Financial Time Series II

Topic 4: Non-stationary Processes and ARIMA models

Hung Chen
Department of Mathematics
National Taiwan University
4/1/2002

OUTLINE

1. Non-stationary Processes
   - Non-stationarity in variance
   - Box-Cox transformation
   - Non-stationarity in mean
   - Eliminate the trend term by differencing
2. ARIMA Models
   - Description
   - Properties
   - Data Modeling
   - Forecasting
3. Regression with Time Series Errors

Nonstationary Time Series Models

- Weakly stationary implies that the mean, variance and autocovariances of the process are invariant under time translation
- Figure 2.14 plots monthly observations from January 1965 December 1990 of the FTA (Financial times-Actuaries) All Share index.
  - It shows that the series to exhibit a prominent upward, but not linear, trend, with pronounced and persistent fluctuations about it, which increase in variability as the level of the series increases.
  - non-stationarity in variance:
    Write a time series as the sum of a **non-stochastic** mean level and a random error component:

$$X_t = \mu_t + \epsilon_t, \tag{1}$$

and we suppose that the variance of the errors is functionally related to the mean level μ_t by

$$V(X_t) = V(\epsilon_t) = h^2(\mu_t)\sigma^2,$$

where h is some known function.

– Box and Cox (1964) class of power transformations

$$g(x_t) = \frac{x_t^{\lambda} - 1}{\lambda}$$

and

$$\lim_{\lambda \to 0} \frac{x_t^{\lambda} - 1}{\lambda}$$
$$= \lim_{\lambda \to 0} \frac{\exp(\lambda \ln x_t) - 1}{\lambda} = \ln x_t.$$

Idea: Consider

$$g(x_t) \approx g(\mu_t) + (x_t - \mu_t) g^{'}(\mu_t)$$

and

$$V(g(x_t)) \approx [g^{'}(\mu_t)]^2 h^2(\mu_t) \sigma^2.$$

Choose $g^{'}(\mu_t) = 1/h(\mu_t)$ to stabilize the variance.

– Apply a logarithmic transformation to FTA all Share index.

– Figure 2.14 indicates that the transformation linearize the trend and stabilize the variance.

– When $h(\mu_t) = \mu_t$, $g(\mu_t) = \ln \mu_t$.
The natural logarithms of x_t can be used to stabilize the variance.

– The use of logarithms is a popular (why?) transformation for financial time series, a constant variance is rarely completely induced by this transformation alone. More to be seen in Chapters 4 and 7.

- non-stationarity in mean:
  How do we model the non-constant mean level in (1)?
  - Figure 2.14 indicates that the transformation linearize the trend and stabilize the variance.
  - Assume that the mean evolves as a polynomial of order d in time.
  - $\{x_t\}$ is decomposed into a trend component, given by the polynomial, and a stochastic, stationary, but possibly autocorrelated, zero mean error component.

$$x_t = \sum_{j=0}^{d} \beta_j t^j + \phi(B)a_t. \qquad (2)$$

Note that $E(\epsilon_t) = \phi(B)E(a_t) = 0$ and hence

$$E(X_t) = E(\mu_t) = \sum_{j=0}^{d} \beta_j t^j.$$

– Consider the linear trend ($d = 1$)

$$x_t = \beta_0 + \beta_1 t + a_t. \qquad (3)$$

Lagging (3) one period and subtracting this from (3) yields

$$x_t - x_{t-1} = \beta_1 + (a_t - a_{t-1}). \qquad (4)$$

Let $w_t = x_t - x_{t-1} = (1 - B)x_t = \triangle x_t$. Then

$$w_t = \triangle x_t = \beta_1 + \triangle a_t,$$

which is stationary ($E(W_t) = \beta_1$) but not invertible $MA(1)$ process.

– Differencing:

$$\begin{aligned} \triangle &= 1 - B \quad \text{the first difference operator} \\ \triangle^d &= (1 - B)^d \quad \text{the } d\text{th difference operator} \end{aligned}$$

– $\triangle^d \Sigma_{j=0}^{d} \beta_j t^j = d!\beta_d$.
Any polynomial trend of degree d can be reduced to a constant by application of the operator.

– Suggestion: Given any sequence $\{x_t\}$ of data, apply the operator $\triangle$ repeatedly until we find a sequence $\{\triangle^d x_t\}$ which can plausibly be modelled as a realization of a stationary process.

– A series $\{x_t\}$ is nonstationary but its dth differenced series $\{(1-B)^d x_t\}$ for some integer $d \geq 1$, is stationary.

– Typically, $d = 1$ or 2.
Note that $\triangle^2 x_t = x_t - 2x_{t-1} + x_{t-2}$.

– Example

$$\begin{aligned} x_t &: \{26.8, 34.7, 25.4, \ldots, 38.1, 39.5\} \\ (1-B)x_t &: \{7.9, -9.3, \ldots, 1.4\} \end{aligned}$$

ARIMA Models

- A series may need first differencing d times to attain stationarity and the obtained series may itself be autocorrelated.
- Suppose this autocorrelation can be modeled by an $ARMA(p, q)$ process.
- The model for the original series is of the form

$$\phi(B) \triangle^d x_t = \theta_0 + \theta(B) a_t, \qquad (5)$$

  where $\theta_0 = d!\beta_d$. It is said to be an autoregressive-integrated-moving average process of orders p, d and q, or $ARIMA(p, d, q)$.
- X_t is said to be integrated of order d, denoted $I(d)$.
- In finance, price series are commonly believed to be nonstationary, but the log return series, $r_t = \ln(p_t) - \ln(p_{t-1})$ is stationary.
  - In this case, the log price series is unit-root nonstationary and, hence, can be treated as an ARIMA process.

  - The AR polynomial has a characteristic root at 1.

- An ARIMA model has long memory because ψ_i coefficients in its MA representation do not decay over time, implying that the past shock a_{t-i} of the model has a permanent effect on the system.

Forecasting using ARIMA models:

Given a realization $\{x_t\}_{1-d}^{T}$ from a general $ARIMA(p, d, q)$ process

$$\phi(B) \triangle^d x_t = \theta_0 + \theta(B) a_t.$$

How do we forecast a future value X_{T+h}?

- Let

$$\begin{aligned}\alpha(B) &= \phi(B) \triangle^d \\ &= \left(1 - \alpha_1 B - \alpha_2 B^2 - \cdots - \alpha_{p+d} B^{p+d}\right).\end{aligned}$$

- Denote a minimum mean square error (MMSE) forecast based on the data up to time T by $f_{T,h}$. Then

$$\begin{aligned}f_{T,h} &= E(\alpha_1 X_{T+h-1} + \alpha_2 X_{T+h-2} + \cdots \\ &\quad + \alpha_{p+d} X_{T+h-p-d} + \theta_0 + a_{T+h} - \theta_1 a_{T+h-1} \\ &\quad - \cdots - \theta_q a_{T+h-q} | x_T, x_{T-1}, \cdots).\end{aligned}$$

- Note that

$$E(X_{T+j} | x_T, x_{T-1}, \cdots) = \begin{cases} x_{T+j}, & j \leq 0 \\ f_{T,j}, & j > 0 \end{cases}$$

and

$$E(a_{T+j} | x_T, x_{T-1}, \cdots) = \begin{cases} a_{T+j}, & j \leq 0 \\ 0, & j > 0 \end{cases}$$

- Algorithm:
  - Replace past expectations ($j \leq 0$) by known values, x_{T+j} and a_{T+j}.
  - Replace future expectations ($J > 0$) by forecast values, $f_{T,j}$ and 0.

Examples:

1st Example:

- $AR(2)$ model

$$(1 - \phi_1 B - \phi_2 B^2)x_t = \theta_0 + a_t$$

Hence, $\alpha(B) = 1 - \phi_1 B - \phi_2 B^2$.

- Note that

$$x_{T+h} = \phi_1 x_{T+h-1} + \phi_2 x_{T+h-2} + \theta_0 + a_{T+h}$$

and

$$f_{T,h} = (\phi_1 + \phi_2) f_{T,h-1} - \phi_2 (f_{T,h-1} - f_{T,h-2}) + \theta_0.$$

- By repeated substitution, we have

$$\begin{aligned} f_{T,h} &= \theta_0 \sum_{j=0}^{h-1} (\phi_1 + \phi_2)^j + (\phi_1 + \phi_2)^h x_T \\ &\quad - \phi_2 \sum_{j=0}^{h-1} (\phi_1 + \phi_2)^j (f_{T,h-1-j} - f_{T,h-2-j}) \end{aligned}$$

where $f_{T,0} = x_T$ and $f_{T-1} = x_{T-1}$.

- As $h \to \infty$,

$$f_{T,h} = \frac{\theta_0}{1 - \phi_1 - \phi_2} = E(X_t) = \mu$$

since $\phi_1 + \phi_2 < 1$ and $|\phi_2| < 1$.

- The best forecast of a future observation with large lead time is eventually the mean of the process.

2nd Example:

- $ARIMA(0,1,1)$ model

$$\triangle x_t = (1 - \theta B) a_t.$$

Hence, $\alpha(B) = 1 - B$.

- Note that

$$\begin{aligned} x_{T+h} &= x_{T+h-1} + a_{T+h} - \theta a_{T+h-1}, \\ f_{T,1} &= x_T - \theta a_T \end{aligned}$$

and, for $h > 1$,

$$f_{T,h} = f_{T,h-1}.$$

- Note that

$$a_T = (1 - B)(1 - \theta B)^{-1} x_T$$

and

$$\begin{aligned} f_{T,h} &= (1 - \theta)(1 - \theta B)^{-1} x_T \\ &= (1 - \theta)(x_T + \theta x_{T-1} + \theta^2 x_{T-2} + \cdots). \end{aligned}$$

- The forecast for all future values of x is an exponentially weighted moving average of current and past values.

Regression with Time Series Errors

In many situations, the relationship between two time series is of major interest.

- Example 1: Consider the market model in finance that relates the return of an individual stock to the return of a market index.
  - Refer to Examples 7.1 and 7.2.
  - Consider weekly observations on the London Stock Exchange FTSE 100 index and the (logarithmic) prices of the company Legal & General from January 1984 to December 1993.
- Example 2: Consider the term structure of interest rates in which the evolution over time of the relationship between interest rates with different maturities is investigated.
  - Consider two U.S. weekly interest rate series.
  - r_{1t}: the 1-year treasury constant maturity rate
  - r_{2t}: the 3-year treasury constant maturity rate

  - Both series have 1967 observations from January 05, 1962 to September 10, 1999.
  - The data can be obtained from gsbwww.uchicago.edu/fac

- The relationship can be analyzed by the model

$$r_{1t} = \alpha + \beta r_{2t} + e_t, \tag{6}$$

where r_{1t} and r_{2t} are two time series and e_t is the error term.

- Quite often, the error term e_t is not a white noise series.

We now use a data example to illustrate a regression analysis with time series errors.

- Figure 6: It shows the time plots of the two interest rates.
  Solid line: 1-year rate; Dashed line: 3-year rate

- Figure 7(a): Plot r_{1t} versus r_{3t}
  It shows that the two interest rates are highly correlated.
  The fitted model is

$$r_{3t} = 0.911(\pm.032) + 0.924(\pm.004) r_{1t} + e_t, \tag{7}$$

with $R^2 = 95.8\%$ and $\hat{\sigma}_e = 0.538$.

- Figure 8 gives the time plot and ACF of the residuals of equation (7). The sample ACF of the residuals shows the pattern of a unit-root nonstationary time series.

- The unit-root behavior of the interest rates leads to the consideration of change series of interest rates. Let $c_{1t} = (1 - \triangle)r_{1t}$ and $c_{3t} = (1 - \triangle)r_{3t}$. Figure 9 gives time plots of change series and Figure 7(b) gives the scatter plot.

- Consider the linear regression $c_{3t} = \alpha + \beta c_{1t} + e_t$. The fitted model is

$$c_{3t} = 0.0002(\pm.0015)+0.7811(\pm.0075)c_{1t}+e_t, \quad (8)$$

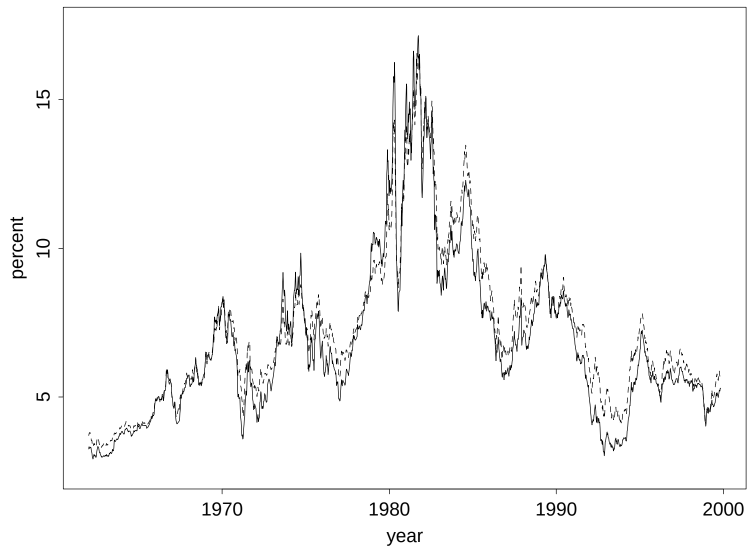
with $R^2 = 84.8\%$ and $\hat{\sigma}_e = 0.0682$.

- Figure 10 shows the time plot and sample ACF of the residuals of (8). The ACF indicates existence of serial correlations in the residuals of (8), but at a much weaker level.

- Modify the model (8) by assuming

$$e_t = a_t - \theta_1 a_{t-1}$$

where $\{a_t\}$ is assumed to be a white noise series.
Use an $MA(1)$ model to capture the serial dependence of the error term.



percent
5
10
15
1970
1980
1990
2000
year

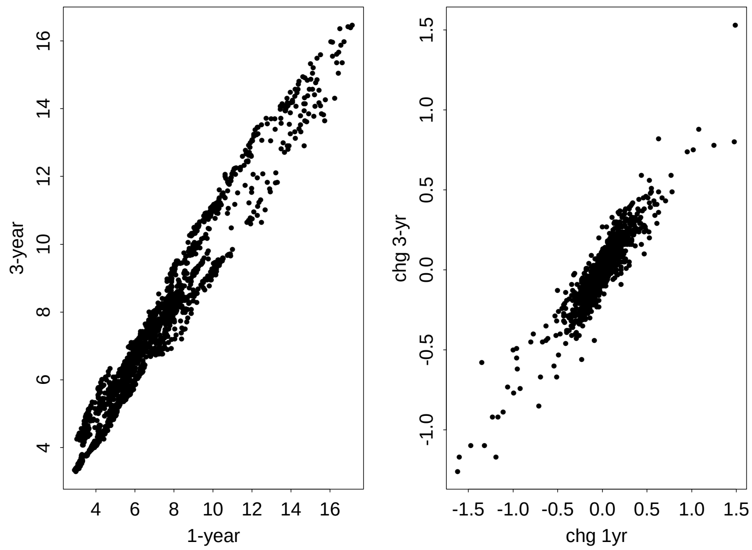


Figure 7: Scatter plots of U.S. weekly interest rates from January 5, 1962 to September 10, 1999. (a) 3-year rate versus 1-year rate. (b) Change in 3-year rate versus change in 1-year rate

.

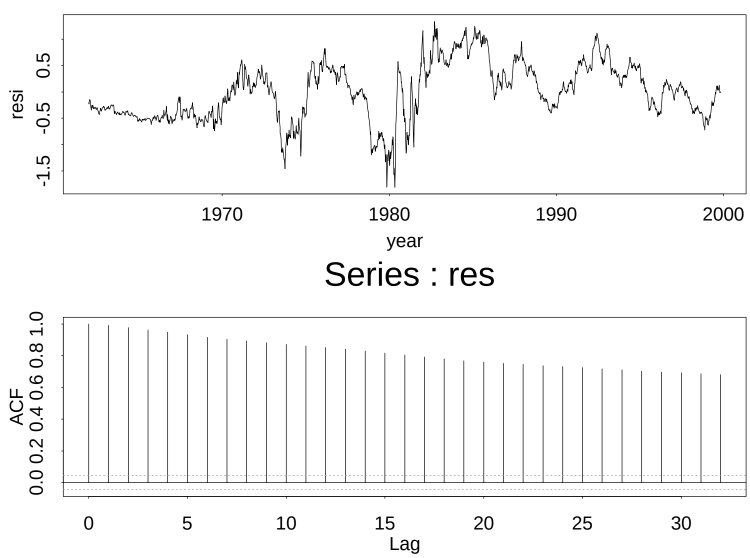
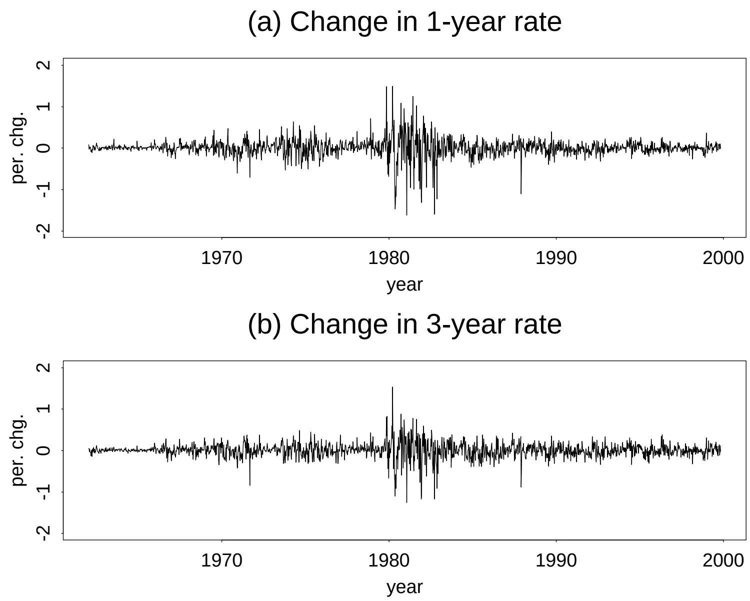
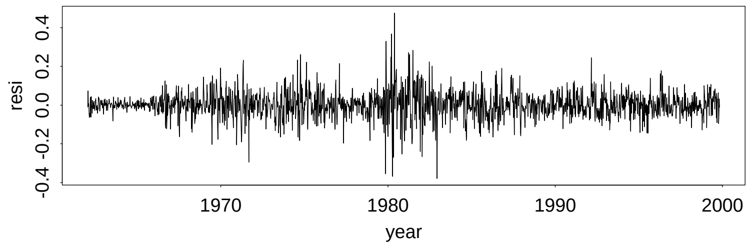
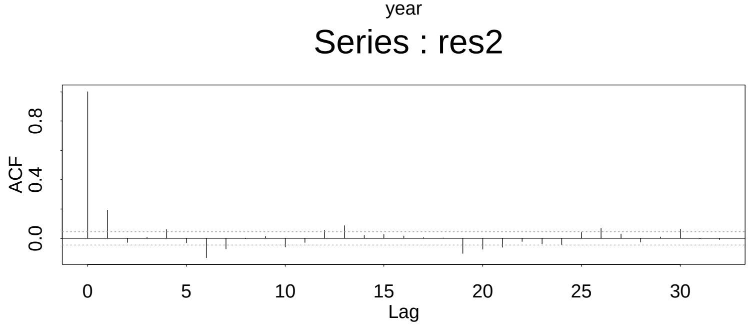


Figure 8: Residual series of linear regression (37) for U.S. weekly interest rates. (a) Time plot, (b) Sample ACF.


(a) Change in 1-year rate
per. chg.
2
1
0
-1
-2
1970
1980
1990
2000
year
(b) Change in 3-year rate
per. chg.
2
1
0
-1
-2
1970
1980
1990
2000
year


resi
-0.4 -0.2 0.0 0.2 0.4
1970 1980 1990 2000
year


Series : res2
ACF
0.0 0.4 0.8
0 5 10 15 20 25 30
Lag