits asymmetric increasing and decreasing patterns.

Smooth transition AR (STAR) Model

- A criticism of the SETAR model is that its conditional mean equation is not continuous.
- The thresholds $\triangle_j$'s are the discontinuity points of the conditional mean function μ_t.
- Introduce smooth transitions between regimes.
- A time series x_t is said to follow a 2-regime $STAR(o)$ model if it satisfies

$$\begin{aligned} x_t &= c_0 + \sum_{i=1}^{p} \phi_{0,i} x_{t-i} \\ &\quad + F\left(\frac{x_{T-d} - \triangle}{s}\right)\left(c_0 + \sum_{i=1}^{p} \phi_{1,i} x_{t-i}\right) + a_t \end{aligned}$$

where

  - d is the delay parameter,
  - $\triangle$ and s are parameters, representing the location and scale parameters of model transition,
  - $F(\cdot)$ is a smooth transition function. It often assumes one of the three forms, logistic, exponential, or a cumulative distribution function.

– The conditional mean of a STAR model is a weighted linear combination between the following two equations:

$$\begin{aligned}\mu_{1t} &= c_0 + \sum_{i=1}^{p} \phi_{0,i} x_{t-i} \\ \mu_{2t} &= (c_0 + c_1) + \sum_{i=1}^{p} (\phi_{0,i} + \phi_{1,i}) x_{t-i}.\end{aligned}$$

The weights are determined in a continuous manner by $F((x_{T-d} - \triangle)/s)$

Markov Switching Model

- Markov switching autoregressive (MSA) model:
  - Hamilton (1989)
    - ∗ Want to model long swings in the growth rate of output.
    - ∗ Find evidence for discrete switches in the growth rate at business cycle frequencies.
    - ∗ Output growth was modeled as the sum of a discrete Markov chain and a Gaussian autoregression:

$$Y_t = Z_t + X_t,$$

where

$$\begin{aligned} Z_t &= \alpha_0 + \alpha_1 S_t, \quad S_t = 0 \text{or } 1 \\ X_t &= \phi_1 X_{t-1} + \cdots + \phi_4 X_{t-4} + \sigma V_t \end{aligned}$$

$V_t \sim N(0,1)$, and

$$\begin{aligned} P(S_t = 1 | S_{t-1} = 1) &= p, \\ P(S_t = 0 | S_{t-1} = 0) &= q \end{aligned}$$

  - The transition is driven by a hidden two-state Markov chain.

– Time series: x_t satisfies

$$t = \begin{cases} c_1 + \Sigma_{i=1}^{p} \phi_{1,i} x_{t-i} + a_{1t} & \text{if } s_t = 1, \\ c_2 + \Sigma_{i=1}^{p} \phi_{2,i} x_{t-i} + a_{2t} & \text{if } s_t = 1, \end{cases}$$

where s_t assumes values in $\{1, 2\}$ and is a first-order Markov chain with transition probabilities

$$\begin{aligned} P(s_t = 2 | s_{t-1} = 1) &= w_1, \\ P(s_t = 1 | s_{t-1} = 2) &= w_2 \end{aligned}$$

The innovational series $\{a_{1t}\}$ and $\{a_{2t}\}$ are sequences of iid random variables with mean zero and finite variance and are independent of each other.

– A small w_i means that the model tends to stay longer in state i.
$1/w_i$ is the expected duration of the process to stay at state i.

- A MSA model uses a hidden Markov chain to govern the transition from one conditional mean function to another.

Fundamental Concepts in Pricing Theory

- An American call option is a contract giving its owner the right to buy a fixed amount of a specified security at a fixed price at any time on or before a given date.
- Once bought an option can usually be sold to someone else before the given date, termed the expiration date.
- The fixed price is termed the exercise price.
- Options which can only be exercised on the expiration date are said to be European.
- An option to sell is called a put option.
- The value of an option depends on the exercise price, the expiration date, and the stochastic process governing the price of the underlying security.

Single-Period Option Pricing Models

Risk-Neutral Probability Assignments

- Single trading period:
  Time set (trading dates): $T = \{0, T\}$
- No complete information about the time evolution of the risky asset (S_t).
  Model the value S_t at some future date T as a random variable defined on some probability space $(\Omega, \mathcal{F}, P)$.
- Contingent claim: H which is a function of S_T
  potential liability inherent in a derivative security
- How do we find the fair price H_0 of the option?
- Estimate H by its average discounted value, i.e. $E(\beta H)$ using some discount factor β.
- How do we determine the probability measure P?
- Is there any natural choice?
  Does P (the assignment of probabilities to

every possible event) depend on investors' risk-preferences?

- Can we get a preference-free version of the option price?

Example 1

- $\beta = 1$: the riskless interest rate r is set at 0.
- Any portfolio fixed at time 0 is held until time T.
- Suppose $S_0 = 10$ at time 0.
  S_T takes one of only two possible values, 20 and 7.5, with probability p and $1 - p$, respectively.
- Consider a European call option $H = (S_T - K)^+$ with strike price $K = 15$.
- What is the investor's attitude to risk (specification of p)?
- Look for a risk-neutral probability assignment $(q, 1 - q)$ (i.e., $E_Q(S_T) = S_0$).
  $E_Q(\Delta S) = 0$ where $\Delta S = S_T - S_0$.
  one-step martingale
- The above requirement leads to

$$10 = 20q + 7.5(1 - q)$$

so that $q = 0.2$ and $\pi(H) = 5q = 1$.

- Consider the hedge portfolio approach to pricing.
  - a portfolio (θ_0, θ_1) of cash and stock
  - Determine what initial capital is needed for this portfolio to have the same time T value as H in all contingencies.
  - Value of portfolio under 0 discount rate:

    $$V_t = \theta_0 + \theta_1 S_t, \quad t = 0, T.$$

    θ_1: number of shares of stock held during the time
  - Gain from trade: $G = \theta_1 \Delta S$
  - $V_T = V_0 + G$.
    Note that

    $$\begin{aligned} V_0 &= E_Q(V_0) = E_Q(V_T - \theta_1 \Delta S) \\ &= E_Q(V_t). \end{aligned}$$

  - Find the hedge (θ_0, θ_1) which replicates the option.

    $$\begin{aligned} 5 &= \theta_0 \times 1 + \theta_1 \times (20), \\ 0 &= \theta_0 \times 1 + \theta_1 \times (7.5). \end{aligned}$$

    $\theta_0 = -3$, $\theta_1 = 0.4$

– At time 0, sell the option to obtain capital of 1 dollar, and borrow 3 dollars, in order to invest the sum of 4 dollars in shares.

Binomial Model

- S_1 takes on $(1+b)S_0$ and $(1+a)S_0$ with probabilities p and $1-p$. Here $a < r < b$.
- Find θ and V_0 so that

$$P(\beta H = V_0 + \theta \Delta \bar{S}) = 1,$$

where β is the discount factor and $\bar{S} = \beta S_1 - S_0$.

- Solve the following equations

$$\begin{aligned} \beta h^a &= V_0 + \theta(\beta(1+b)S_0 - S_0), \\ \beta h^b &= V_0 + \theta(\beta(1+a)S_0 - S_0). \end{aligned}$$

Here h^a and h^b are the values of H when $S_1 = (1+a)S_0$ and $S_1 = (1+b)S_0$, respectively.

- We get

$$\theta = \frac{h^b - h^a}{(b-a)S_0} = \frac{\delta V}{\delta S}.$$

the rate of change in V relative to the change in the stock price

θ: the delta of the contingent claim

- For Eurpoean call option with strike K,
  - K is between $(1+a)S_0$ and $(1+b)S_0$
  - riskless interest rate $r > a$ (Then $\beta = (1+r)^{-1}$)
  - $H = (S_1 - K)^+$:
    $h^a = 0$ and $h^b = (1+b)S_0 - K$
    We have

$$\theta = \frac{S_0(1+b) - K}{S_0(b-a)}$$

$$V_0 = \frac{1}{1+r}\frac{r-a}{b-a}[S_0(1+b) - K]$$

  - The variability of the stock price S is

$$\sigma^2 = (b-a)^2 p(1-p).$$

    σ is called the volatility of the stock.
  - Risk-neutral probability assignment:

$$q = \frac{\beta^{-1} - S_0(1+a)}{S_0(1+b) - S_0(1+a)}$$

$$1 - q = \frac{S_0(1+b) - \beta^{-1}S_0}{S_0(1+b) - S_0(1+a)}.$$

A General Single-Period Model

- Stock price takes the (known) value S_0 at time 0 and the random value S_1 at time 1.
- Use discount factor β to express all values in terms of time-0 prices.
- $\bar{X}$ and $\bar{S}_1$ stand for βS and βS_1.
- S and contingent claim H are both taken to be random variables.
- Goal: Hedge against the obligation to honor the claim.
  Pay out $H(w)$ at time 1. (Assume P is known in advance.)
- Portfolio at time 0: θ shares of stock and η_0 units of cash.
- Initial value of this portfolio: $V_0 = \eta_0 + \theta S_0$
- Place the cash in the savings account.
  It accrues interest at some positive rate r and reaches $\beta^{-1}\eta_0$ by time 1.
- Wish this portfolio to have value $V_1 = H$ at time 1 or $\bar{V}_1 = \bar{H}$. How do we find the fair price H_0 of the option?

- Suppose we have access to external funds to pay out H at time 1. (i.e., Adjust η_0 to $\eta_1 = H - \theta S_1$ without any restriction.
Then $V_1 = \theta S_1 + \eta_1 = H$.

- Remains to choose θ and V_0 to determine hedging strategy (θ, η).

- Cost: the initial investment C_0 and $\Delta C = C_1 - C_0 = \eta_1 - \eta_0$.

- Note that

$$\begin{aligned} \Delta \bar{C} &= \beta C_1 - C_0 = \beta(V_1 - \theta S_1) - (V_0 - \theta S_0) \\ &= \bar{H} - (V_0 + \theta \Delta \bar{S}). \end{aligned}$$

where $\Delta \bar{S} = \beta S_1 - S_0$.

- The above equality means that the discounted cost increment $\Delta \bar{C}$ is the difference between the discounted claim $\bar{H}$ and the discounted price increment $\Delta \bar{S}$.

- Determine θ and V_0 to minimize the risk function

$$R = E((\Delta \bar{C})^2) = E((\beta H - (V_0 + \theta \Delta \bar{S}))^2).$$

The regression estimates are

$$\theta = \frac{Cov(\bar{H}, \Delta\bar{S})}{V(\Delta\bar{S})}$$
$$V_0 = E(\bar{H}) - \theta E(\Delta\bar{S}).$$

- Note that $E(\Delta\bar{C}) = 0$.
  The average discounted cost remains constant at V_0.
- The minimal risk is

$$\begin{aligned} R_{\min} &= V(\bar{H}) - \theta^2 V(\Delta\bar{S}_0) \\ &= Var(\bar{H})(1-\rho^2) \end{aligned}$$

  where $\rho = \rho(\bar{H}, \bar{S}_1)$ is the correlation coefficient.
  The intrinsic risk of the claim H cannot be completely eliminated unless $|\rho| = 1$.
- In general models, we cannot expect all contingent claims to be attainable by some hedging strategy that eliminates all the risk-where this is possible, we call the model complete.

Multi-Period Binomial Models

- Trading dates: $0, 1, 2, \ldots, T$ for some fixed integer T

- Price of the stock: $S_0, S_1, S_2, \cdots, S_T$
  $S_t = (1+b)S_{t-1}$ with probability p
  $S_t = (1+a)S_{t-1}$ with probability $1-p$

- $\beta = (1+r)^{-1}$,
  $r > 0$: the riskless interest rate, $a < r < b$

- Consider the current value of H at time $T-1$.
  (One period before expiration)
  Now consider this as the initial value of a claim in the single-period model.

- Assume $H = (S_T - K)^+$.
  Writing h^b for the value of H if $S_T = (1+b)S_{T-1}$ and h^a similarly.
  Current valueof H is given by $E_Q(H/(1+r))$ where the measure Q is given by $(q, 1-q)$ as

$$
\begin{aligned}
q &= \frac{\beta^{-1} - S_0(1+a)}{S_0(1+b) - S_0(!+a)} \\
1-q &= \frac{S_0(1+b) - \beta^{-1}S_0}{S_0(1+b) - S_0(!+a)}.
\end{aligned}
$$

$$
V_{T-1} = (1+r)^{-1}(qh^b + (1-q)h^a)
$$

with

$$q = \frac{(1+r)S_{T-1} - (1+a)S_{T-1}}{(1+b)S_{T-1} - (1+a)S_{T-1}} = \frac{r-a}{b-a}.$$

- Two-Period Trading:
  Apply this analysis to the value V_{T-2} of the call H at time $T-2$.

- S_{T-2} can take one of the three values $(1+b)^2 S$, $(1+a)(1+b)S$, and $(1+a)^2 S$ at time T.
  The call H must have one of three values at that time.
  Write them as h^{aa}, h^{ab}, and H^{bb}.

General Discrete-Time Market Model

- Information structure
  - Time set: $T = \{0, 1, 2, \ldots, T\}$
    $0, 1, 2, \ldots, T$: admissible trading date
    T: trading horizon
  - $(\Omega, \mathcal{F}, P)$: all possible states of the market
  - In example 1, Ω is a finite probability space (i.e., has a finite number of points w each with $P(\{w\}) > 0$).
    Then, $\mathcal{F} = 2^{\Omega}$ and every subset of Ω is $\mathcal{F}$-measurable.
  - The information available to the investors is given by an increasing (finite) sequence of sub-σ-fields of $\mathcal{F}$.
  - Filtration: $(\mathcal{F}_t)_{t \in T}$
    an increasing family of σ-field

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$$

    Here $\mathcal{F}_0$ contains only sets of P-measure 0 or 1.

- $\mathcal{F}_t$: It contains the information available to our investors at time t.
  Assume the investors learn without forgetting and inside trading is not possible.

- Market model
  - S_t^0: a riskless (i.e., nonrandom) bond (or bank account)
  - S_t^i, $1 \leq i \leq d$: d risk (i.e., random) stocks
  - d: the dimension of the market model

- Value processes
  - The discount factor: $\beta_t = 1/S_t^0$
    It is the sum of money we need to invest in bonds at time 0 in order to have 1 unit at time t.
  - The securities $S^0, S^1, \cdots, S^d$ are traded at times $t \in T$.
  - An investor's portfolio: $\theta_t = (\theta_t^i)_{0 \leq i \leq d}$

- Market assumptions

- Self-Financing strategies

ARCH and Asset Pricing

- Maximize expected utility over uncertain future events.
- Engle and Bollerslev (1986, Econometric Reviews)
- A representative agent allocates his wealth, W_t, between shares of a risky asset q_t at a price p_t and those of a risk-free asset x_t, whose price is set equal to 1.
- The shares of the risky asset will be worth y_{t+1} each at the end of the period (if there are no dividends, then $y_{t+1} = p_{t+1}$).
- The risk-free asset will be worthy $r_t x_t$, where r_t denotes one plus the risk-free rate of interest.
- Mean-variance utility function:

$$\max_{W_t = x_t + p_t q_t} \left\{ 2E_t(q_t y_{t+1} + r_t x_t) - \gamma_t V_t(q_t y_{t+1}) \right\}.$$

- Solution:

$$p_t = r_t^{-1} E_t(y_{t+1}) - \gamma_t q_t r_t^{-1} V_t(y_{t+1}). \quad (3)$$

- Suppose $q_t = q$, $\gamma_t = \gamma$, and $r_t = r$. (3) describes the asset pricing model.

Estimation in stochastic volatility models

Suppose that $\{x_t\}_1^t$ is generated by the product process (1).

- $x_t = \mu + \sigma_t U_t$ with $EU_t = 0$ and $V(U_t) = 1$.
- Assume σ_t and U_t are independent and $E|U_t| = \delta$. Then

$$\begin{aligned} ES_t &= E(x_t-\mu)^2 = E(\sigma_t^2)E(U_t^2) \\ &= E(\sigma_t^2) = E^2(\sigma_t) + V(\sigma_t), \\ EM_t &= E|x_t - \mu| = E(\sigma_t)\delta. \end{aligned}$$

- Estimate based on the principle of plug-in: $\bar{x} = n^{-1}\Sigma_{t=1}^n x_t$, $\bar{M} = n^{-1}\Sigma_{t=1}^n |x_t - \bar{x}|$, and $\bar{S} = n^{-1}\Sigma_{t=1}^n (x_t - \bar{x})^2$
- $\hat{E}(\sigma_t) = \bar{M}/\delta$ and $\hat{V}(\sigma_t) = \bar{S} - (\bar{M}/\delta)^2$
- Those estimates depend on δ. A wrong specification of the distribution of U_t may lead to wrong estimate.
  If $U_t \sim N(0,1)$, δ is around 0.798.
- Many series have low values for the ratio $\hat{E}(\sigma_t)/\hat{V}(\sigma_t)$. It implies that U_t does not have normal distribution.