Mean Field Equations, Hyperelliptic Curves, and Modular Forms

6/30, 2014 at NCTS (joint project with C.S. Lin & C.L. Chai)

MFE : $\Delta u + e^u = \rho\,\delta_0$ $\quad \rho \in \mathbb{R}_{\geq 0}$ on a flat torus $E = \mathbb{C}/\Lambda$

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$\tau = \omega_2/\omega_1 \in \mathbb{H}$

More generally, on a compact R.S. (M, g) :

$$\Delta u + \rho\left(\frac{h e^u}{\int h e^u} - \frac{1}{|M|}\right) = 4\pi \sum_{j=1}^{N} \alpha_j \left(\delta_{a_j} - \frac{1}{|M|}\right)$$

Fact : The topological degree for MFE is defined for $\rho \notin 8\pi\mathbb{N}$

ρ : o——o——o——o——o ···· $\quad 8\pi n < \rho < 8\pi(n+1)$

d_ρ = 1 2 3 4 ··· $\quad d_\rho = n+1$, indep of τ.

Chen-Lin : PDE method,

Chai-Lin-W : Algebraic method.

Q : How about $\rho = 8n\pi$?

Denote by $\rho = 8\pi\eta$, $\eta \in \mathbb{R}_{\geq 0}$

Liouville theory: On simply connected domain $\Omega \subset \mathbb{C} \setminus \Lambda$

$$u(z) = \log \frac{8\,|f'(z)|^2}{(1+|f(z)|^2)^2}$$

f mero. on Ω

a developing map of u

simple zero / pole outside Λ

Schwartzian derivative

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2} u_z^2$$

analytic

indep of f

$\tilde{f} = \dfrac{af+b}{cf+d}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$

$\Rightarrow$ Solution u $\longleftrightarrow$ Collection $\{(\Omega, f)\}$ with $PSU(2)$ monodromy

when $\rho = 4\pi\ell$

i.e. $\eta = \frac{\ell}{2} \in \frac{1}{2}\mathbb{N}$ $\Rightarrow$

$$\begin{array}{ccc} \pi_1(E^\times) \simeq \mathbb{Z} * \mathbb{Z} & \longrightarrow & PSU(2) \\ \downarrow & \nearrow & \\ \pi_1(E) \simeq \mathbb{Z} \oplus \mathbb{Z} & & \end{array}$$

abelian quotient

Get global f mero. on $\mathbb{C}$

$\operatorname{ord}_{z=0} f'(z) = \ell$

May choose f st. it belongs to 2 types : $(\rho = 4\pi \ell)$

Type I : $\begin{cases} f(z+\omega_1) = -f(z) \\ f(z+\omega_2) = \dfrac{1}{f(z)} \end{cases}$

(projective) monodromy group

$PM \simeq K_4$ Klein's 4-group

$:= \mathbb{Z}/2 \times \mathbb{Z}/2$

Type II : $\begin{cases} f(z+\omega_1) = e^{2i\theta_1} f(z) \\ f(z+\omega_2) = e^{2i\theta_2} f(z) \end{cases}$

$PM \subset U(1) \simeq S^1$

Thm (Lin-W) Assume that solution u exists. Then

u has type I developing map f $\Longleftrightarrow$ $\ell = 2n+1$ (odd)

" II " $\ell = 2n$, ie $\rho = 8\pi n$

Rmk : Type II solutions must exists in families $f \mapsto e^\lambda f$

$$u_\lambda(z) := \log \frac{8\pi |e^\lambda f'(z)|^2}{(1+|e^\lambda f(z)|^2)^2} \qquad \lambda \in \mathbb{R}$$

As $\lambda \to +\infty$, $u_\lambda(z) \to +\infty$ at zeros of f , $u_\lambda(z) \to -\infty$ otherwise

Relation to Lamé equations

$$u(z) \sim \frac{\rho}{2\pi} \log|z| = 4\eta \cdot \log|z| = 2\eta\left(\log z + \log \bar{z}\right), \text{ smooth on } E^\times$$

$$\Rightarrow \quad S(f) = u_{zz} - \tfrac{1}{2} u_z^2 = \frac{1}{z^2}\left(-2\eta - 2\eta^2\right) + O(1)$$

$$= -2\left(\eta(\eta+1)\wp(z) + B\right) \quad \text{for some } B \in \mathbb{C}$$

i.e. $f = \frac{y_1}{y_2}$ for 2 independent solutions to $y'' - \frac{1}{2} S(f) y = 0$:

$$L_{\eta,B}\, y := y'' - \left(\eta(\eta+1)\wp(z) + B\right) y = 0$$

MFE $\longleftrightarrow$ Lamé eq'n with projective unitary monodromy

$\eta = n + \frac{1}{2}$ $\leftrightarrow$ $PM \cong K_4$ $\Longleftrightarrow$ All sol's are logarithmic free

Brioschi-Halphen : $\exists$ polynomial $P_n(B) \in \mathbb{Q}[g_2, g_3][B]$, $\deg_B P_n = n+1$

19th century $\quad P_n(B) = 0 \Longleftrightarrow L_{n+\frac{1}{2}, B}$ log-free.

Eg. $P_0(B) = B$, $\quad P_1(B) = B^2 - \frac{3}{4} g_2$, $\quad P_2(B) = B^3 - 7 g_2 B + 20 g_3$.

$\eta = n \in \mathbb{N}$ Main issue : The structure of solutions depends on τ !

Type II condition $f(z+w_i) = e^{2\sqrt{-1}\theta_i} f(z) \Rightarrow f$ = abelian integral of 2nd kind

$g := (\log f)' = \frac{f'}{f}$ is elliptic on E

Let $a_1, \cdots, a_n$ zeros of f, $b_1, \cdots, b_n$ poles of f

$\Rightarrow g$ has simple poles at a_i, residue $=1$, at b_i, residue $=-1$

g has zero only at $0 \in E$, $\mathrm{ord}_{z=0}\, g = \ell = 2n$.

Prop : (1) g is even, hence $\{a_1, \cdots, a_n\} = \{-b_1, \cdots, -b_n\}$

(2) $g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}$, $f = \exp \int g$

(3) $a = \{a_i\}$ satisfies $\sum_{i=1}^{n} \wp'(a_i)\, \wp(a_i)^r = 0$ for $r = 0, 1, \cdots, n-2$

Def : $X_n \subset \mathrm{Sym}^n E$ is the Liouville curve $\sum_{i=1}^{n} y_i x_i^r = 0$ $0 \le r \le n-2$

where $(x_i, y_i) = (\wp(a_i), \wp'(a_i))$, $y_i^2 = 4x_i^3 - g_2 x_i - g_3$

Lamé point of view:

Hermite-Halphen ansatz: for $a = (a_i)$, $a_i \in \mathbb{C} \setminus \Lambda$, $[a_i] \in E^\times$ distinct

$$w_a(z) := e^{z \sum_{i=1}^n \zeta(a_i)} \prod_{i=1}^n \frac{\sigma(z-a_i)}{\sigma(z)}$$

w_a is a solution to $L_{n,B}$ for some $B \in \mathbb{C}$ $\iff$ $a \in Y_n$ Lamé curve

$$Y_n := \Big\{ a \in \mathrm{Sym}^n E^\times \;\Big|\; \sum_{j \neq i} \big(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\big) = 0$$

$$a_i \neq a_j \ \forall i \neq j, \quad i = 1, \cdots, n \Big\}$$

$$\sum_{j \neq i} \frac{y_i + y_j}{x_i - x_j} = \sum_{j \neq i} \frac{\wp'(a_i) + \wp'(a_j)}{\wp(a_i) - \wp(a_j)}$$

X_n sm, Y_n may be singular

Lemma: $X_n \subset Y_n$, $Y_n \setminus X_n$ are those $a \in Y_n$ st "$a = -a$": Lamé functions

Thm (Hyperelliptic structure on $X_n \subset Y_n \subset \overline{X_n}$) $\pi : Y_n \twoheadrightarrow \mathbb{C}$ by

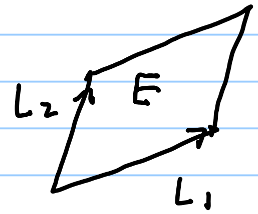
(i) $Y_n \ni a \longmapsto B_a := (2n-1) \sum_{i=1}^n \wp(a_i)$ gives the B in Lamé eq'n

(ii) π is 2 to 1 on X_n. $Y_n \setminus X_n$ is defined by $\ell_n(B) = 0$, $\deg \ell_n = 2n+1$.

(iii) π extends to $\overline{X_n} \longrightarrow \mathbb{P}^1$ with $\pi^{-1}(\infty) = 0^n$ a smooth pt.

The remaing equation $f(z+\omega_i) = e^{2\sqrt{-1}\,\theta_i} f(z)$ $\quad i=1,2$, ie.

$$\int_{L_i} g = \sum_{j=1}^{n} \int_{L_i} \frac{\wp'(a_j)}{\wp(w)-\wp(a_j)}\,dw = 2\sum_{j=1}^{n} \left(\omega_i\,\zeta(a_j) - \eta_i\,a_j\right) \in \sqrt{-1}\cdot\mathbb{R}$$



Green function : $\quad -\Delta G = \delta_0 - \frac{1}{|E|}$; $\int_E G = 0$

Hecke function : $\quad Z(t\omega_1 + s\omega_2\,;\tau) = \zeta(t\omega_1+s\omega_2) - t\,\eta_1 - s\,\eta_2$

(for $t, s \in \frac{1}{N}\mathbb{Z}$, Z is $\Gamma(N)$ modular of weight 1 in τ)

Prop : $-4\pi G_z = Z$ where $z = t\omega_1 + s\omega_2$, so NOT holo. in $z = x+iy$

pf : $\tau = a+bi$, $q = e^{2\pi i\tau}$, $\vartheta(z;\tau) = -i\sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac12)^2} e^{(2n+1)\pi i z}$

$\Rightarrow G = -\frac{1}{2\pi}\log|\vartheta| + \frac{y^2}{2b} + C(\tau)$. Using $(\log\vartheta)_z = \zeta(z) - \eta_1 z$

$\Uparrow$

$\sigma(z) = e^{\eta_1 z^2/2}\,\frac{\vartheta(z)}{\vartheta'(0)}$

and Legendre relation :

$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ $\quad\square$

Via Green function G, $\omega_1 \zeta(z) - \eta_1 z = \omega_1 \left(\zeta(z) - t\eta_1 - s\eta_2 \right) - 2\pi i s$

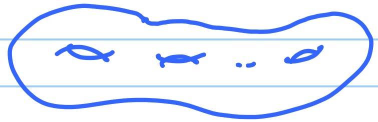
$$\omega_2 \zeta(z) - \eta_2 z = \omega_2 \left(\zeta(z) - t\eta_1 - s\eta_2 \right) + 2\pi i t$$

$\int_{L_i} g \in \sqrt{-1}\,\mathbb{R}$, $i = 1, 2$ $\iff$ $-4\pi \omega_i \sum_{j=1}^{n} G_z(a_j) \in \sqrt{-1} \cdot \mathbb{R}$ $\quad a_j = t_j \omega_1 + s_j \omega_2$

$\iff \sum_{j=1}^{n} \nabla G(a_j) = 0$ since ω_1, ω_2 are linearly indep.

For such $a \in X_n$, $\frac{1}{2\pi\sqrt{-1}} \int_{L_i} g$ gives $2\left(-\sum s_j, \sum t_j\right)$, coor. of $a_1 + \cdots + a_n$

How to study $\sum_{j=1}^{n} \nabla G(a_j) = 0$ for $a \in \bar{X}_n$?



$\bar{X}_n$

Def'n : Let $\sigma : \bar{X}_n \longrightarrow E$, $a \longmapsto \sum_{i=1}^{n} a_i$

$z_n \in K(\bar{X}_n)$ by $z_n(a) = \zeta(\sigma(a)) - \sum_{i=1}^{n} \zeta(a_i)$

Thm (Lin-W 2013) σ is a branched covering with $\deg \sigma = \frac{1}{2} n(n+1)$

Moreover, z_n is a primitive generator of $K(\bar{X}_n) / K(E)$.

Namely, $\exists$ irreducible polynomial $W_n(z) \in \mathbb{C}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][z]$ of degree $\frac{1}{2}n(n+1)$ defines $\sigma : \bar{X}_n \longrightarrow E$.

Examples : $C^2 = \ell_n(B)$ & $W_n(z)$

$n=1$ $\quad C^2 = 4B^3 - g_2 B - g_3$, $\quad W_1(z) = z$

$n=2$ $\quad C^2 = \frac{2^2}{3^4}(B^2 - 3g_2)\prod_{i=1}^{3}(B + 3e_i)$, $\quad W_2(z) = z^3 - 3\wp(\sigma)z - \wp'(\sigma)$

$n=3$ $\quad C^2 = \frac{2^2}{3^4 5^4} B \prod_{i=1}^{3}\left(B^2 - 6e_i B + 15(3e_i^2 - g_2)\right)$,

$$W_3(z) = z^6 - 15\wp z^4 - 20\wp' z^3 + \left(\frac{27}{4}g_2 - 45\wp^2\right)z^2 - 12\wp'\wp z - \frac{5}{4}\wp'^2$$

Def'n (New Pre-modular forms) $\quad Z_n(\sigma; \tau) := W_n(Z(\sigma; \tau))$

i.e. $\Gamma(N)$ modular in $\tau \in \mathbb{H}$ for $\sigma \in E[N]$.

$n=1$:

$Z_1 \equiv Z = -4\pi G_{\bar{z}}$ $\quad$ zero of $Z \Longleftrightarrow$ critical pt of $G \Longleftrightarrow$ sol. u to MFE

This generalizes to all $n \in \mathbb{N}$:

(1) Sol. u to MFE $\Delta u + e^u = 8n\pi\delta_0$ on E_τ

(2) Periods integrals $\frac{1}{2\pi\sqrt{-1}}\int_{L_i} g \in \mathbb{R}$ $i=1,2$

(3) Green equation $\sum_{i=1}^{n} \nabla G(a_i) = 0$ on the hyperelliptic curve $X_n \ni a$

(4) Coincidence equation $z_n(a) = Z(\sigma(a))$ for $a \in X_n$

(5) Zeros of pre-modular form $Z_n(\sigma;\tau) := W_n(Z)$ with $\sigma \notin E_\tau[2]$

(3) $\Leftrightarrow$ (4) : $-4\pi \sum \nabla G(a_i) = \sum Z(a_i)$ $\qquad a_i = t_i\omega_1 + s_i\omega_2$

$$= \sum\left(\zeta(t_i\omega_1 + s_i\omega_2) - t_i\eta_1 - s_i\eta_2\right)$$

$$= \zeta(\sigma(a)) - z_n(a) - \left(\sum t_i\right)\eta_1 - \left(\sum s_i\right)\eta_2$$

$$= Z(\sigma(a)) - z_n(a) .$$

(4) $\Leftrightarrow$ (5) is harder $\qquad a = -a \Rightarrow \sigma(a) \in E[2]$ considered as trivial solutions.

Rmk : Lamé with finite monodromy corr to $\sigma \in E[N]$, modular form case

The good thing here is the "deformations in both σ and τ"

Results for $n=1$: Pair $\pm p$ of extra critical pts of G

i.e. $p \notin E[2] \longleftrightarrow$ sol $\{u_\lambda\}$ of MFE for $\rho = 8\pi$

Thm (Lin-W 2006) For any τ, G has either 3 or 5 critical points

i.e. Sol. u, if exists, is unique up to scaling

Idea of pf : Method of continuity in $\rho \in [4\pi, 8\pi]$

Linearized eq at u_ρ
$$\begin{cases} \Delta\varphi + e^{u_\rho}\varphi = 0 & \text{on } E \\ \varphi(-z) = \varphi(z) \end{cases}$$

$4\pi \longrightarrow 8\pi$

$d\rho = 1$

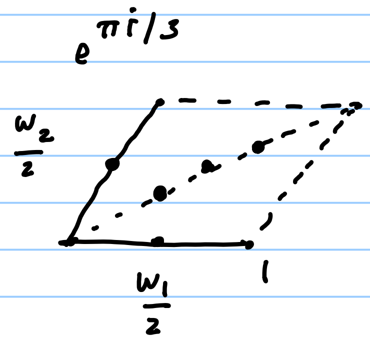
unique sol at $\rho = 4\pi$

Moser symmetrization $\Rightarrow$ non-degenerate ($\varphi \equiv 0$).

Eg. $\tau \in i\mathbb{R}$ Rectangular tori, 3 critical pts (Maximal principle)

$\tau = e^{\pi i/3}$ 5 critical points ($\mathbb{Z}/3$ symmetry)

extra pts $p = \pm \frac{1}{3}(w_1 + w_2)$.



Let $\Omega_3 \subset M_1$, G has 3 critical pts

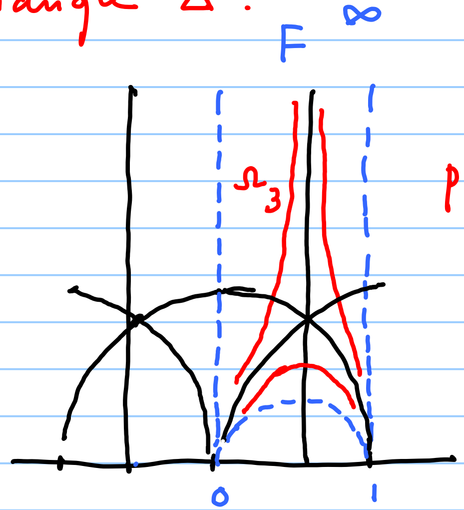
$\Omega_5 \subset M_1$ " 5 " (open set) .

Thm (Lin–W 2012) Ω_5 is simply connected

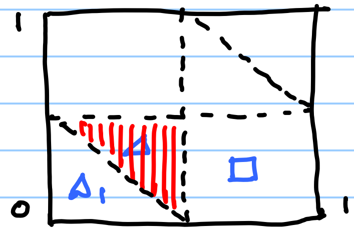
$C = \partial\Omega_5 \cup \{\infty\}$ is a smooth curve in $\overline{M_1} \cong S^2$

$\tau \in C \iff \frac{1}{2}\omega_i$ is a degenerate critical point of G for some $i \in \{1, 2, 3\}$

Moreover, for $\pm p \in \Omega_5$, $p = t\omega_1 + s\omega_2$, $p \mapsto (t, s) \in [0,1]^2$ is $1-1$ onto the triangle Δ.



$p \longmapsto (t, s)$



Idea of pf: Argument principle on F, for

$Z_{t,s}(\tau) := Z(t + s\tau; \tau)$ for $t, s \notin \frac{1}{2}\mathbb{Z}$

(i) Hecke (1926): $Z_{t,s}$ has no zeros/poles at cusps $0, 1, \infty$

(ii) $Z_{t,s}$ has no zeros along ∂F

$\Rightarrow$ # of zeros of $Z_{t,s}$ is constant in each region Δ_1, Δ, $\square$

(iii) $\forall\ \tau \in \mathbb{H}$, $Z_{\frac{1}{6},\frac{1}{6}}(\tau) \neq 0 \Rightarrow$ no sol for $(t,s) \in \Delta_1$

pf: Let $z = \frac{1}{6}(\omega_1+\omega_2) = \frac{1}{6}\omega_3$, $u = \frac{1}{3}\omega_3$, then

$$0 \neq \frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z+u) + \zeta(z-u) - 2\zeta(z)$$

$$= \underline{\zeta(\tfrac{1}{2}\omega_3)} + \zeta(-\tfrac{1}{6}\omega_3) - 2\zeta(\tfrac{1}{6}\omega_3) = -\underline{3}\left(\zeta(\tfrac{1}{6}\omega_1+\tfrac{1}{6}\omega_2) \underline{-\tfrac{1}{6}\eta_1 - \tfrac{1}{6}\eta_2}\right).$$

(iv) Similarly, $Z_{\frac{3}{4},\frac{1}{4}}(\tau) \neq 0 \Rightarrow$ no sol for $(t,s) \in \square$

(v) $Z_{t,s}$ has no sol if $(t,s) \notin \Delta$ $\left(\{(t,s) \mid Z_{t,s} \text{ has finite sol}\} \text{ is open}\right)$

(vi) Will show $Z_{\frac{1}{3},\frac{1}{3}}$ has a unique sol $\tau = e^{\pi i/3}$ in Δ:

pf: Let $Z(\tau) = \prod' Z_{\frac{k_1}{3},\frac{k_2}{3}}(\tau)$ over $0 \le k_1, k_2 \le 2$ $(k_1,k_2,3)=1$ i.e. $\neq (0,0)$

$$\nu_\infty(Z) + \frac{1}{2}\nu_i(Z) + \frac{1}{3}\underline{\nu_\rho(Z)} + \sum \nu_p(Z) = \frac{8}{12} = \frac{2}{3}$$

wt $Z = 9-1 = 8$

All others vanish, since $Z_{\frac{2}{3},\frac{2}{3}}(\rho) = 0$ too.

Corollary: Precise result on finite monodromy problem of Lamé eq'n beyond enumeration (Beuker, Waall, Dahmen etc)

Similar and more interesting stories for general $n \in \mathbb{N}$, via

$$
\begin{tikzcd}
\bar{X}_n \arrow[r, "\sigma", "\frac{1}{2}n(n+1)\,:\,1"'] \arrow[d, "2:1\quad B"'] & E \\
\mathbb{P}^1 &
\end{tikzcd}
$$

and pre-modular forms $Z_n(\sigma; \tau)$

For deformations in $\rho \in \mathbb{R}_{\geq 0}$, non-degeneracy fails for $\rho > 8\pi$
we still have

Thm (Chai-Lin-W) Let u_k be a blow-up sequence wrt $\rho_k \to 8\pi n$
if $\rho_k \neq 8\pi n$ for $k \gg 0$, then the blow-up set $p = \{p_1, \ldots, p_m\}$
must satisfy $m = n$ and $p = -p$, ie. $p \in Y_n \setminus X_n$, a branch point

Question: Is this theory an analytic local Calabi–Yau theory?
W_a (or f) is related to "complex elliptic genera". End