Quantum Leray–Hirsch

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A general framework to determine $g = 0$ GW invariants: From I to J.

Let $\tau = \sum_\mu \tau^\mu T_\mu \in H(X)$, $g_{\mu\nu} = (T_\mu, T_\nu)$, $T^\mu = \sum g^{\mu\nu} T_\nu$.

$$
\begin{aligned}
J^X(\tau, z^{-1}) &= 1 + \frac{\tau}{z} + \sum_{\beta \in NE(X), n, \mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau, \cdots, \tau \right\rangle_{0,n+1,\beta} \\
&= e^{\frac{\tau}{z}} + \sum_{\beta \neq 0, n, \mu} \frac{q^\beta}{n!} e^{\frac{\tau_1}{z} + (\tau_1.\beta)} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau_2, \cdots, \tau_2 \right\rangle_{0,n+1,\beta},
\end{aligned}
$$

where $\tau = \tau_1 + \tau_2$ with $\tau_1 \in H^2(X)$.

Witten's *dilaton, string, and topological recursion relation* in 2D gravity $\Longleftrightarrow$ Givental's symplectic space reformulation of GW theory.

Let $H := H(X)$, $\mathcal{H} := H[z, z^{-1}]]$, $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[\![z^{-1}]\!]$.
$\mathcal{H} \cong T^*\mathcal{H}_+$ gives a canonical symplectic structure on $\mathcal{H}$.

$$\mathbf{q}(z) = \sum_{\mu} \sum_{k=0}^{\infty} \mathbf{q}_k^{\mu} T_{\mu} z^k \in \mathcal{H}_+.$$

The natural coordinates on $\mathcal{H}_+$ are $\mathbf{t}(z) = \mathbf{q}(z) + 1z$ (dilaton shift), with $\mathbf{t}(\psi) = \sum_{\mu,k} t_k^{\mu} T_{\mu} \psi^k \in \mathcal{H}_+$ the general descendent insertion.

Let $F_0(\mathbf{t})$ be the generating function. The one form dF_0 gives a section of $\pi : \mathcal{H} \to \mathcal{H}_+$. Givental's *Lagrangian cone* $\mathcal{L}$ = the graph of dF_0.

The existence of $\mathbb{C}^*$ action on $\mathcal{L}$ is due to the dilaton equation $\sum \mathbf{q}_k^{\mu} \partial / \partial \mathbf{q}_k^{\mu} F_0 = 2F_0$. Thus $\mathcal{L}$ is a cone with vertex $\mathbf{q} = 0$.

$-zJ : H \to z\mathcal{H}_-$ is a section over $\tau \in H \cong -1z + H \subset \mathcal{H}_+$.

Let $L_{\mathbf{f}} = T_{\mathbf{f}}\mathcal{L}$ be the tangent space of $\mathcal{L}$ at $\mathbf{f} \in \mathcal{L}$ and $L_\tau = L_{(\tau, dF_0(\tau))}$.

(i) $zL \subset L$ and so $L/zL \cong \mathcal{H}_+/z\mathcal{H}_+ \cong H$ has rank $N := \dim H$.

(ii) $L \cap \mathcal{L} = zL$, considered as subspaces inside $\mathcal{H}$.

(iii) L is the tangent space at every $\mathbf{f} \in zL \subset \mathcal{L}$. Moreover, $T_{\mathbf{f}} = L$ implies that $\mathbf{f} \in zL$. Thus zL is the *ruling* of the cone.

(iv) The intersection of $\mathcal{L}$ and the affine space $-1z + z\mathcal{H}_-$ is parameterized by its image $-1z + H \cong H \ni \tau$ under π.

$$-zJ(\tau, -z^{-1}) = -1z + \tau + O(1/z)$$

is the function of τ whose graph is the intersection.

(v) The set of all directional derivatives $z\partial_\mu J = T_\mu + O(1/z)$ spans $L \cap z\mathcal{H}_- \cong L/zL$.

Let $R = \mathbb{C}[\widehat{NE(X)}]$ be the ground (Novikov) ring.

Denote $a = \sum q^{\beta} a_{\beta}(z) \in R\{z\}$ if $a_{\beta}(z) \in \mathbb{C}[z]$. All discussions are only as *formal germs* around the neighborhood of $\mathbf{t} = \mathbf{0}$ ($\mathbf{q} = -1z$).

Lemma

$z\nabla J = (z\partial_{\mu} J^{\nu})$ forms a matrix whose column vectors $z\partial_{\mu} J(\tau)$ generates the tangent space L_{τ} of the Lagrangian cone $\mathcal{L}$ as an $R\{z\}$-module.

In fact, by TRR, $z\nabla J$ is the fundamental solution matrix of the Dubrovin connection on $TH = H \times H$:

$$\nabla^{z} = d - \frac{1}{z} d\tau^{\mu} \otimes \sum_{\mu} T_{\mu} *_{\tau} .$$

Namely we have the quantum differential equation (QDE)

$$z\partial_{\mu} z\partial_{\nu} J = \sum \tilde{C}_{\mu\nu}^{\kappa}(\tau, q) z\partial_{\kappa} J.$$

Let $\bar{p} : X \to S$ be a smooth toric bundle with fiber divisor $D = \sum t^i D_i$. $H(X)$ is a free over $H(S)$ with finite generators $\{D^e := \prod_i D_i^{e_i}\}_{e \in \Lambda}$. Let $\bar{t} := \sum_s \bar{t}^s \bar{T}_s \in H(S)$. $H(X)$ has basis $\{T_{\mathbf{e}} = T_{(s,e)} = \bar{T}_s D^e\}_{\mathbf{e} \in \Lambda^+}$.

Denote by $\partial_{\bar{T}_s} \equiv \partial_{\bar{t}^s}$ the $\bar{T}_s$ directional derivative on $H(S)$,

$$\partial^{\mathbf{e}} = \partial^{(s,e)} := \partial_{\bar{t}^s} \prod_i \partial_{t^i}^{e_i},$$

and the *naive quantization*

$$\hat{T}_e \equiv \partial^{z\mathbf{e}} \equiv \partial^{z(s,e)} := z\partial_{\bar{t}^s} \prod_i z\partial_{t^i}^{e_i} = z^{|e|+1} \partial^{(s,e)}.$$

As usual, the $T_{\mathbf{e}}$ directional derivative on $H(X)$ is denoted by $\partial_{\mathbf{e}} = \partial_{T_{\mathbf{e}}}$. This is a special choice of basis T_μ (and ∂_μ) of $H(X)$.

$\partial^{z\mathbf{e}}$ and $z\partial_{\mathbf{e}}$ are very different, but they are also closely related.

Let $\bar{p} : X \to S$ be a split toric bundle quotient from $\bigoplus \mathscr{L}_\rho \to S$. The hypergeometric modification of J^S by the $\bar{p}$-fibration takes the form

$$I^X(\bar{t}, D, z, z^{-1}) := \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I^{X/S}_\beta (z, z^{-1}) J^S_{\beta_S}(\bar{t}, z^{-1}),$$

where $I^{X/S}_\beta = \prod_{\rho \in \triangle_1} 1 / \prod_{m=1}^{(D_\rho + \mathscr{L}_\rho).\beta} (D_\rho + \mathscr{L}_\rho + mz)$ comes from fiber localization, and the product is *directed* when $(D_\rho + \mathscr{L}_\rho).\beta \leq -1$.

In general positive z powers may occur in I^X. Nevertheless for each $\beta \in NE(X)$, the power of z in $I^{X/S}_\beta(z, z^{-1})$ is bounded above by a constant depending only on β. I is defined only on the subspace

$$\hat{t} := \bar{t} + D \in H(S) \oplus \bigoplus_i \mathbb{C} D_i \subset H(X).$$

Theorem (J. Brown 2009)

$(-z) I^X(\hat{t}, -z)$ lies in the Lagrangian cone $\mathcal{L}$ of X.

Definition (GMT)

For each $\hat{t}$, say $zI(\hat{t})$ lies in L_τ of $\mathcal{L}$. The correspondence

$$\hat{t} \mapsto \tau(\hat{t}) \in H(X) \otimes R$$

is called the *generalized mirror transformation*.

Proposition (BF)

(1) *The GMT:* $\tau = \tau(\hat{t})$ *satisfies* $\tau(\hat{t}, q = 0) = \hat{t}$.
(2) *Under the basis* $\{T_{\mathbf{e}}\}_{\mathbf{e} \in \Lambda^+}$, *there exists an invertible* $N \times N$ *matrix-valued formal series* $B(\tau, z)$, *the Birkhoff factorization, such that*

$$\left(\partial^{z\mathbf{e}} I(\hat{t}, z, z^{-1})\right) = \left(z \nabla J(\tau, z^{-1})\right) B(\tau, z),$$

where $(\partial^{z\mathbf{e}} I)$ *is the* $N \times N$ *matrix with* $\partial^{z\mathbf{e}} I$ *as column vectors. The first column vectors are* I *and* J *respectively (string equation).*

Proof

$zI \in \mathcal{L} \Rightarrow z\partial I \in T\mathcal{L} = L$. Then $z(z\partial)I \in zL \subset \mathcal{L}$ and so $z\partial(z\partial)I \in L$. Inductively, $\partial^{z\mathbf{e}} I \in L$. The factorization $(\partial^{z\mathbf{e}} I) = (z\nabla J)B(z)$ follows.

From $\hat{t} = \sum \bar{t}^s \bar{T}_s + \sum t^i D_i$, it is easy to see that

$$\partial^{z\mathbf{e}} e^{\hat{t}/z} = T_{\mathbf{e}} e^{\hat{t}/z}, \qquad z\partial_{\mathbf{e}} e^{t/z} = T_{\mathbf{e}} e^{t/z}.$$

Hence, modulo $NE(X)$, $\partial^{z\mathbf{e}} I(\hat{t}) \equiv T_{\mathbf{e}} e^{\hat{t}/z}$, $z\partial_{\mathbf{e}} J(\tau) \equiv T_{\mathbf{e}} e^{\tau/z}$.

To prove (1), modulo all q^β's we have

$$e^{\hat{t}/z} \equiv \sum_{\mathbf{e} \in \Lambda^+} B_{\mathbf{e},1}(z) T_{\mathbf{e}} e^{\tau(\hat{t})/z}.$$

Thus

$$e^{(\hat{t} - \tau(\hat{t}))/z} \equiv \sum_{\mathbf{e}} B_{\mathbf{e},1}(z) T_{\mathbf{e}},$$

which forces that $\tau(\hat{t}) \equiv \hat{t}$ and $B_{\mathbf{e},1}(z) \equiv \delta_{T_{\mathbf{e}},1}$. Then we also have $B(\tau, z) \equiv I_{N\times N}$. In particular B is invertible. This proves (2).

Theorem (BF/GMT)

There is a unique, recursively determined, scalar-valued differential operator

$$P(z) = \sum_{\mathbf{e}\in\Lambda^+} C_{\mathbf{e}} \partial^{z\mathbf{e}} = 1 + \sum_{\beta\in NE(X)\setminus\{0\}} q^\beta P_\beta(t^i, \bar{t}^s, z; z\partial_{t^i}, z\partial_{\bar{t}^s}),$$

with P_β polynomial in z, such that $P(z)I = 1 + O(1/z)$. Moreover,

$$J(\tau(\hat{t}), z^{-1}) = P(z)I(\hat{t}, z, z^{-1}),$$

with $\tau(\hat{t})$ being determined by the $1/z$ coefficient of the right-hand side.

Proof. We construct $P(z)$ by induction on $\beta \in NE(X)$. We set $P_\beta = 1$ for $\beta = 0$. Suppose that $P_{\beta'}$ has been constructed for all $\beta' < \beta$. We set $P_{<\beta}(z) = \sum_{\beta'<\beta} q^{\beta'} P_{\beta'}$. Let

$$A_1 = z^{k_1} q^\beta \sum_{\mathbf{e}\in\Lambda^+} f^{\mathbf{e}}(t^i, \bar{t}^s) T_{\mathbf{e}}$$

be the top z-power term in $P_{<\beta}(z)I$. If $k_1 < 0$ then we are done.

Otherwise we remove it via the "naive quantization"

$$\hat{A}_1 := z^{k_1} q^{\beta} \sum_{\mathbf{e} \in \Lambda^+} f^{\mathbf{e}}(t^i, \bar{t}^s) \partial^{z\mathbf{e}}.$$

In $(P_{<\beta}(z) - \hat{A}_1)I = P_{<\beta}(z)I - \hat{A}_1 I$, the term A_1 is removed since

$$\hat{A}_1 I(q=0) = \hat{A}_1 e^{\hat{t}/z} = A_1 e^{\hat{t}/z} = A_1 + A_1 O(1/z).$$

All the newly created terms have curve degree $q^{\beta''}$ with $\beta'' > \beta$ in $NE(X)$. Thus we keep on removing the new top z-power term A_2, which has $k_2 < k_1$. The process stops in k_1 steps and we define P_β by

$$q^{\beta} P_{\beta} = -\sum_{1 \le j \le k_1} \hat{A}_j.$$

By induction we get $P(z) = \sum_{\beta \in NE(X)} q^{\beta} P_{\beta}$ as expected.

Q: Is it possible to get explicit forms/analytic properties of P or B?

From now on we work with the *projective local model* of a split P^r flop $f : X \dashrightarrow X'$ with bundle data (S, F, F'), where

$$F = \bigoplus_{i=0}^{r} L_i \quad \text{and} \quad F' = \bigoplus_{i=0}^{r} L'_i.$$

The contraction $\psi : X \to \bar{X}$ has exceptional loci $\bar{\psi} : Z = P_S(F) \to S$ with $N = N_{Z/X} = \bar{\psi}^* F' \otimes \mathscr{O}_Z(-1)$. Similarly we have $Z' \subset X', N'$.

The local model $\bar{p} : X = P_Z(N \oplus \mathscr{O}) \xrightarrow{p} Z \xrightarrow{\bar{\psi}} S$ is a *double projective bundle*. Leray–Hirsch $\Longrightarrow$ for h, ξ being the *relative hyperplane classes*,

$$H(X) = H(S)[h, \xi] / (f_F, f_{N\oplus\mathscr{O}}),$$

where the Chern polynomials take the form (we identify L with $c_1(L)$)

$$f_F = \prod_{i=0}^{r} a_i := \prod (h + L_i), \qquad f_{N\oplus\mathscr{O}} = b_{r+1} \prod_{i=0}^{r} b_i := \xi \prod (\xi - h + L'_i).$$

The graph correspondence $\mathscr{F} = [\bar{\Gamma}_f] \in A(X \times X')$ induces an isomorphism $\mathscr{F} : H(X) \cong H(X')$ in the group level: for $\bar{t} \in H(S)$,

$$\mathscr{F}\bar{t}h^i\xi^j = \bar{t}(\mathscr{F}h)^i(\mathscr{F}\xi)^j = \bar{t}(\xi' - h')^i\xi'^j, \qquad i \leq r.$$

$\mathscr{F}$ also preserves the Poincaré pairing, but not the ring structure.

Theorem (LLW 2010)

$\mathscr{F}$ induces an isomorphism of quantum rings $QH(X) \cong QH(X')$ under analytic continuations in the Kähler moduli formally defined by

$$\mathscr{F}q^\beta = q^{\mathscr{F}\beta}, \qquad \beta \in NE(X).$$

Let γ, ℓ be the fiber line class in $X \to Z \to S$. Then $\mathscr{F}\gamma = \gamma' + \ell'$, but $\mathscr{F}\ell = -\ell' \notin NE(X')$. So analytic continuations are necessary.

Li–Ruan 2000 ($r = 1$, $\dim X = 3$), LLW 2006 (simple P^r flop in any dimension, $S = \mathrm{pt}$), LLW 2008 (simple flop, any $g \geq 0$).

Any $\beta \in A_1(X)$ is of the form $\beta = \beta_S + d\ell + d_2\gamma$ where $\beta_S \in A_1(S)$ is identified with its *canonical lift* in $A_1(Z)$ with $(\beta_S.h) = 0 = (\beta_S.\xi)$.

h, ξ are dual to ℓ, γ hence $\beta.h = d$, $\beta.\xi = d_2$.

Lemma (Minimal lift and I-minimal lift)

- *Given a primitive class $\beta_S \in NE(S)$, $\beta = \beta_S + d\ell + d_2\gamma \in NE(X)$ if and only if*
$$d \geq -\mu \quad \textit{and} \quad d_2 \geq -\nu,$$
where $\mu = \max_i\{(\beta_S.L_i)\}$, $\mu' = \max_i\{(\beta_S.L_i')\}$, and $\nu = \max\{\mu + \mu', 0\}$.
- *Consequently, β is $\mathscr{F}$-effective (i.e. $\beta \in NE(X)$ and $\mathscr{F}\beta \in NE(X')$) if and only if*
$$d + \mu \geq 0 \quad \textit{and} \quad d_2 - d + \mu' \geq 0.$$

We define $\beta \in NE(X)$ to be I-effective, resp. $\mathscr{F}I$-effective, by the above inequalities without assuming β_S to be primitive.

Back to the hypergeometric modification of $\bar{p} : X \to S$:

$$I^X = I(\hat{t}; z, z^{-1}) = \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S} J_{\beta_S}^S(\bar{t}),$$

where $D = t^1 h + t^2 \xi$ is the fiber divisor and $\bar{t} \in H(S)$.

For a split projective bundle $\bar{\psi} : P = P(F) \to S$, $F = \bigoplus_{i=0}^r L_i$

$$I_\beta^{P/S} = \prod_{i=0}^r \frac{1}{\prod_{m=1}^{\beta.(h+L_i)} (h + L_i + mz)}; \qquad \prod_{m=1}^s := \prod_{m=-\infty}^s / \prod_{m=-\infty}^0 .$$

The product in $m \in \mathbb{Z}$ is directed so that for each i with $\beta.(h + L_i) \leq -1$, the subfactor is in the numerator containing $h + L_i$ (corresponding to $m = 0$). Hence $I_\beta^{P/S} = 0$ if $d + \mu < 0$.

Remark: The relative factor comes from the equivariant Euler class of $H^0(C, T_{P/S}|_C) - H^1(C, T_{P/S}|_C)$ at the moduli point $[C \cong P^1 \to X]$. It counts only the contribution from β_S in *generic positions*.

Now $I_\beta^{X/S} = I_\beta^{Z/S} I_\beta^{X/Z}$ is given by

$$\prod_{i=0}^{r} \frac{1}{\prod_{m=1}^{\beta.a_i}(a_i + mz)} \prod_{i=0}^{r} \frac{1}{\prod_{m=1}^{\beta.b_i}(b_i + mz)} \frac{1}{\prod_{m=1}^{\beta.\xi}(\xi + mz)}.$$

(Recall $a_i = h + L_i$, $b_i = \xi - h + L_i'$.) Although $I_\beta^{X/S}$ makes sense for any $\beta \in N_1(X)$, it is non-trivial only if $\beta \in NE^I(X)$.

Proposition (Picard–Fuchs system on X/S)

$\Box_\ell I^X = 0$ and $\Box_\gamma I^X = 0$, where

$$\Box_\ell = \prod_{j=0}^{r} z\partial_{a_j} - q^\ell e^{t^1} \prod_{j=0}^{r} z\partial_{b_j}, \qquad \Box_\gamma = z\partial_\xi \prod_{j=0}^{r} z\partial_{b_j} - q^\gamma e^{t^2}.$$

Here ∂_v is the directional derivative: $v = \sum v^i T_i \in H^2 \Rightarrow \partial_v = \sum v^i \partial_{t^i}$.

Similarly $I^{X'}$ is a solution to

$$\Box_{\ell'} = \prod_{j=0}^{r} z\partial_{a'_j} - q^{\ell'} e^{-t^1} \prod_{j=0}^{r} z\partial_{b'_j}, \qquad \Box_{\gamma'} = z\partial_{\xi'} \prod_{j=0}^{r} z\partial_{b'_j} - q^{\gamma'} e^{t^2+t^1},$$

where the dual coordinates of h' and ξ' are $-t^1$ and $t^2 + t^1$ (since $\mathscr{F}(t^1 h + t^2 \xi) = t^1(\xi' - h') + t^2\xi' = (-t^1)h' + (t^2 + t^1)\xi'$).

Proposition ($\mathscr{F}$-invariance of PF ideal)

$$\mathscr{F}\langle \Box_\ell^X, \Box_\gamma^X \rangle \cong \langle \Box_{\ell'}^{X'}, \Box_{\gamma'}^{X'} \rangle.$$

Proof. Since $\mathscr{F} a_j = \mathscr{F}(h + L_i) = \xi' - h' + L_i = b'_j$ and $\mathscr{F} b_j = a'_j$ for $0 \le j \le r$. It is clear that

$$\mathscr{F}\Box_\ell = -q^{-\ell'} e^{t^1} \Box_{\ell'},$$

and $\mathscr{F}\Box_\gamma = z\partial_{\xi'} \prod_{j=0}^{r} z\partial_{a'_j} - q^{\gamma'+\ell'} e^{t^2} = z\partial_{\xi'} \Box_{\ell'} + q^{\ell'} e^{-t^1} \Box_{\gamma'}$.

The Picard–Fuchs system on X and X' are indeed equivalent under $\mathscr{F}$. Both $I = I^X$ and $I' = I^{X'}$ satisfy this system, but in different coordinate charts "$|q^\ell| < 1$" and "$|q^\ell| > 1$" on the Kähler moduli.

However, I and I' are not the same solution under analytic continuations. Nor do J and J', since the general descendent invariants are not $\mathscr{F}$-invariant. Nevertheless we will see that $B(z)$ and $\tau(\hat{t})$, hence $*_t$, are correct objects to admit $\mathscr{F}$-invariance.

By QDE, the cyclic $\mathscr{D}$ module $\mathscr{M}_J = \mathscr{D}J$ is *holonomic* of length $N = \dim H$ with basis $z\partial_\mu J$. For $\mathscr{M}_I = \mathscr{D}^z I$. The BF/GMT $(\partial^{z\mathbf{e}} I) = (z\nabla J)B$ implies that $\mathscr{M}_I$ is also holonomic of length N.

The idea is to go backward: To find $\mathscr{M}_I$ first and then transform it to $\mathscr{M}_J$. While the derivatives along the fiber directions are determined by the PF, we still need to control derivatives along the base direction.

Write $\bar{t} = \sum \bar{t}^i \bar{T}_i$. This is achieved by *lifting* the QDE on $QH(S)$

$$z\partial_i z\partial_j J^S = \sum_k \bar{C}_{ij}^k(\bar{t})\, z\partial_k J^S$$

to $H(X)$. Write $\beta_S \equiv \bar{\beta}$ and $\bar{C}_{ij}^k(\bar{t}, \bar{q}) = \sum_{\bar{\beta} \in NE(S)} \bar{C}_{ij,\bar{\beta}}^k(\bar{t})\, q^{\bar{\beta}}$, then

$$z\partial_i z\partial_j J^S_{\bar{\beta}} = \sum_{k,\bar{\beta}_1} \bar{C}^k_{ij,\bar{\beta}_1}\, z\partial_k J^S_{\bar{\beta}-\bar{\beta}_1}.$$

For $\bar{\beta} \in NE(S)$, its I-minimal lift in $NE(X)$ is denoted by $\bar{\beta}^I$. Then

$$\begin{aligned} z\partial_i z\partial_j I &= \sum_\beta q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S} z\partial_i z\partial_j J^S_{\bar{\beta}} \\ &= \sum_{k,\beta,\bar{\beta}_1} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S} \bar{C}^k_{ij,\bar{\beta}_1}\, z\partial_k J^S_{\bar{\beta}-\bar{\beta}_1} \\ &= \sum_{k,\bar{\beta}_1} q^{\bar{\beta}_1^I} e^{D.\bar{\beta}_1^I} \bar{C}^k_{ij,\bar{\beta}_1} z\partial_k \sum_\beta q^{\beta-\bar{\beta}_1^I} e^{\frac{D}{z} + D.(\beta-\bar{\beta}_1^I)} I_\beta^{X/S} J^S_{\bar{\beta}-\bar{\beta}_1}. \end{aligned}$$

Theorem (Quantum Leray–Hirsch)

(1) *(I-Lifting) The QDE on $QH(S)$ can be lifted to $H(X)$ as*

$$z\partial_i\, z\partial_j I = \sum_{k,\bar\beta} q^{\bar\beta^I} e^{(D.\bar\beta^I)} \bar C^k_{ij,\bar\beta}(\bar t)\, z\partial_k D_{\bar\beta^I}(z) I,$$

where $D_{\bar\beta^I}(z)$ is an operator depending only on $\bar\beta^I$. Any other lifting is related to it modulo the Picard–Fuchs system.

(2) *Together with the Picard–Fuchs $\Box_\ell$ and $\Box_\gamma$, they determine a first order matrix system under the naive quantization basis:*

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a(z,q), \qquad \text{where } t^a = t^1, t^2 \text{ or } \bar t^i.$$

(3) *For $\bar\beta \in NE(S)$, its coefficients in C_a are polynomial in $q^\gamma e^{t^2}$, $q^\ell e^{t^1}$ and $\mathbf{f}(q^\ell e^{t^1})$, and formal in $\bar t$. Here $\mathbf{f}(q) := q/(1-(-1)^{r+1}q)$ is the "origin of analytic continuation" satisfying $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$.*

(4) *The system is $\mathscr{F}$-invariant, though in general $\mathscr{F}\bar\beta^I \neq \bar\beta^{I'}$.*

Ideas involved in the proof of (2) and (3).

The Picard–Fuchs system generated by $\Box_\ell$ and $\Box_\gamma$ is a perturbation of the Picard–Fuchs (hypergeometric) system associated to the (toric) fiber by operators in base divisors.

The fiberwise toric case is a GKZ system, which by the theorem of Gelfand–Kapranov–Zelevinsky is a holonomic system of rank $(r+1)(r+2)$, the dimension of cohomology space of a fiber. It is also known that the GKZ system admits a Gröbner basis reduction to the holonomic system.

We apply this result in the following manner: We would like to construct a $\mathscr{D}$ module with basis $\partial^{z\mathbf{e}}$, $\mathbf{e} \in \Lambda^+$. We apply operators $z\partial_{t^1}, z\partial_{t^2}$ and first order operators $z\partial_i$'s to this selected basis.

Notice that

$$\Box_\ell = (1-(-1)^{r+1}q^\ell e^{t^1})(z\partial_{t^1})^{r+1} + \cdots ,$$
$$\Box_\gamma = (z\partial_{t^2})^{r+2} + \cdots .$$

This is where **f** appears. The Gröbner basis reduction allows one to reduce the differentiation order in $z\partial_{t^1}$ and $z\partial_{t^2}$ to smaller one. In the process higher order differentiation in $z\partial_i$'s will be introduced.

Using part (1), the *I*-lifting, the differentiation in the base direction with order higher than one can be reduced to one by introducing more terms with strictly larger effective classes in $NE(S)$.

A careful induction will conclude the proof. In fact in the current special case coming from ordinary flops, neither the GKZ theorem nor the Gröbner basis were needed.

Finally we will construct a gauge transformation B to eliminate all the z dependence of C_a in the $\mathscr{F}$-invariant system

$$z\partial_a(\partial^{z\mathbf{e}}I) = (\partial^{z\mathbf{e}}I)C_a. \tag{1}$$

B is nothing more than the Birkhoff factorization matrix

$$\partial^{z\mathbf{e}}I(\hat{t}) = (z\nabla J)(\tau)B(\tau) \tag{2}$$

valid at the generalized mirror point $\tau = \tau(\hat{t})$. Substituting (2) into (1), we get

$$z\partial_a(\nabla J)B + z(\nabla J)\partial_a B = (\nabla J)BC_a,$$

hence

$$z\partial_a(\nabla J) = (\nabla J)(-z\partial_a B + BC_a)B^{-1} =: (\nabla J)\tilde{C}_a. \tag{3}$$

We must notice the subtlety in the meaning of $\tilde{C}_a(\hat{t})$.

Let $\tau = \sum \tau^\mu T_\mu$. Write the QDE as

$$z\partial_\mu (\nabla J)(\tau) = (\nabla J)(\tau)\tilde{C}_\mu(\tau),$$

then

$$z\partial_a (\nabla J) = \sum_\mu \frac{\partial \tau^\mu}{\partial t^a} z\partial_\mu (\nabla J) = (\nabla J) \sum_\mu \tilde{C}_\mu \frac{\partial \tau^\mu}{\partial t^a},$$

hence

$$\tilde{C}_a(\hat{t}) \equiv \sum_\mu \tilde{C}_\mu(\tau(\hat{t})) \frac{\partial \tau^\mu}{\partial t^a}(\hat{t}). \tag{4}$$

In particular, $\tilde{C}_a$ is independent of z. And (3) is equivalent to

$$\tilde{C}_a = B_0 C_{a;0} B_0^{-1} \tag{5}$$

($B_0^{-1} := (B^{-1})_0$, coefficient matrix of z^0) and the cancellation equation

$$z\partial_a B = BC_a - B_0 C_{a;0} B_0^{-1} B. \tag{6}$$

Analyze $B = B(z)$ by induction on $w := (\bar{\beta}, d_2) \in W$. The initial condition is the extremal ray case $B_{w=(0,0)} = \mathrm{Id}$.
Suppose that $B_{w'}$ satisfies $\mathscr{F} B_{w'} = B'_{w'}$ for all $w' < w$. Then

$$z\partial_a B_w = \sum_{w_1+w_2=w} B_{w_1} C_{a;w_2} - \sum_{w_1+w_2+w_3+w_4=w} B_{w_1,0} C_{a;w_2,0} B^{-1}_{w_3,0} B_{w_4}.$$

Write $B_w = \sum_{j=0}^{n(w)} B_{w,j}\, z^j$. Then in the RHS all the B terms have strictly smaller degree than w except

$$B_w C_{a;(0,0)} - C_{a;(0,0)} B_w + B_{w,0} C_{a;(0,0)} - C_{a;(0,0)} B^{-1}_{w,0}$$

which has maximal z degree $\leq n(w)$. By descending induction on j, the z degree, we get

$$\partial_a (\mathscr{F} B_{w,j} - B'_{w,j}) = 0.$$

The functions involved are all formal in $\bar{t}$ and analytic in t^1, t^2, and without constant term ($B_{w=(0,0)} = \mathrm{Id}$). Hence $\mathscr{F} B_{w,j} = B'_{w,j}$. Done.

We have proved that for any $\hat{t} = \bar{t} + D \in H(S) \oplus \mathbb{C}h \oplus \mathbb{C}\xi$,

$$\mathscr{F}B(\tau(\hat{t})) \cong B'(\tau'(\hat{t})),$$

hence the $\mathscr{F}$-invariance of $\tilde{C}_a(\hat{t}) = B_0 C_{a;0} B_0^{-1}$: Explicitly

$$\tilde{C}_{a\nu}^{\kappa} = \sum_{n \geq 0, \mu} \frac{q^{\beta}}{n!} \frac{\partial \tau^{\mu}(\hat{t})}{\partial t^a} \langle T_{\mu}, T_{\nu}, T^{\kappa}, \tau(\hat{t})^n \rangle_{\beta}.$$

The case $T_{\nu} = 1$ leads to non-trivial invariants only for 3-point classical invariant ($n = 0$) and $\beta = 0$, and also $\mu = \kappa$. Since κ is arbitrary, we have thus proved the $\mathscr{F}$-invariance of $\partial_a \tau$. Then

$$\partial_a(\mathscr{F}\tau - \tau') = \mathscr{F}\partial_a \tau - \partial_a \tau' = 0.$$

Again since $\tau(\hat{t}) = \hat{t}$ for $(\bar{\beta}, d_2) = (0, 0)$, this proves

$$\mathscr{F}\tau = \tau'.$$

END