Analytic continuations of
QH under ordinary flops

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at HDAG

Given $F, F' \to S$ sm. proj $/\mathbb{C}$. vb of rk $= r+1$

$$\begin{tikzcd} \mathcal{O}(-1,-1) \arrow[r] & E = Z \times_S Z' \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\bar{\varphi}'"] & \\ N \arrow[r] & Z = P(F) \arrow[dr, "\bar{\psi}"'] & Z' = P(F') \arrow[dl, "\bar{\psi}'"] \\ & S & \end{tikzcd}$$

$N := \bar{\varphi}^* F' \otimes \mathcal{O}_Z(-1)$

Def'n: A flop $\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\psi"'] & & X' \arrow[dl, "\psi'"] \\ & \bar{X} & \end{tikzcd}$ with such local str on exceptional loci is a P^r-flop of type (S, F, F')

Known: $\bar{\Gamma}_f \in A(X \times X')$ identifies Chow motives ✓ & poincaré pairing but not cup prod.

Gromov-Witten: $F_g^X(t) = \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}^X \sim \int_{[\bar{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t$

$t \in H(X)$, $e_i : \bar{M}_{g,n}(X,\beta) \longrightarrow X$.

formal function in $t \in H(X) \cong H(X')$ under $\mathcal{F} := [\bar{\Gamma}_f]_*$

but diff $\beta \in NE(X)$ since $\mathcal{F}\ell = -\ell'$ on extremal rays.

Def'n: Analytic conti under $\mathcal{F}$ (or $\mathcal{F}$-inv.) is defined on expressions, rational in q^ℓ, st. $\mathcal{F} F(q^\beta) = F'(q^{\mathcal{F}\beta})$.

Example: $f(q) := \dfrac{q}{1-(-1)^{r+1} q} = q + (-1)^{r+1} q^2 + \cdots$

then $f(q) + f(q^{-1}) = (-1)^r$. $\left(\text{ie. } E(q) := \sum_{d=-\infty}^{\infty} q^d \sim 0\right)$

Quantum product: let $t = \sum t^i T_i$, $\{T_i\}$ basis of $H = H(X)$, dual basis T^i

$$T_i *_t T_j = \sum_k \partial^3_{ijk} F_0^X(t)\, T^k = \sum_{n,\beta} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle^X_{0,n+3,\beta} T^k$$

$t=0, \beta=0 \to$ cup product

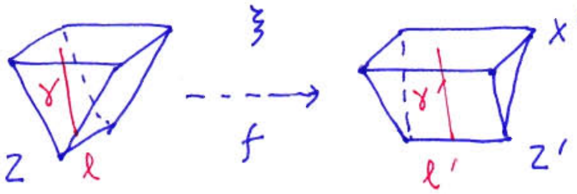
Thm (LLW): For split P^r flops, the q-prod is $\mathcal{F}$-inv.
not needed for quantum corrections.

History: Witten (1991) local P^1 flop (C-Y 3folds)
A. Li & Y. Ruan (2000) global 3-folds via degeneration
Lee-Lin-W arXiv:math/0608370, simple P^r flop
$S = pt$.
$\exists$ reduction to proj local models.
Also all $g \geq 0$ $(S = pt)$ ILLW arXiv:0804.3816.

Local model : "double" proj bundle, has $\mathbb{C}^\times$ action

$$\begin{tikzcd} P^{r+1} \arrow[r] & X = P_Z(N \oplus \mathcal{O}) \arrow[d, "p"] & \xi := c_1(\mathcal{O}_X(1)) \quad X \supset E \\ P^r \arrow[r] & Z = P_S(F) \arrow[d, "\bar\Psi"] & h := c_1(\mathcal{O}_Z(1)) \\ & S & \end{tikzcd}$$

γ $\quad \ell$



$F = \oplus_{i=0}^r L_i$, $F' = \oplus_{i=0}^r L_i'$; $\prod_{i=0}^r (h + L_i) = 0 = \xi . \prod_{i=0}^r (\xi - h + L_i')$

Big J function (1-descendent gen fcn.) $J(\tau) := 1 + \frac{\tau}{z} + \sum_{\beta, n, i} \frac{q^\beta}{n!} T_i \langle \frac{T^i}{z(z-\psi)}, \tau^n \rangle_{0,n+1,\beta}$

$$= e^{\frac{\tau}{z}} + \sum_{\beta \neq 0, n, i} e^{\frac{\tau_0 + \tau_1}{z}} \frac{e^{(\tau_1 . \beta)} q^\beta}{n!} T_i \langle \frac{T^i}{z(z-\psi)}, \tau_*^n \rangle_{0,n+1,\beta}$$

here $\tau = \tau_0 + \tau_1 + \tau_* \in H^0 \oplus H^2 \oplus H^{\text{other}}$

Givental translates Witten's (SE + DE + TRR) into Lagrangian cone $\mathcal{L}$.

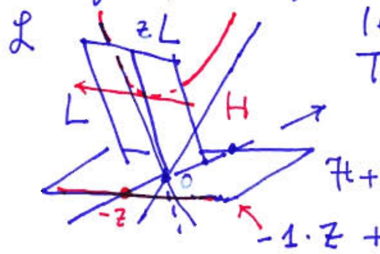
$\mathcal{H} := H[z, z^{-1}]] \simeq T^* \mathcal{H}_+$ when $\mathcal{H}_+ = H[z]$, $\mathcal{H}_- = z^{-1} H[[z^{-1}]]$

$$\begin{tikzcd} T^*\mathcal{H}_+ \arrow[d, "\pi"] \\ \mathcal{H}_+ \arrow[u, bend right, "d\mathcal{D}_0"'] \end{tikzcd}$$

(formal loop space)

$\mathcal{D}_0$ descendent potential

$\mathcal{L} :=$ graph of $d\mathcal{D}_0$



$-1 \cdot z + H \simeq H \ni \tau$

let $L_a = T_a \mathcal{L}$, on $\mathbf{t} = \sum t_k^i T_i z^k$ $\leftarrow \psi$

Then (1) $zL \subset L$, $L/zL \simeq H$

(2) $L \cap \mathcal{L} = zL$ the ruling of $\mathcal{L}$

(3) $z J(\tau)$ generates $\mathcal{L}$.

(4) $\langle \partial_\mu J \rangle \subset L$, which $\xrightarrow{\sim} L/zL$

Hypergeometric modification:

$$\tilde{I} = \tilde{I}(t^h, t^\xi, t^S, z, z^{-1}) = e^{\frac{t^h h + t^\xi \xi}{z}} \sum_{\substack{\beta \in NE(X) \\ \beta_S + d\ell + d_2 \gamma = \beta}} q^\beta e^{d t^h + d_2 t^\xi} I_\beta^{X/S} . \bar{p}^* J_{\beta_S}^S (t^S)$$

$\bar{p} : X \to S$

where $$I_\beta^{X/S} = \prod_{i=0}^r \frac{1}{\prod_{m=0}^{\beta(h+L_i)}{}^{+} (h + L_i + mz)} \frac{1}{\prod_{m=0}^{\beta(\xi - h + L_i')}{}^{+} (\xi - h + L_i' + mz)} \cdot \frac{1}{\prod_{m=0}^{d_2}{}^{+} (\xi + mz)}$$

in the sense of directed product. r

Brown: $\tilde{I}$ lies in $\mathcal{L}$.

i.e. $\exists B(z)$ st. $\tilde{I}(z, z^{-1}, t) = B(z) \, z \nabla J(z^{-1}, \tau)$ for some $\tau(t)$.

$B(z) = 1 + \cdots$

BF poly in z in NE(X)-top. GMT

Prop: $J(\tau) = \sum_{i,j,k} C_{ijk}(z, q) z\partial_h^i z\partial_\xi^j z \nabla_{t_k^S} \tilde{I}(t) = \tilde{I} + \cdots$ for $\tau = \tau(t)$ determined by $\frac{1}{z}$.

idea of pf: Q diff eq'n $\Rightarrow z\partial_h^i z\partial_\xi^j z \nabla_{t_k^S} \sim z \nabla_{h^i \xi^j t_k^S}$.

General Plan about the pf of Thm:

(1) By

$$\begin{tikzcd}
N_1(X) \arrow[rr, "\mathfrak{F}"] \arrow[dr, "\bar{p}_* \oplus d_2"'] & \circlearrowright & N_1(X') \arrow[dl, "\bar{p}'^* \oplus d_2"] \\
& N_1(S) \oplus \mathbb{Z} &
\end{tikzcd}$$

Analytic conti on a generating series $\langle A \rangle$ is equiv. to analytic conti of its fiber series $\langle A \rangle_{\beta_2, d_2}$, sum over $d\ell$ with $\beta \in NE(X)$.

(2) For extremal series $\beta_2 = 0$, $d_2 = 0$,

$$\tilde{I} = e^{\frac{t^h h + t^\xi \xi}{z}} \sum_{d=1}^{\infty} q^{d\ell} e^{d t^h} \prod_{i=0}^{r} \frac{\prod_{m=-d+}^{0+} (\xi - h + L_i' + mz)}{\prod_{m=0^+}^{d+} (h + L_i + mz)} \quad \frac{\left((-1)^{d-1}(d-1)!\right)^{r+1}}{z^{r+1}(d!)^{r+1}} \prod_{i=0}^{r} (\xi - h + L_i') \times \left(1 + \frac{*}{z} \cdots\right)$$

$$\Rightarrow \tilde{I}(t) = J(\tau) \text{ and } \tau = t \qquad = \frac{(-1)^{(r+1)(d-1)} \textcircled{H}_{r+1}}{\underline{\underline{z^{r+1}}}\, d^{r+1}} \left(1 + O(z^{-1})\right)$$

restricted to $t = t^h h + t^\xi \xi$.

- No BF & GMT are needed.
- $\sum_{d=1}^{\infty} \pm \frac{q^d}{d^{r+1}}$ is NOT $\mathfrak{F}$-inv. Need to go to $\boxed{h = \ell_k + \text{3-pt}}$ fcn's by Reconstruction and get $\sum_{d=1}^{\infty} d^k q^d$ which corrects $(k=0)$ the constant term given by cup product.

(*) $\boxed{\text{Fact: for } P(d) \text{ poly},\ \sum P(d) q^d = P(\underset{\overset{\shortparallel}{q\frac{d}{dq}}}{\delta}) \sum q^d.}$

(3) For non-extremal series, may have $\tilde{I}(\underline{\underline{z}}, z^{-1})$.

- Notice the formal symmetry between $I^{X/S}$ and $I^{X'/S}$ in particular $\mathfrak{F}(I^{X/S} . \xi) = I^{X'/S} . \xi'$

Rmk: The highest z-power is $z^{-\lambda}$
$\lambda = (c_1(F) + c_1(F')).\beta_S + d_2(r+2)$
if $c_1 + c_1' \geq 0$ (say $F' = F^*$)
then No BF/GMT.

Thus $J(\tau).\xi a = c(z, q; z\partial)\, \tilde{I}(t).\xi a$

$$\overset{\shortparallel}{\sum_{\beta} *} \left\langle \frac{\xi a}{z(z-\psi)} \right\rangle_{\beta} + \left\langle \frac{\xi a}{z(z-\psi)}, \underline{\tau(t)} \right\rangle_{\beta_1 \ulcorner \beta} + \left\langle \frac{\xi a}{z(z-\psi)}, \underline{\tau(t)}, \underline{\tau(t)} \right\rangle_{\beta_2 \ulcorner \beta_1} + \cdots$$

$$\underline{\tau(t)} = \underset{\text{divisor}}{t} + q^{\beta_1'}(\cdots) \qquad \text{All } \beta_1, \beta_1' \neq 0.$$

By induction, the problem reduced to analytic conti of GMT: $\tau(t)$ & $\tau'(t)$ if $\exists\, \xi$ insertion.

Claim: Each fiber series has poly coeffi as (*).

Example 1. $(S, F, F') = (\mathbb{P}^1, \mathcal{O}\oplus\mathcal{O}(-4), \mathcal{O}\oplus\mathcal{O}(1))$

$\beta = d\ell + sb$, $b = [S]$ (so $d_2 = 0$)

$$d_n^{vir} = c_1(X).\beta + \dim X + n - 3 = -s + n + 1 \geq n$$

$s = 0$, div eq'n $\Rightarrow$ $\langle hp\rangle_0^X = \sum_{d\geq 1}\frac{1}{d^2} q^{d\ell}$

$s = 1$, div eq'n $\Rightarrow$ $\langle - \rangle_1^X = \langle p\rangle_1^X$

$$I_{1,d} = \text{"}\frac{\prod_{-d}^{0}(\xi - h + mz)\prod_{-d+1}^{0}(\xi - h + p + mz)}{\prod_{0}^{1}(p+mz)^2\prod_{0}^{d}(h+mz)\prod_{0}^{d-4}(h-4p+mz)}\text{"} \quad \boxed{\begin{array}{l}\text{directed}\\ \text{product}\\ \text{with } +\end{array}}$$

$$d \text{ large} \Rightarrow z\text{-deg} = (d - \underline{\underline{1}}) + (d - 1 - \underline{\underline{1}}) - 2 - d - (d-4) = \underline{-1 \text{ (good)}}$$

but $d = 0$ $\Rightarrow$

$$I_{1,0} = \frac{\prod_{-4}^{0}(h-4p+mz)}{\prod_{0}^{1}(p+mz)^2\prod_{0}^{1}(\xi - h + p + mz)} = -6(h-4p) + O(z^{-1})$$

Birkhoff factorization & GMT

let $q = q^{\ell}$, $\bar{q} = q^{b}$, $t = t_0 + t_1 \in H^0 \oplus H^2$, $\tilde{I}(t) = e^{\frac{t}{z}} I$,

$$\Rightarrow \tilde{I}(t) + 6\bar{q}\, z\, \partial_{h-4p}\tilde{I}(t) = 1 + O(z^{-1})$$

$$J(\tau) = \quad =: e^{t/z}\left(1 + \frac{I_1^+}{z} + \frac{I_2^+}{z^2} + O(z^{-3})\right)$$

$\Rightarrow$ $\tau = t + I_1^+ \sim$ classes with deg ≥ 2.

Take away $e^{t/z}$

$$\Rightarrow e^{I_1^+/z} + \sum_{\beta\neq 0}\frac{q^{\beta} e^{(t_1,\beta)}}{n!} T^{\mu}\left\langle \frac{T_\mu}{z(z-\psi)}, I_1^+, \dots, I_1^+\right\rangle_{0,n+1,\beta}$$

$$= 1 + \frac{I_1^+}{z} + \frac{1}{z^2}\left(I_2 + 6\bar{q}\left((h-4p)I_2 + z\,\partial_{h-4p} I_3\right)\right).$$

Calculation:

$$\langle p\rangle_1^X = \bar{q}\sum_{d\geq 0} p.\overset{-1+\frac{6}{d^2}}{\overset{\shortparallel}{I_{1,d;2}}} + 6\bar{q}\sum_{d\geq 1}\left(hp.\overset{\frac{1}{d^2}}{\overset{\shortparallel}{I_{0,d;2}}} + dp.\overset{d\cdot\frac{-2}{d^3}}{\overset{\shortparallel}{I_{0,d;3}}}\right)$$

$$= \bar{q}(\underline{-6 - 6q} - \textstyle\sum_{d\geq 2} q^d) , \quad \Rightarrow \langle p\rangle_1^{X'} - f\langle p\rangle_1^X$$

$$\langle p\rangle_1^{X'} = \bar{\xi}(\underline{-5 - 5q'^{-1}} + \textstyle\sum_{d\geq 1} q'^d) . \qquad = \sum_{d\in\mathbb{Z}} q'^d = E(q').$$

f-ef range

WDVV reduction to get ξ insertion

e.g. 3 pt function $\langle a, b, t^i h^j \rangle_{\beta_S, d_2=0}$, $a, b \in H(Z)$

$\forall\, c, d \in H(X)$

assume analytic conti for $\beta_S' < \beta_S$ in $NE(S)$.

Apply WDVV to β over $(\beta_S, d_2 = 1)$:

$$\Sigma_{i,j} \langle a, b, \underline{t_i \xi^j} \rangle_{\beta_S, 0} \langle \underbrace{t_i^* H_{r-j} \Theta_{r+1}}_{\text{dual basis}}, \xi c, \xi d \rangle_{0, d_2=1} = R_{c,d}$$

all smaller β_S or $d_2 \neq 0$ moved to $\mathcal{J}$-inv.

intend to form a $N \times N$ invertible system,

$N = \dim H(S) \cdot (r+1)$,

$$d^{vir}_{\substack{2nd\\ \text{serie}}} = \underset{1}{d_2}(r+2) + (2r+1+s) + (\overset{3}{n} - 3)$$

Choice of (c, d): $c = c_{k,\ell} = t_k \xi^\ell$, $d = h^r$

$\{c_{k,\ell}\}$ partial ordered by $|t_k|$, then ℓ.

$\langle t_i^* H_{r-j} \Theta_{r+1}, t_k \xi^{\ell+1}, \xi h^r \rangle_{0,1} \neq 0$ ONLY if $|t_k| + \ell = |t_i| + j$

Notice the bundle structure:
$$\begin{tikzcd} \overline{M}_{0,n} \text{(simple case)} \arrow[r] & \overline{M}_{0,n}(X, \beta) \arrow[d] \\ & S \end{tikzcd}$$
$(\text{for } \beta = \underset{=0}{\beta_S} + d\ell + d_2 \gamma)$

- $|t_k| > |t_i| \Rightarrow |t_i^*| + |t_k| > s \Rightarrow$ inv. $= 0$
- $|t_k| = |t_i| \Rightarrow \ell = j$ & $t_k = t_i$ to avoid trivial inv.
- $|t_k| < |t_i|$: upper triangular region (OK.)

Calculation: For diagonal fiber serie:

$$\langle t_i^* H_{r-j} \Theta_{r+1}, t_i \xi^{j+1}, \xi h^r \rangle_{0,1} = \langle h^{r-j} (\xi - h)^{r+1}, \xi^{j+1}, \xi h^r \rangle^{\text{simple}}_{d_2=1} = \begin{cases} (-1)^j q \bar{q} & 0 \le j \le r-1 \\ (1 - (-1)^{r+1} q) \bar{q}, & j \end{cases}$$

All $\neq 0$.

This generalizes to $n \geq 3$ pt functions.

Conclusion: $\mathcal{J}$-inv of GW is reduced to $\mathcal{J}$-inv of GMT.

i.e. analytic conti