Critical Points of Green Functions on Flat Tori

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  and $\int_T G(z,w)\, dA = 0$, where δ_w is the Dirac measure with singularity at $z = w$.

- Because of the translation invariance of $\triangle_z$, we have $G(z,w) = G(z-w,0)$ and it is enough to consider *the Green function* $G(z) := G(z,0)$. Asymptotically

$$G(z) = -\frac{1}{2\pi} \log|z| + o(|z|^2).$$

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$$\vartheta_1(z;\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi iz}.$$

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- The structure of G, especially its critical points and critical values, will be the fundamental objects that interest us. $\nabla G(z) = 0 \Longleftrightarrow$

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi} \left((\log \vartheta_1)_z + 2\pi i \frac{y}{b} \right) = 0.$$

- Recall $\wp(z) = 1/z^2 + \cdots$, $\zeta(z) = -\int^z \wp = 1/z + \cdots$. and $\sigma(z) = \exp \int^z \zeta(w)\,dw = z + \cdots$ is entire, odd with a simple zero on lattice points and

$$\sigma(z + \omega_i) = -e^{\eta_i(z + \frac{1}{2}\omega_i)}\sigma(z)$$

with $\eta_i = \zeta(z + \omega_i) - \zeta(z) = 2\zeta(\frac{1}{2}\omega_i)$ the quasi-periods.

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- Indeed

$$\sigma(z) = e^{\eta_1 z^2/2}\frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Hence $\zeta(z) - \eta_1 z = (\log\vartheta_1(z))_z$.

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- Let $z = t\omega_1 + s\omega_2$. By Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, $\nabla G(z) = 0$ if and only if

$$G_z = -\frac{1}{4\pi}\Big(\zeta(t\omega_1 + s\omega_2) - (t\eta_1 + s\eta_2)\Big) = 0.$$

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- **Question:** How many critical points can G have in T?

- The 3 half periods are trivial critical points. Indeed,

$$G(z) = G(-z) \Rightarrow \nabla G(z) = -\nabla G(-z).$$

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  For rectangular tori T: $(\omega_1, \omega_2) = (1, \tau = bi)$, $\frac{1}{2}\omega_i, i = 1, 2, 3$ are precisely all the critical points.

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  - However, it is very difficult to study the critical points from the "simple equation" $\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2$ directly.

- **In PDE**, the geometry of $G(z, w)$ plays fundamental role in the non-linear mean field equations (= Liouville equation with singular RHS): On a flat torus T it takes the form ($\rho \in \mathbb{R}_+$)

$$\triangle u + \rho e^u = \rho \delta_0.$$

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- It is the mean field limit of Euler flow in statistic physics.
- It is related to the self-dual condensation of abelian Chern-Simons-Higgs model (Nolasco and Tarantello 1999).
- **In Arithmetic Geometry**, $G(z, w)$ also appears in the Arakelov geometry as the intersection number of two sections z and w of the arithmetic surface $\mathcal{T} \to \operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ at the ∞ fiber $\mathcal{T}_\infty =$ Riemann surface T.

- When $\rho \notin 8\pi\mathbb{N}$, it has been proved by C.-C. Chen and C.-S. Lin that the Leray-Schauder degree is

$$d_\rho = k + 1 \quad \text{for} \quad \rho \in (8k\pi, 8(k+1)\pi),$$

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Theorem (Existence Criterion)

For $\rho = 8\pi$, the mean field equation on a flat torus $T = \mathbb{C}/\Lambda$:

$$\triangle u + \rho e^u = \rho \delta_0$$

has solutions if and only if the G has more than 3 critical points. Moreover, each extra pair of critical points $\pm p$ corresponds to an one parameter family of solutions u_λ, where $\lim_{\lambda\to\infty} u_\lambda(z)$ blows up precisely at $z \equiv \pm p$.

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  where f, called a developing map of u, is meromorphic in Ω.
- It is straightforward to show that

  $$\mathcal{S}(f) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2}u_z^2.$$

  I.e., any developing map f of u has the same Schwartz derivative.

- Thus for any two developing maps f and $\tilde{f}$ of u, there exists $S = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in PSU(2)$ such that $\tilde{f} = Sf := \dfrac{pf - \bar{q}}{qf + \bar{p}}$.

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- Geometrically the Liouville equation is simply the prescribing Gauss curvature equation in the new metric $g = e^u g_0$ over D, where g_0 is the Euclidean flat metric on $\mathbb{C}$:

$$K_g = -e^{-u} \triangle u = \rho. \qquad (1)$$

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- It is then clear the inverse stereographic projection $\mathbb{C} \to S^2_{1/\sqrt{\rho}} \backslash N$

$$(X, Y, Z) = \frac{1}{\sqrt{\rho}} \Big(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \Big)$$

provides solutions to (1).

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- The conformal factor is then the one as expected:

$$e^u = \frac{4|f'|^2}{\rho(1+|f|^2)^2}.$$

- Given Λ, for $\rho = 4\pi l, l \in \mathbb{N}$, by analytic continuing the f's among simply connected domains via $PSU(2), f$ is glued into a meromorphic function on $\mathbb{C}$. (Not yet on $T = \mathbb{C}/\Lambda$.)

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- Let $z = e^{2\pi i w} : \mathbb{H} \to \Delta^{\times}$ and let $F(w) = f(z) = f(e^{2\pi i w})$. Then

  $$F(w+1) = SF(w)$$

  for some $S \in PSU(2)$. Up to a conjugation, we may start with another f so that

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- Now let $\Psi(w) = e^{-2i\theta w}F(w)$. Then

$$\Psi(w+1) = e^{-2i\theta(w+1)}F(w+1) = e^{-2i\theta w}F(w) = \Psi(w).$$

Hence $\Psi(w)$ comes from a meromorphic function $\psi(z)$ on $\Delta^\times$.

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$$\int_{\Delta^\times} \frac{4|f'|^2}{(1+|f|^2)^2}\, dA = \rho \int_{\Delta^\times} e^u \, dA < \infty,$$

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- This implies that ψ is meromorphic on the whole Δ.
- For $\rho = 4\pi l$ with $l \in \mathbb{N}$, the asymptotic of u at $z = 0$ is given by

$$u(z) \sim 2l \log |z|$$

since $\rho/2\pi = 2l$.

- Let $n = \operatorname{ord}_{z=0} \psi \in \mathbb{Z}$ and $\psi = z^n g$. Then $f = z^a g$ with $a = n + \theta/\pi$ and

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- The asymptotic is then given by

$$u(z) \sim 2(|a| - 1) \log |z|.$$

  In particular, $|a| = l + 1 \in \mathbb{N}$, which forces $\theta = 0$ because $0 \leq \theta < \pi$. Moreover $f = z^{\pm(l+1)} g$ is meromorphic at $z = 0$.

- **First constraint from the double periodicity:**

$$f(z+\omega_1) = S_1 f, \quad f(z+\omega_2) = S_2 f$$

with $S_1 S_2 = \pm S_2 S_1$.

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  (1) Let $f(z)$ has a pole at z_0.

  If $z_0 \equiv 0 \pmod{\Lambda}$ then the order $r = l + 1$.
  If $z_0 \not\equiv 0 \pmod{\Lambda}$ then $r = 1$.

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(2) Let $f(z) = a_0 + a_r(z - z_0)^r + \cdots$ be regular at z_0.

If $z_0 \equiv 0 \pmod{\Lambda}$ then $r = l + 1$.
If $z_0 \not\equiv 0 \pmod{\Lambda}$ then $r = 1$.

- May assume that $S_1 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}$, $S_2 = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix}$, then

  $$f(z+\omega_1) = e^{2i\theta_1} f(z), \quad f(z+\omega_2) = S_2 f(z).$$

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- The essential object to consider is the logarithmic derivative

$$g(z) = (\log f(z))' = \frac{f'(z)}{f(z)}.$$

Any zero/pole of f gives a simple pole of g. The residue is $+1/-1$ outside Λ.

- **Type I (Topological) Solutions:**

$$f(z+\omega_1) = -f(z), \qquad f(z+\omega_2) = \frac{1}{f(z)}.$$

Then $g = (\log f)'$ is elliptic on $T' = \mathbb{C}/\Lambda'$, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ with

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  Then $g = (\log f)'$ is elliptic on $T' = \mathbb{C}/\Lambda'$, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ with

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- For $\rho = 4\pi l$, since g must have zeros, we get $f(z) = f(0) + a_{l+1}z^{l+1} + \cdots$ with $f(0) \neq 0$ and g has its only zeros at $z = 0, \omega_2 \bmod \Lambda'$, both of order l.

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- So g has $2l$ simple poles coming from $p_1, \ldots, p_l$ (simple zeros of f) and $q_1, \ldots, q_l$ (simple poles of f) on T'. May set

$$q_i \equiv p_i + \omega_2, \quad i = 1, \ldots, l.$$

The first condition forces that $\sum p_i \equiv \frac{1}{2}\omega_1 \pmod{\Lambda}$.

- Using elliptic functions on T' and the addition theorem,

$$\begin{aligned} g(z) &= \sum_{i=1}^{l} (\zeta(z-p_i) - \zeta(z-p_i-\omega_2)) + l\eta_2/2 \\ &= -\frac{1}{2}\sum_{i=1}^{l} \frac{\wp'(z-p_i)}{\wp(z-p_i)-e_2} \qquad (e_i := \wp(\tfrac{1}{2}\omega_i')). \end{aligned}$$

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- Then $0 = g(0) = g''(0) = g^{(4)}(0) = \cdots$ leads to that all odd symmetric function of slopes $s(p_i)$'s are zero. This leads to the evenness of solutions.

- The remaining condition $0 = g'(0) = g'''(0) = g^{(5)}(0) = \cdots$ leads to the polynomial equations of $\wp(p_i)$'s using the half period formula on $T' = \mathbb{C}/\mathbb{Z}\omega_2 + \mathbb{Z}2\omega_2$:

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(2) *For $\rho = 4\pi l$ with $l = 2k + 1$ ($k \geq 0$), f has simple zeros at $\omega_1/2$ and $\pm p_i$ for $i = 1, \ldots, k$.* **When $k = 0$ ($\rho = 4\pi$), $\exists!$ solution.**

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  (3) *The equation is algebraically completely integrable: For $x_i := \wp(p_i) - e_2$ and $\tilde{x}_i := \wp(q_i = p_i + \omega_2) - e_2$,*

$$\sum_{i=1}^{k} x_i^m - \sum_{i=1}^{k} \tilde{x}_i^m = c_m, \quad x_m \tilde{x}_m = \mu, \qquad m = 1, \ldots, k.$$

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  Then $l = 2k$ since $\sum \operatorname{res}_{p_i} g = \sum(\pm 1) = 0$.

- **Periods integrals.** Let L_1, L_2 be the fundamental 1-cycles. Then

$$F_i(p) := \int_{L_i} \Omega(\xi, p)\, d\xi,$$

where $p \not\equiv \frac{1}{2}\omega_i \pmod{\Lambda}$ and

$$\begin{aligned}\Omega(\xi, p) &= A\,\frac{\sigma^2(\xi)}{\sigma(\xi - p)\sigma(\xi + p)} = \frac{\wp'(p)}{\wp(\xi) - \wp(p)} \\ &= 2\zeta(p) - \zeta(p + \xi) - \zeta(p - \xi).\end{aligned}$$

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Lemma (Periods Integrals and Critical Points)

Let $p = t\omega_1 + s\omega_2$, then up to $4\pi i\mathbb{N}$,

$$F_1(p) = 2(\omega_1\zeta(p) - \eta_1 p) = 2(\zeta - t\eta_1 - s\eta_2)\omega_1 - 4\pi i s,$$
$$F_2(p) = 2(\omega_2\zeta(p) - \eta_2 p) = 2(\zeta - t\eta_1 - s\eta_2)\omega_2 + 4\pi i t.$$

- E.g. when $\rho = 8\pi$ ($l = 2$), $p_1 = p$, $p_2 = -p$, $g(z) = \Omega(z, p)$ and

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

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For $\rho = 8\pi$, the mean field equation $\triangle u + \rho e^u = \rho \delta_0$ on a flat torus has at most one solution **up to scaling**.

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  - We were unable to prove it from the critical point equation.

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- ## Theorem

For $\rho \in [4\pi, 8\pi]$, Let u be a solution of $\triangle u + \rho e^u = \rho\delta_0$, $u(-z) = u(z)$ in T (so $\int_T e^u = 1$.) Then the linearized equation at u:

$$\begin{cases} \triangle \varphi + \rho e^u \varphi = 0 \\ \varphi(z) = \varphi(-z) \end{cases} \quad \text{in } T$$

is non-degenerate, i.e. it has only trivial solution $\varphi \equiv 0$.

Sketch of the main idea:

Use $x = \wp(z)$ as two-fold covering map $T \to S^2 = \mathbb{C} \cup \{\infty\}$ and require $\wp$ being an isometry:

$$e^{u(z)}|dz|^2 = e^{v(x)}|dx|^2 = e^{v(x)}|\wp'(z)|^2|dz|^2.$$

Namely we set

$$v(x) := u(z) - \log|\wp'(z)|^2 \quad \text{and} \quad \psi(x) := \varphi(z).$$

There are four branch points on $\mathbb{C} \cup \{\infty\}$, $p_0 = \wp(0) = \infty$ and $p_j = e_j := \wp(\omega_j/2)$ for $j = 1, 2, 3$. Since $\wp'(z)^2 = 4\prod_{j=1}^3 (x - e_j)$, then

$$\begin{cases} \triangle v + \rho\, e^v = \sum_{j=1}^3 (-2\pi)\delta_{p_j} \\ \triangle \psi + \rho\, e^v \psi = 0 \end{cases} \quad \text{in } \mathbb{R}^2$$

At infinity let $y = 1/x$. The isometry reads as

$$e^{u(z)}|dz|^2 = e^{w(y)}|dy|^2 = e^{w(y)}\frac{|\wp'(z)|^2}{|\wp(z)|^4}|dz|^2,$$

$$w(y) = u(z) - \log\frac{|\wp'(z)|^2}{|\wp(z)|^4} \sim \left(\frac{\rho}{2\pi} - 2\right)\frac{1}{2}\log|y|.$$

Thus $\rho \geq 4\pi$ implies that p_0 is a singularity with non-negative α_0.

By replacing u by $u + \log\rho$ etc., we may (and will) replace the ρ in the left hand side by 1 for simplicity. The total measure on T and $\mathbb{R}^2$ are then given by

$$\int_T e^u\,dz = \rho \leq 8\pi \quad \text{and} \quad \int_{\mathbb{R}^2} e^v\,dx = \frac{\rho}{2} \leq 4\pi.$$

The proof is then reduced to:

Theorem (Symmetrization Lemma)

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and let v be a solution of

$$\triangle v + e^v = \sum_{j=1}^{N} \alpha_j \delta_{p_j}$$

in Ω. Suppose that the first eigenvalue of $\triangle + e^v$ is zero on Ω with φ the first eigenfunction. If the isoperimetric inequality with respect to $ds^2 = e^v |dx|^2$:

$$2\ell^2(\partial\omega) \geq m(\omega)(4\pi - m(\omega))$$

holds for all level domains $\omega = \{\varphi \geq t\}$ with $t \geq 0$, then

$$\int_\Omega e^v \, dx \geq 2\pi.$$

Moreover, the isoperimetric inequality holds if there is only one negative α_j and $\alpha_j = -1$.

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▸ Theorem (Moduli dependence**)

(1) *Let $\Omega_3 \subset \mathcal{M}_1 \cup \{\infty\} \cong S^2$ (resp. Ω_5) be the set of tori with 3 (resp. 5) critical points, then $\Omega_3 \cup \{\infty\}$ is closed containing $i\mathbb{R}$, Ω_5 is open containing the vertical line $[e^{\pi i/3}, i\infty)$.*

(2) *Both Ω_3 and Ω_5 are simply connected with $C := \partial\Omega_3 = \partial\Omega_5$ homeomorphic to S^1 containing ∞.*

(3) *Moreover, the extra critical points are split out from some half period point when the tori move from Ω_3 to Ω_5 across C.*