COHOMOLOGY THEORY OF BIRATIONAL GEOMETRY

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CONTENTS

§1. Classical Topology

Poincaré: Analysis Situs ~1900

$$\underset{\text{triangulable space}}{X} \longmapsto \underset{\text{Homology Groups}}{H_k(X)}, \quad k = 0, 1, 2 \ldots$$

Euler-Poincaré characteristic ← add definition & functoriality

$$\chi(X) = h_0(X) - h_1(X) + h_2(X) - \cdots$$ ← add Euler sum

Poincaré duality (for oriented triangulated closed mfd)

$$H_k(X) \otimes H_{n-k}(X) \longrightarrow \mathbb{Z}$$

$(\alpha, \beta) \mapsto \alpha \cap \beta$ is a perfect (dual) pairing

Hopf-Poincaré index formula (dynamical system)

X smooth oriented closed mfd

v tangent v.f. with isolated zeros

$$\chi(X) = \mathrm{Index}(v) := \sum_{v(p)=0} \mathrm{index}_p(v)$$

Poincaré conjecture

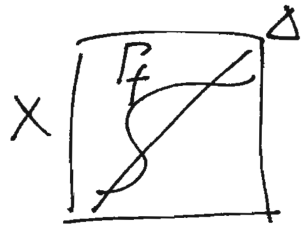
$$\dim X = 3, \quad \pi_1(X) = 0 \Rightarrow X = S^3.$$

Lefschetz: "Topology" ~1930 ← Künneth formula, product structure, H^* ring

$f: X \to X$ continuous function

$$|\mathrm{Fix}(f)| = |\Gamma_f \cdot \Delta| = \sum_i (-1)^i \, \mathrm{Tr}\big(f_*: H_i(X) \to H_i(X)\big)$$

$\sigma_i \in H_*(X)$ basis, σ_i^* dual basis



$$\Delta = \sum_i \sigma_i^* \otimes \sigma_i$$

$$\Gamma_f = \sum_i f_*(\sigma_i) \otimes \sigma_i^*$$

$$\Rightarrow \Gamma_f \cdot \Delta = \sum_i (-1)^{(\deg \sigma_i)^2} f_*(\sigma_i) \cdot \sigma_i^*$$

$$\Delta: X \to X \times X$$

QED.

de Rham – Hodge (1950 ICM) – Kodaira (1955)

X C^∞ mfd

$$\Lambda^0(X) \xrightarrow{d_0} \Lambda^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(X) \longrightarrow 0 \quad ; \; d^2 = 0$$

$$H^k_{DR}(X) := \ker d_k / \operatorname{Im} d_{k-1} \cong H^k(X)$$

Harmonic forms $\longleftrightarrow$ coh. classes

$$\Lambda^{k-1}(X) \underset{d^*}{\overset{d}{\rightleftarrows}} \Lambda^k(X) \underset{d^*}{\overset{d}{\rightleftarrows}} \Lambda^{k+1}(X)$$

$d^* = *d*$ dual of d

$$\Delta = dd^* + d^*d : \Lambda^k \to \Lambda^k$$

$$\Delta\omega = 0 \iff d\omega = 0, \; d^*\omega = 0 \quad \text{elliptic PDE}$$

$$\mathbb{H}^k(X) \cong H^k_{DR}(X)$$

$* : \mathbb{H}^k \to \mathbb{H}^{n-k}$ gives Poincaré duality

Hodge Theory for Kähler mfds

X cpt Kähler mfd

$$\Lambda^k = \oplus_{p+q=k} \Lambda^{p,q} \leftarrow dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots d\bar{z}_{j_q}$$

$$d = \partial + \bar{\partial}, \quad \Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

$$\mathbb{H}^k = \bigoplus_{p+q=k} \mathbb{H}^{p,q}$$

In fact (Dolbeault) $\mathbb{H}^{p,q} \cong H^q(X, \Omega^p_X)$

$$\begin{pmatrix}\text{Topological} \\ \text{cohomology}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{coherent} \\ \text{cohomology}\end{pmatrix}$$

- Kodaira vanishing
- Kodaira – Serre duality
- Weak Lefschetz $H \hookrightarrow X$ hyperplane
- Hard Lefschetz

$$\mathbb{H}^{n-k}(X) \xrightarrow[\sim]{\cdot [H]^k} \mathbb{H}^{n+k}(X)$$

§2. Birational Geometry

Birational Geometry

X integral variety, $K(X)$ field of rational functions

eg. $X = \{ f_1 = 0, f_2 = 0, \cdots f_r = 0 \}$

$R =$ regular functions $= k[x_1, \cdots, x_n] / (f_1, \cdots, f_r)$

rational function $= \{ f/g \}$ $f, g \in R$

X, X' birational $\iff K(X) = K(X')$

geometrically,

$$X \overset{f}{\dashrightarrow} X'$$

f, f^{-1} are defined and isom on Zariski open set

Hironaka (1962–1964) Ann. Math

Every K can be the function field of a smooth proj variety if $k = \mathbb{C}$. ← add resol. of singularity.

Castelnuovo (1893) Italian school.

X smooth surface, $C \subset X$ a (-1) curve st. $C \cong \mathbb{P}^1$ (ie. $C.C = -1$), then $\exists$ projective morphism

$$\varphi : X \longrightarrow \bar{X}$$

with $\bar{X}$ smooth and $\varphi(C) = pt$. $\varphi|_{X-C} \cong$.

X minimal if no such contraction exists. (keep on contracting (-1) curves)

If $\Gamma(X, K_X^{\otimes r}) \neq 0$ for some $r \in \mathbb{N}$, $K_X := \Omega_X^2$

then $\exists !$ minimal model of $K(X)$.

Property of (-1) curves:

$$K_C = (K_X + C)|_C \quad \text{adjunction formula (Cauchy int. formula)}$$

$\deg K_C = K_X . C + C^2 = K.C - 1$

so C rational $\iff K.C = -1 < 0$

Mori : 1979 existence of $\mathbb{P}^1$ (Hartshorne conjecture)
1982 cone theorem Ann. Math
1988 Flip theorem

Theorem: Minimal model exists for 3-dim'l cpx proj variety with $\Gamma(X, K^{\otimes r}) \neq 0$ for some $r \in \mathbb{N}$ in the category of $\mathbb{Q}$ factorial terminal varieties, $K := \Omega^n_X$

Rmk 1: If $\Gamma(X, K^{\otimes r}) = 0 \ \forall\, r \in \mathbb{N}$ then
X is unirulled (ie. $\sim \mathbb{P}^1 \times Y$), Miyaoka-Mori

Rmk 2: Minimal model is not unique, but they are all related by a sequence of flops. Kawamata - Kollár

Cone Theorem:

If K is not nef, then for the numerical class of 1-cycles

$$\overline{NE(X)} = \overline{NE(X)} \cap K_{(\geq -\varepsilon)} + \sum_i \mathbb{R}_+ [C_i]$$

where $\sum_i \mathbb{R}_+ [C_i] = NE(X) \cap K_{(K-\varepsilon)}$

D nef $\Leftrightarrow D.C \geq 0 \ \forall C$

C_i called extremal rays. and $\exists$ divisor D_i st
$\varphi : |mD_i| : X \longrightarrow X' \subset \mathbb{P}^N$ is bpf and
$\varphi(c) = pt \Leftrightarrow [C] = [C_i]$.

Definition of minimal model:

Let X be $\mathbb{Q}$-Gorenstein
ie. rK_X is Cartier (line bundle)
$\varphi : Y \longrightarrow X$ be a resolution of singularity

$$K_Y =_{\mathbb{Q}} \varphi^* K_X + \sum a_i E_i \qquad a_i \in \mathbb{Q}$$

$a_i > 0$ terminal
$a_i \geq 0$ canonical
$a_i > -1$ log-terminal ($\Rightarrow \varphi^*\Omega$ int'ble in classical sense)

X is minimal if X is terminal and K_X is nef.

Problem: what's the relation between minimal models?

§3 Semi-Classical Topology

Weil (1949) Number of solutions of equations over finite fields.

- Alexandroffer-Weil
  general Gauss-Bonnet Formula

$$\chi(X) = \int_X K \qquad K: \text{Gauss curvature}$$

- Chern-Weil Theory

$$\begin{array}{ll} E & \text{vector bundle}, \ D \ \text{connection} \\ \downarrow & R \text{"="} D^2 \ \text{curvature} \in \Omega^2(\mathrm{End}\, E) \\ X & \det\left(I - \frac{i}{2\pi} R\right) = 1 + c_1 + c_2 + \cdots \in H^*_{DR}(X) \end{array}$$

Chern (1945) Ann Math. $\chi(X) = \int_X c_n$.
making use of the Hopf-Poincaré index formula

- Weil conjectures

$$X \subset \mathbb{P}^N_k, \quad k = \mathbb{F}_q \quad \text{finite field}$$

$$F: X \longrightarrow X \quad \text{Frobenius} \quad x \longmapsto x^q$$

$$X(\mathbb{F}_{q^k}) = \mathrm{Fix}(F^k)$$

$$N_k = |X(\mathbb{F}_{q^k})| = \sum_i (-1)^i \,\mathrm{Tr}\left(F_*^k : H_i(X) \to H_i(X)\right)$$

Zeta function $Z(X,t) := \sum_{k \geq 1} N_k \frac{t^k}{k}$

$$\Rightarrow \quad Z(X,t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2n}(t)}$$

$P_k(t) = \det(1 - tF_*)$ char. poly on $H_i(X)$

Grothendieck (1960~) motive?
étale coh. $H^i(X_{\acute{e}t}, \mathbb{Z}_\ell)$, F conti in étale top.
Deligne (1970~)
Weil-Riemann Hypothesis $P_k(\alpha) = 0 \Rightarrow |\alpha| = q^{-k/2}$

Deligne's Mixed Hodge Theory (1970 ~)

X cpx alg. v.

$H^k_c(X)$, $H^k(X)$ has funct'l mixed Hodge structure

motivic property $U \hookrightarrow X$, $Z = X - U$

$$\cdots \to H^k_c(U) \to H^k_c(X) \to H^k_c(Z) \to H^{k+1}_c(U) \to \cdots$$

exact seq of MHS's.

Goresky - MacPherson (1980), Cheeger (1978)

X cpx alg. (projective) v.

$\exists$ sub cpx $IC^\cdot(X)$ of $C^\cdot(X)$ st.

$IH_\cdot(X)$ satisfies the Poincaré duality property

* Cheeger: L^2-cohomology. conjecture.

Deligne - Gabber - Bernstein - Beilinson [BBD] (1980)

étale generalization of $IH_\cdot(X_{\acute{e}t})$

still satisfies the Weil-Riemann Hypothesis

Frobenius acts on $IH^k(X_{\acute{e}t})$ with

pure weight $q^{-k/2}$.

The corresponding LFT is NOT CLEAR !

Decomposition theorem

$$\begin{array}{ll} Y & \text{proper projective} \\ \downarrow \varphi & \Rightarrow \quad \underline{\varphi_* IC_\cdot(Y) = \bigoplus_\alpha IC_\cdot(X_\alpha, L_\alpha)} \\ X & \end{array}$$

L_α local system on X_α

Saito (1988, IHES. Publ. Math)

$IH^k(X)$ has functorial pure Hodge structure

Does not solve the Cheeger conjecture.

L^2-story is not clear.

§4. K-partial ordering and the Meta Theorem

MRL 1997

K-partial ordering (Wang. JDG 1998)

(Main observation)

Theorem: $f : X \dashrightarrow X'$ birational map, st X, X' has canonical sing. Suppose the exceptional locus $Z \subset X$ is proper and K_X is nef along Z. Then

$$X \leq_K X'$$

i.e. for any common resolution

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

, Y smooth,

$\varphi^* K_X \leq \varphi'^* K_{X'}$.

Moreover, if X' is terminal, then codim $Z \geq 2$.

Idea of pf:

$$\varphi^* K_X + F + G =_{\mathbb{Q}} K_Y =_{\mathbb{Q}} \varphi'^* K_{X'} + F' + G'$$

F, F' both φ, φ' exceptional

G : φ-exc not φ'

G' : φ'-exc not φ

Want to prove $F - F' - G' \geq 0$ (ie. $F \geq F'$, $G' = 0$)

$$\varphi'^* K_{X'} =_{\mathbb{Q}} \varphi^* K_X + G + (F - F' - G')$$

take generic hyperplane section H of Y $(n-2)$ times

$$H^{n-2} \cdot \varphi'^* K_{X'} =_{\mathbb{Q}} H^{n-2} \cdot \varphi^* K_X + \zeta + (\xi - \xi' - \zeta')$$

$\cap$ with b (if $b \neq 0$), B: φ'-exc $\qquad a - b = H^{n-2} \cdot (A-B)$

$$\underset{0}{\overset{\|}{B \cdot H^{n-2} \cdot \varphi'^* K_{X'}}} =_{\mathbb{Q}} \underset{0}{\overset{\vee\!/}{B \cdot H^{n-2} \cdot \varphi^* K_X}} + \underset{0}{\overset{\vee\!/}{b \cdot \zeta}} + \underset{0}{\overset{\vee\!/}{b \cdot a}} - \underset{0}{\overset{\vee}{b^2}}$$

by Hodge index thm on surfaces. $\square$

Cor. $\underline{X, X' \text{ minimal} \Rightarrow X =_K X'}$.

A Meta Theorem

$$\begin{array}{lll} Y & K_Y = \Omega_Y^n \ni \varphi^*\Omega = \mathrm{Jac}(\varphi)\, dy_1 \wedge \cdots \wedge dy_n \\ \downarrow \varphi & \\ X & K_X = \Omega_X^n \ni \Omega = dx_1 \wedge \cdots \wedge dx_n \end{array}$$

the formula $K_Y = \varphi^* K_X + E$

means that $\operatorname{div}(\mathrm{Jac}(\varphi)) = E$

so K-equivalence $\Rightarrow$ have the same Jacobian factor for holo. top form

How to relate invariants of X and X' ?

I. Geometric situation lead to the conclusion of K-equivalence (done).

II. Suitable integration / measure theory attached to a variety. such that the correspondi[ng] Jacobian factor is determined by $\mathrm{Jac}(\varphi)$

III. Topological / geometric interpretation of the integral.

$$\begin{tikzcd}
\int_Y \mathrm{Jac}(\varphi)\, d\mu \arrow[rrrr, bend left, equal] \arrow[d, equal] & & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & & \int_Y \mathrm{Jac}(\varphi')\, d\mu \arrow[d, equal] \\
\int_X d\mu \arrow[d, equal] & X \arrow[rr, dashed, leftrightarrow, "f"] & & X' & \int_{X'} d\mu \arrow[d, equal] \\
\text{invariant of } X & & & & \text{invariant of } X'
\end{tikzcd}$$

§5. Classical Integration Theory

1996

Integration Theory via Differential Geometry (Wang)

for X almost cpx mfd. Chern-Weil Theory represents c_i as differential forms $\in \Lambda^{2i}(X)$

Invariants :

chern numbers $c^I = c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n}$

st. $1 \cdot i_1 + 2 \cdot i_2 + \cdots + n \cdot i_n = n$

eg. $\int_X c_n = \chi(X)$ Gauss-Bonnet-Chern

$\int_X Td = \chi(X, \mathcal{O}_X)$ Riemann-Roch

$\int_X L = \sigma(X)$ signature thm (if n even) Hirzebruch

or more generally, the elliptic genera.

$$X \overset{f}{\longleftarrow\!\!\dashrightarrow} X'$$

$$U = X - Z \cong X' - Z' = U'$$

- If can construct Kähler metric ω on X, ω' on X' such that $\omega|_U = \omega'|_{U'}$ then done!
  But this is impossible unless $X \cong X'$.
- Alternative :
  construct degenerate Kähler metrics ω, ω' with $\omega|_U = \omega'|_{U'}$ Kähler metrics. Then

eg.

$$\int_X c_n(\omega) = \int_{X'} c_n(\omega') \quad *.$$

singular chern form

if semi-conti is indeed ~~conti~~

$$\int_X c_n(\omega_t) \qquad\qquad \int_{X'} c_n(\omega'_t)$$

$$\| \qquad\qquad\qquad \|$$

$$\chi(X) \qquad\qquad\qquad \chi(X')$$

$\omega_t \to \omega$ smoothing of metrics.

such ω, ω' can be constructed using the Calabi conj by Yau. But the degeneracy can't be controlled well enough.

Hader - Narashiham 70 —
Denef - Loeser 90

p-adic integral, Batyrev 96 Wang 97'

consider the embedding of fields, S = defining ring of $X \leftarrow Y \rightarrow X'$.

$$\begin{array}{ccccc} S \subset F & \longrightarrow & \mathbb{Q}_p \supset \mathbb{Z}_p = R \\ m. & & p\mathbb{Z}_p \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_p & \end{array}$$

$\underline{\mathbb{Q}_p \text{ has trdeg} = \infty}$

This induces

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & & & & \bar{Y} \arrow[dl, "\bar\varphi"'] \arrow[dr, "\bar\varphi'"] & \\ X & & X' \ /R \arrow[rr, "\text{reduction}", "\text{mod } p"'] & & \bar{X} & & \bar{X}' \ /\mathbb{F}_p \end{tikzcd}$$

Idea: Want to use Deligne's soln of the Weil conjecture. if can show that

$$\underline{|\bar{X}(\mathbb{F}_{p^k})| = |\bar{X}'(\mathbb{F}_{p^k})| \quad \forall k}$$

then $Z(\bar{X}, t) = Z(\bar{X}', t)$, hence the same Betti numbers and also the same eigenvalues for the Frobenius!

The p-adic structure allows to count points:

Theorem: (Weil)

Let U smooth R scheme ($R/p \cong \mathbb{F}_q$)

Ω nowhere zero r-pluri canonical form

i.e. $\Omega = \psi(z)(dz_1 \wedge \dots \wedge dz_n)^{\otimes r}$ locally $|\psi|_p = 1$

Then
$$\underline{\int_{U(R)} |\Omega|^{1/r} = \frac{|\bar{U}(\mathbb{F}_q)|}{q^n}} \quad *$$

Rmk 1: the RHS is indep of the choosen Ω, hence the integral can be glue together to count $\bar{X}(\mathbb{F}_q)$

Rmk 2: Let X be a $\mathbb{Q}$-Gorenstein R-scheme, Then $X(R)$ has finite p-adic measure
$\Longleftrightarrow$ X has at most log-terminal singularities.

§6. Motivic Integration

Grothendieck ring of Alg. varieties.

k field char $k = 0$

$\mathcal{M}$ = Grothendieck ring of alg v. $/k$

$= \{ \text{alg. v} \} / \sim$

$[S] = [S']$ if S isom S'

$\underline{[S] = [S - S'] + [S']}$ if $S' \subset S$ is closed

the ring of "numerical motives"

$[S] \cdot [S'] = [S \times S']$

Let $\mathbb{L} = [\mathbb{A}^1_k]$ the Lefschetz motive

$\mathcal{M}_{loc} := \mathcal{M}[\mathbb{L}^{-1}]$

Nash space of arcs. (1960–70)

X alg. v.

$\mathcal{L}(X)$ = the scheme of germs of arcs on X

$= \varprojlim \mathcal{L}_n(X)$

$\mathcal{L}_n(X) = \mathrm{Mor}_{k\text{-scheme}}(\mathrm{Spec}\, k[t]/t^{n+1}, X)$

arc space up to degree n in X

$\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$ projection

stability

$\exists\, n \in \mathbb{N}$ st. $\pi_{m+1}(\mathcal{L}(X)) \longrightarrow \pi_m(\mathcal{L}(X))$

is a piece-wise trivial fibration with fiber $\mathbb{A}^d_k$

so $\underline{[\pi_m \mathcal{L}(X)] = [\pi_n \mathcal{L}(X)] \cdot \mathbb{L}^{(m-n)d}}$

Nash has expected that

N_A (stable range) of a singular subset A, $\pi^{-1}(A) \subset \mathcal{L}(X)$

should be related to the descrepancy in Hironaka's

resolution $f: Y \to X$

$$K_Y = f^* K_X + \sum e_i E_i$$

Motivic integration (Kontsevich, Denef-Loeser) 1997–98

$\alpha : \mathcal{L}(X) \longrightarrow \mathbb{Z}$ a simple function

i.e. all $\alpha^{-1}(n)$ are stable subsets.

$\mathcal{M}_{loc}$

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} := \sum_{n \in \mathbb{Z}} \mathbb{L}^{-n} \underbrace{[\pi_m(\alpha^{-1}(n))]\, \mathbb{L}^{-(m+1)d}}_{m \geq \text{ stable range of } \alpha^{-1}(n)}$$

This can be extend to semi-algebraic subset of $\mathcal{L}(X)$, hence get a measure-integral theory

$$\mu : B(X) \longrightarrow \widehat{\mathcal{M}}$$

eg. if X is smooth, then $\mu(X) = \int_{\mathcal{L}(X)} \mathbb{L}^0 = [X] \cdot \mathbb{L}^{-d}$

change of variable (Hard):

$\varphi : Y \longrightarrow X$ proper birational, Y smooth

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\alpha \circ \varphi - \operatorname{ord} \varphi^*(\Omega_X^d)}$$

Jacobian factor

K-equivalence $\Rightarrow$ equivalence of motivic volume

If
$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$
s.t. $\varphi^* K_X = \varphi'^* K_{X'}$

(in the $\mathbb{Q}$-Gorenstein case)

then $\mu(X) = \int_{\mathcal{L}(X)} 1 = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord} \varphi^*(\Omega_X^d)} = \mu(X')$

what's the geometric meaning of $\mu(X)$?

Theorem: Birat'l smooth minimal models are equivalent in the Grothendieck ring $\mathcal{M}$.

Hodge realization

Deligne had put a functorial MHS on cpt supp. coh. of cpx alg. varieties.

If $Z \subset X$ closed $U = X - Z$, then

$$\cdots \to H_c^k(U) \to H_c^k(X) \to H_c^k(Z) \to H_c^{k+1}(U) \to \cdots$$

is an exact sequence of MHS's.

Apply the Euler functor $\chi^{p,q} : Alg \to \mathbb{Z}$

$$\chi_c^{p,q}(W) := \sum_i (-1)^i \, h^{p,q}(H_c^i(W))$$

get $\chi_c^{p,q}(X) = \chi_c^{p,q}(U) + \chi_c^{p,q}(Z)$

hence $\chi_c^{p,q}$ factors through $\mathcal{M} \to \mathbb{Z}$

Corollary:

if $X \cong_K X'$ and smooth, then X and X' have the same Hodge numbers

Pf: $\chi_c^{p,q}(X) = h^{p,q}(X)$ if X is smooth, hence $h^{p,q}(X) = \chi^{p,q}(X) = \chi^{p,q}(X') = h^{p,q}(X')$ ※.

Étale realization

From

$$\cdots \to H_c^k(U_{\acute{e}t}) \to H_c^k(X_{\acute{e}t}) \to H_c^k(Z_{\acute{e}t}) \to H_c^{k+1}(U_{\acute{e}t}) \to \cdots$$

only get $\chi_c(X) = \chi_c(U) + \chi_c(Z)$

hence X and X' have the same Euler number.

a result weaker than the p-adic integral!

Rmk: Even the Hodge realization does not cover the p-adic result about eigenvalues.

§7 Open Problems

Problems and Discussions

I. Singular case (terminal minimal models)

p-adic integral, motivic integral work well, but lack of step 3. geometric meaning of $\mu(X)$

- what's the weighted counting of the p-adic int.?
- motivic $\int_{\mathcal{L}(X)} \mathbb{L}^0 \neq [X]$ in $\mathcal{M}$!

~~conjecture I.~~ X, X' birational ter. minimal models $\Rightarrow$ $IH(X)$, $IH(X')$ same Betti numbers. and same Hodge numbers.

II. Construct the "minimal invariants" directly from K. Since we are considering invariants of K (B. Mazur) should look at $\underline{H^i_{ét}(\operatorname{Spec} K, \mu_n)}$

? minimal motive

From the Kummer seq: $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$

get
$$H^0(K, \mathbb{G}_m) \xrightarrow{n} H^0(K, \mathbb{G}_m) \to H^1(K, \mu_n) \to \underset{\substack{\parallel \\ 0}}{H^1(K, \mathbb{G}_m)}$$

$$0 \to H^2(K, \mu_n) \longrightarrow \underset{\substack{\parallel \\ \text{coh. } Br(K)}}{H^2(K, \mathbb{G}_m)} \xrightarrow{n} H^2(K, \mathbb{G}_m)$$

Hilbet 90.

so $H^1_{ét}(\operatorname{Spec} K, \mu_n) \cong K^\times / K^{\times n}$ is OK.

$H^2_{ét}(\operatorname{Spec} K, \mu_n) \cong {}_n H^2(K, \mathbb{G}_m)$ not ok if $(n,p)=1$. (eg. $\dim X = 2$)

In fact, it is hard to identify $H^i(K, \mathbb{G}_m)$ geometrically. ~~Grothendieck's theory of Brauer groups~~ only shows

"$\boxed{b_2 - \underbrace{\rho}_{\text{picard}}}$" is birat'l inv

which is not interesting (transcendal cycles only).

Rmk: $h^0(\Omega^p_X)$ are tricial bi-rat'l inv's (Hartogs Lemma)

III. Natural Morphisms between $H(X)$ and $H(X')$

from

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

get $Y \xrightarrow[\Phi]{} X \times X'$

$$\Phi(Y) \in H_n(X \times X')$$
$$\| \text{ Künneth}$$
$$\bigoplus_{p+q=n} H_p(X) \otimes H_q(X')$$
$$\cup$$
$$\bigoplus_{q \geq 0} \mathrm{Hom}(H_q(X), H_q(X'))$$

Get <u>cohomological correspondence</u>

$$\Phi : H_*(X) \longrightarrow H_*(X') \quad \text{and} \quad \Phi^t : H_*(X') \longrightarrow H_*(X)$$

<u>conjecture II</u>: Φ is an isomorphism? (over $\mathbb{Q}$)

Notice that $H^*(X)$ and $H^*(X')$ can not have the same ring structure (even in $\dim = 3$)

<u>Importancy</u>:

if $K_X = 0$ (Calabi-Yau), $T_X \otimes \Omega^1_X \to K$

$$H^1(X, T_X) \cong H^1(X, \Omega_X^{n-1}) = H^{n-1,1}(X)$$

Knowing the morphism may help the study of "birational moduli space"

"Morphic" motivic integration ??

in the Bernstein - Beilinsson - Deligne - Gabber decomp. thm of G-M int. coh. $f : Y \to X$

$$f_* IC^\cdot(Y) = \bigoplus_\alpha IC^\cdot(X_\alpha, L_\alpha)$$

can one determine X_α, L_α in terms of the Jacobian $\mathrm{Jac}(f)$? effectively ??