K-equivalence in Birational Geometry

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K-equivalence Relation

Definition 1 - Two $\mathbb{Q}$-Gorenstein varieties X and X' are K-equivalent, $(X =_K X')$, if $\exists$ smooth Y and birational morphisms

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \phi'} & \\ X & & & & X' \end{array}$$

such that $\phi^* K_X =_{\mathbb{Q}} \phi'^* K_{X'}$.

Theorem 2 - *If X and X' are birational terminal varieties with K_X and $K_{X'}$ nef along the exceptional loci then $X =_K X'$.*

A Geometric Hueristic

For manifolds, this is equivalent to

$$-\partial\bar{\partial}\log(\phi^*\omega)^n = -\partial\bar{\partial}\log(\phi'^*\omega')^n + \partial\bar{\partial}f.$$

where ω (resp. ω') is a Kähler forms on X (resp. X'). That is,

$$(\phi'^*\omega')^n = e^f(\phi^*\omega)^n.$$

Can one rotate $\phi^*\omega$ to $\phi'^*\omega'$ while fixing the rate of volume degeneracy? Say, via Monge-Amperè equations for $t \in [0,1]$

$$\omega_t^n := (\omega_0 + \partial\bar{\partial}\varphi_t)^n = e^{tf+c(t)}(\phi^*\omega)^n?$$

The Formalism of L^2-cohomology

Let $K_Y = \phi^* K_X + E = \phi'^* K'_X + E$, $Z = \phi(E)$ and $Z' = \phi'(E)$. Then

$$X \backslash Z \cong Y \backslash E \cong X' \backslash Z'.$$

Since $H^i(X, \mathbb{C}) \cong L^i_2(X \backslash Z, \omega) \cong L^i_2(Y \backslash E, \phi^* \omega)$, under the rotation, we should get

$$H^i(X, \mathbb{C}) \cong H^i(X', \mathbb{C}).$$

Problem 3 - However, we are unable to prove enough regularity of ϕ_t. It is known to be C^0 on Y and $C^{1,1}$ transversal to E.

The p-adic Measure

Assume X **and** X' **smooth projective. Take an integral model of the** K**-equivalence diagram** $\mathfrak{X} \to \operatorname{Spec} S$ **with** $F := K(S)$ **a number field.** $\mathfrak{X}_P$ **has good reduction** $\forall' P \in \operatorname{Spec} S$**. Let** $R = \widehat{S}_P$ **with** $R/P \cong \mathbb{F}_q$**,** $q = p^r$**. Let** U_i**'s be a Zariski open cover of** X_R **such that** $K_{X_R}|_{U_i}$ **is free. Then for a compact open subset** $A \subset U_i(R) \subset X_R(R)$**,**

$$m_X(A) := \int_A |\Omega_i|_p$$

(independent of generator $\Omega_i \in K_{X_R}(U_i)$**).**

Equivalence of Galois Representations

$m(X_R(R)) = m(X'_R(R))$ **by the change of variable formula and** $X =_K X'$. **Since**

$$m(X(R)) = \frac{|X_P(\mathbb{F}_q)|}{q^n},$$

and by Grothendieck-Lefschetz ($\ell \neq p$)**:**

$$|X_P(\mathbb{F}_q)| = \sum_i (-1)^i \mathrm{Tr}(Fr_q : H^i_{et}(X_P, \mathbb{Q}_\ell)),$$

we conclude by Deligne's theorem and the Cĕbotarev density, as $\mathrm{Gal}(\bar{F}/F)$ **modules**

$$H^i_{et}(X_{\bar{F}}, \mathbb{Q}_\ell)^{ss} \cong H^i_{et}(X'_{\bar{F}}, \mathbb{Q}_\ell)^{ss}.$$

Equivalence of Hodge Numbers

Let P **has** char $k_P = p$, $K := \widehat{F}_P$, $G = \mathrm{Gal}(\bar{K}/K)$ **and** $\mathbb{C}_p = \widehat{\bar{K}}$. **By base change theorem**

$$H^j_{et}(X_{\bar{K}}, \mathbb{Q}_p)^{ss} \cong H^j_{et}(X'_{\bar{K}}, \mathbb{Q}_p)^{ss}.$$

By Faltings' *Hodge-Tate decomposition*:

$$\bigoplus_i \mathbb{C}_p \bigotimes_K H^{m-i}(X_K, \Omega^i)(-i) \cong \mathbb{C}_p \bigotimes_{\mathbb{Q}_p} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p),$$

Here $(i) := \otimes(\varprojlim \mu_{p^n})^{\otimes i}$. **Since** $\mathbb{C}_p^G = K$ **and** $\mathbb{C}_p(i)^G = 0$ **for** $i \neq 0$, **we get***

$$h^{i,m-i} = \dim_K(\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)^{ss}(i))^G.$$

So $h^{p,q}(X) = h^{p,q}(X')$.

A Key Example: The Filling-in Problem

Theorem 4 (Wang) *Let $\mathcal{X} \to \Delta$ be a smoothing of a minimal Gorenstein 3-fold $\mathcal{X}_0$. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$, up to any finite base change.*

Such $\mathcal{X}' \to \Delta$ must be terminal Gorenstein. So $\mathcal{X} =_K \mathcal{X}'$, $\mathcal{X}_0 \sim \mathcal{X}'_0$ and $\mathcal{X}_0$ is not $\mathbb{Q}$-factorial. Consider a projective small morphism $X \to \mathcal{X}_0$ with X $\mathbb{Q}$-factorial minimal. Then $X \sim \mathcal{X}_0 \sim \mathcal{X}'_0$ and so X is smooth and

$$H^*(X) \cong H^*(\mathcal{X}'_0) \cong H^*(\mathcal{X}'_t) \cong H^*(\mathcal{X}_t).$$

Consider the small transition diagram:

$$\begin{array}{ccccc} X & & & & \\ {\scriptstyle\phi}\downarrow & & & & \\ \mathcal{X}_0 & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_t \end{array}$$

If $\mathcal{X}_0$ **has only ODP, done by** $H_2(\mathcal{X}_t) =$ coker e, $e : \bigoplus_i \mathbb{Z}[C_i] \to H_2(X, \mathbb{Z})$ **since** X **is projective. For cDV, use symplectic deformations to reduce to the ODP case.**

Remark 5 - *Clemens had shown that if* $\mathcal{X}_0$ *is a quintic Calabi-Yau with only an* A_2 *singular point then the punctured family* $\mathcal{X}^\times \to \Delta^\times$ *is* C^∞ *trivial. We had shown that this family admits no smooth projective filling-in.*

Symplectic Deformation of 3D Flops

Index one terminal = isolated cDV = one parameter deformation of RDP. By Friedman, if $p \in V$ is isolated cDV and $C \subset U$ is the exceptional curve, then $\mathrm{Def}(C, U) \hookrightarrow \mathrm{Def}(p, V)$ and both spaces are smooth.

Moreover, One can deform the complex structure of a nbd of C so that C decomposes into $\mathbb{P}^1$'s and the contraction map deforms to nontrivial contractions of these $\mathbb{P}^1$'s to ODP's, while keeping a nbd of these ODP's to remain in $\mathrm{Def}(p, V)$.

We can preform this analytic process for all C's and p's simultaneously in each corresponding small nbd and then patch them together smoothly, or as a deformation of almost complex structures or even symplectically (Wilson).

For smooth flops, we may do this process for $X \to \bar{X}$ and $X' \to \bar{X}$ simultaneously to end up with a birational map which consists of several copies of ordinary $\mathbb{P}^1$-flops.

Main Conjectures

Fix a birational map $f : X \dashrightarrow X'$ **such that** $X =_K X'$**. Let** $T = \phi'_* \circ \phi^*$ **be the correspondence determined by** $\bar{\Gamma}_f \subset X \times X'$**.**

I $T : H^i(X, \mathbb{Q}) \xrightarrow{\sim} H^i(X', \mathbb{Q})$.
II X *and* X' *have isomorphic quantum cohomology rings over the birational Kähler cone.*
III X *and* X' *have canonically isomorphic complex moduli spaces.*
IV X *and* X' *admit symplectic deformations such that* f *deforms into ordinary flops.*

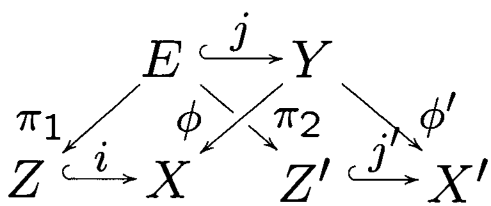
Evidences

3D: Kawamata, Mori and Kollár classified threefold flops. Isomorphism on quantum ring is due to Li and Ruan.

Hyperkähler: Huybrechts showed that birational hyperkähler manifolds X and X' admits deformations $\mathcal{X} \to \Delta$ and $\mathcal{X}' \to \Delta$ such that $\mathcal{X}_t \cong \mathcal{X}'_t$.

So Mukai flops is excluded in IV. Yet it is necessary to include (at least) all ordinary $\mathbb{P}^k$ flops in IV for dimension reason.

Definition 6 ($\mathbb{P}^k$ Flops) **Let $\psi : Z \to S$ be a $\mathbb{P}^k$ bundle inside X of codimension $k+1$, and $N_{Z/X}|_\psi = \mathcal{O}_Z(-1)^{\oplus k+1}$. Then E is a $\mathbb{P}^k \times \mathbb{P}^k$ bundle over S and one may blow down E in another direction $\phi' : Y \to X'$ to get $j' : Z' = \phi'(E) \hookrightarrow X'$. $\psi' : Z' \to S$ is also a $\mathbb{P}^k$ bundle with $N_{Z'/X'}|_{\psi'} = \mathcal{O}_{\mathbb{P}^k}(-1)^{\oplus k+1}$:**



It is not hard to prove I and III for $\mathbb{P}^k$ flops.

Conclusion: Topological Evidences

Let Ω^U be the cobordism ring of stably almost complex manifolds. An R-valued complex genus is a ring homomorphism $\varphi : \Omega^U \to R$. The cobordism class is determined exactly by all chern numbers, i.e. all complex genera. Let I_K be the ideal generated by $X - X'$ for $X =_K X'$. And similarly I_k for $\mathbb{P}^k$ flops.

Theorem 7 (Totaro) $\varphi_{\text{ell}} = (\Omega^U \to \Omega^U / I_1)$.

Theorem 8 (Wang) $I_K = I_1$, so IV is true up to complex cobordism.

Complex Elliptic Genera

An R**-genus** φ **is defined by** $Q(x) \in R[\![x]\!]$ **through Hirzebruch's recipe: for** $c(T_X) = \prod_{i=1}^n (1 + x_i)$ **formally,**

$$\varphi_Q(X) := \prod_{i=1}^n Q(x_i)[X] =: \int_X K_Q(c(T_X)).$$

$K_Q \equiv K_\varphi$ **is the multiplicative sequence. Let** $Q(x) = x/f(x)$**. The CEG** φ_{ell} **is defined by**

$$f(x) = e^{(k+\zeta(z))x} \frac{\sigma(x)\sigma(z)}{\sigma(x+z)},$$

The Change of Variable Formula for CEG

Theorem 9 - *Let φ be the CEG. Then for any algebraic cycle D in X and birational morphism $\phi : Y \to X$ with $K_Y = \phi^* K_X + \sum e_i E_i$, we have*

$$\int_D K_\varphi(c(T_X)) = \int_{\phi^* D} \prod_i A(E_i, e_i + 1)\, K_\varphi(c(T_Y)).$$

Equivalently, a GRH type formula

$$\phi_* \prod_i A(E_i, e_i + 1)\, K_\varphi(c(T_Y)) = K_\varphi(c(T_X)),$$

where the Jacobian factor is defined by

$$A(t, r) = e^{-(r-1)(k+\zeta(z))t} \frac{\sigma(t + rz)\sigma(z)}{\sigma(t + z)\sigma(rz)}.$$

Idea of The Proof

Theorem 10 (Residue Theorem) *For any cycle D in X and for any blowing-up $\phi : Y \to X$ along smooth center Z with exceptional divisor E, one has for any power series $A(t) \in R[\![t]\!]$:*

$$\int_{\phi^* D} A(E)\, K_Q(c(T_Y)) = \int_D A(0)\, K_Q(c(T_X)) + \int_{Z.D} \mathrm{Res}_0\left(\frac{A(t)}{f(t)\prod_{i=1}^{r} f(n_i - t)}\right) K_Q(c(T_Z)).$$

Here n_i's denote the formal chern roots of the normal bundle $N_{Z/X}$.

The proof makes use of deformations to the normal cone to reduce to the case that $X = \mathbb{P}_Z(N \oplus 1)$. Let $p : X \to Z$ with zero section $i : Z \to N \to \mathbb{P}_Z(N \oplus 1)$. Then $N = i^*\mathcal{Q}$ where $\mathcal{Q}$ is the universal quotient bundle in

$$0 \to \mathcal{S} \to p^*(N \oplus 1) \to \mathcal{Q} \to 0.$$

then apply

$$c(T_Y) = \phi^* c(T_X)\, \phi^* c(\mathcal{Q})^{-1}\, (1+E)\, c(\phi^*\mathcal{Q} \otimes \mathcal{O}(-E))$$

to get the main term $\int_D A(0)\, K_Q(c(T_X))$ and localize the remaining to $E \to Z$.

Finally we use Newton formulae for chern classes $\bar{\phi}_* e^k = 0$ for $0 \le k \le r-2$ and $\bar{\phi}_* e^{(r-1)+k} = (-1)^{(r-1)+k} s_k(N)$ for $k \ge 0$ to reduce everything on E down to Z. Here $s(N) = \sum s_k(N)$ st $s(N)c(N) = 1$. **QED**

From it, for φ_Q to admits CVF for one blowing-up we need functional equation

$$\frac{1}{\prod_{i=1}^r f(x_i)} = \sum_{j=1}^r \frac{A(x_j, r)}{f(x_j) \prod_{i \neq j} f(x_i - x_j)}.$$

When $r = 2$, let $A(t) = A(t, 2)$. It is

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}.$$

A lengthy calculation (with help from J.-K. Yu) gives f which defines φ_{ell} and then it is easy to guess $A(t,r)$ for all r. This finishes the proof of CVF for one blowing-up.

By induction we get the CVF for composite of blowing-ups.

The general case for birational morphism $\phi : Y \to X$ follows from it and Wlodarsczyk's weak factorization theorem. Because in $K_Y = \phi^* K_X + e_i E_i$, e_i does not depend on the birational model we choose. QED

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