Quantum Invariance under Flops and Transitions

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August 29, 2008 at Harvard University
In celebration of Professor Yau's 60 birthday

In 1994, Yau suggested the study of finite distance boundary points of the moduli space of Calabi-Yau manifolds, with respect to the Weil-Petersson metric:

$$\omega_{WP} = -\partial\bar{\partial} \log \sqrt{-1}^{n} \int_{X_s} \Omega(s) \wedge \overline{\Omega(s)}.$$

Candelas et. al. 1990: Conifold (ODP) degenerations of Calabi-Yau 3-folds are at finite WP distance (by way of explicit calculations).

—, 1995; MRL 1997:

Schmid's Nilpotent Orbit Theorem $\Longrightarrow$ A CY degeneration $\mathfrak{X} \to \Delta$ is at finite WP distance iff $NF^n_\infty = 0$.

Clemens-Schmid exact sequence $\Longrightarrow$ For a semi-stable CY degeneration $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = \bigcup_{i=0}^m X_i$, $NF^n_\infty = 0$ iff there is a component X_0 with $h^{n,0} \equiv h^0(K) \neq 0$.

Corollary: Degenerations of CY acquiring only canonical singularities are at finite WP distance.

—, MRL 2003: Assuming the MMP in dimension $n+1$, then the converse holds in dimension n. In particular, for $\mathfrak{X} \to \Delta$ being a finite distance degeneration of CY 3-folds, there exists another birational model $\mathfrak{X}' \to \Delta$ such that $\mathfrak{X}'_0$ is a CY with at most canonical singularities.

Interesting Geometries occur at finite WP distance:

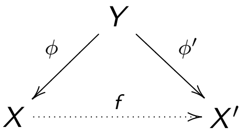
Extremal transitions: $Y \mapsto X$:

$$\begin{array}{l} Y \\ \downarrow \psi \\ W \overset{i}{\rightsquigarrow} \mathfrak{X} \supset \mathfrak{X}_t = X \end{array}$$

where ψ is a crepant (K-equivalent) resolution and i is a smoothing of canonical singularities. Notice that there is a topology change from Y to X.

Flops: Different crepant resolutions Y and Y' of W are related by flops. $h^{p,q}(Y) = h^{p,q}(Y')$, but they are not homotopy equivalent and the classical cohomology rings are not isomorphic.

K-**equivalence**: For birational projective complex d-dimensional manifolds $f : X \dashrightarrow X'$, $X =_K X'$ if $\phi^* K_X = \phi'^* K_{X'}$ for some



eg. birational Calabi-Yau's or minimal models.

Conjecture: There exists $\mathfrak{F} = [\bar{\Gamma}_f] + \sum T_i \in A^d(X \times X')$ which gives isomorphism of Chow motives $[X] \cong [X']$. $\mathfrak{F}$ is orthogonal (preserving the Poincaré pairing) and

$$\mathfrak{F} : QH(X) \cong QH(X')$$

after an analytic continuation over the Kähler moduli. (In general X and X' are not even homotopy equivalent.)

Gromov-Witten invariants: For $\alpha \in H(X)^{\otimes n}$, $\beta \in H_2(X, \mathbb{Z})$

$$\langle \alpha \rangle_{g,n,\beta} = \int_{[M_{g,n}(X,\beta)]^{vir}} ev^* \alpha$$

with $ev = \prod e_i : M_{g,n}(X, \beta) \to X^n$ being the evaluation map.

Big quantum ring: Let $\{T_i\}$ be a basis of $H(X)$ and $t = \sum t_i T_i$,

$$F_g(t) := \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}.$$

The quantum product uses only $g = 0$. Let $\Phi = F_0$,

$$T_i *_t T_j = \sum_k \Phi_{ijk}(t) T^k = \sum_{k,n,\beta} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle_{0,n+3,\beta} T^k,$$

where $g_{ij} = (T_i, T_j)$, $T^j = g^{ij} T_i$ is the dual basis.

Kähler moduli: Let $\mathcal{K}_X^{\mathbb{C}} = H^{1,1}_{\mathbb{R}}(X) \times \mathcal{K}_X$ be the complexified Kähler cone and let $\omega = B + iH \in \mathcal{K}_X^{\mathbb{C}}$. Then

$$q^\beta = e^{2\pi i (\omega, \beta)}, \quad |q^\beta| = e^{-2\pi (H.\beta)} < 1.$$

It is conjectured that $\langle \alpha \rangle = \sum \langle \alpha \rangle_\beta q^\beta$ converges in $\omega \in \mathcal{K}_X^{\mathbb{C}}$.

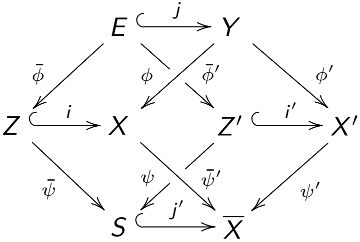
Analytic continuation: For $X =_K X'$ and $X \not\cong X'$, $H^2(X) \cong H^2(X')$ but $\mathcal{K}_X \cap \mathcal{K}_{X'} = \emptyset$ in H^2. If $\mathcal{F}$ preserves the Poincaré pairing, then $\mathcal{F}(T_i *_t T_j) = \mathcal{F}T_i *_{\mathcal{F}t} \mathcal{F}T_j$ is equivalent to

$$\Phi^X_{ijk}(\omega, t) = \Phi^{X'}_{ijk}(\mathcal{F}\omega, \mathcal{F}t).$$

up to analytic continuations in ω from $\mathcal{K}_X^{\mathbb{C}}$ to $\mathcal{K}_{X'}^{\mathbb{C}}$. Since ω and $\mathcal{F}\omega$ are canonically identified and $(\omega, \beta)_X = (\mathcal{F}\omega, \mathcal{F}\beta)_{X'}$, formally this means

$$q^\beta \mapsto q^{\mathcal{F}\beta}.$$

Ordinary $\mathbb{P}^r$ flops: Let F, F' be rank r bundles over S. It is a square



where $Z = \mathbb{P}_S(F)$, $Z' = \mathbb{P}_S(F')$ and $E = Z \times_S Z'$. Moreover

$$N_{E/Y} = \bar{\phi}^* \mathcal{O}_Z(-1) \otimes \bar{\phi}'^* \mathcal{O}_{Z'}(-1),$$

$$N_{Z/X} \cong \mathcal{O}_Z(-1) \otimes \bar{\psi}^* F'.$$

These are the simplest K equivalent maps $f : X \dashrightarrow X'$.

Theorem (Y.-P. Lee, H.-W. Lin, —; 2006–2008)

(1) *For $\mathbb{P}^r$ flops $f : X \dashrightarrow X'$, the graph closure $\mathcal{F} = [\bar{\Gamma}_f]$ induces canonical isomorphism of Chow motives.*

(2) *For simple $\mathbb{P}^r$ flops, the full Gromov-Witten theory in the stable range $2g + n \geq 3$ can be analytic continued to each other under the graph correspondence.*

(3) *For $\mathbb{P}^r$ flops, the Gromov-Witten theory in the stable range $2g + n \geq 3$ attached the the extremal rays are invariant up to analytic continuations.*

(4) *For $\mathbb{P}^r$ flops with split bundles $F = \bigoplus L_i$ and $F' = \bigoplus L'_i$, the big quantum cohomology rings are analytic continuations of each other under the graph correspondence.*

Genus zero theory: The Conjecture for 3-folds was previously solved by A. Li and Y. Ruan in 1998. 3 ingredients of their proof:

(1) Symplectic deformations and decompositions of K equivalent maps into $\mathbb{P}^1$ flops. (Kawamata, Kollár, Friedman.)

(2) Multiple cover formula for $\mathbb{P}^1 = C \subset X$, $N_{C/X} = \mathcal{O}(-1)^{\oplus 2}$:

$$\langle - \rangle_{0,dC}^X = \frac{1}{d^3}.$$

(Aspinwall-Morrison, Voisin, Lian-Liu-Yau.)
Witten 1992: The defect of classical cup product is corrected by the 3-point functions on C.

(3) Relative GW invariants and the degeneration formula. (Li-Ruan, Inoel-Parker, J. Li.) For $\beta \notin \mathbb{Z}[C]$,

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta} = \langle \mathcal{F}\alpha_1, \ldots, \mathcal{F}\alpha_n \rangle_{g,n,\mathcal{F}\beta}.$$

We make progresses on (2) and (3).

The defect of product structure:

$f : X \dashrightarrow X'$ a simple $\mathbb{P}^r$ flop, $S = \mathrm{pt}$, $\mathcal{F} = [\bar{\Gamma}_f]$,
$h =$ hyperplane class of $Z = \mathbb{P}^r$, $h' =$ hyperplane class of Z',
$\ell := [C] = h^{r-1}$ line class in Z (extremal ray) etc.. Then

$$\mathcal{F}[h^s] = (-1)^{r-s}[h'^s].$$

In particular $\mathcal{F}\ell = -\ell'$.

Lemma. *For $\alpha \in A^i(X)$, $\beta \in A^j(X)$, $\gamma \in A^k(X)$ with $i + j + k = \dim X = 2r + 1$,*

$$\mathcal{F}\alpha.\mathcal{F}\beta.\mathcal{F}\gamma = \alpha.\beta.\gamma + (-1)^r(\alpha.h^{r-i})(\beta.h^{r-j})(\gamma.h^{r-k}).$$

Quantum corrections attached to the extremal rays:

$$\dim\,[\overline{M}_{g,n}(X,\beta)]^{virt} = -(K_X.\beta) + (\dim X - 3)(1-g) + n.$$

Theorem. For all $\alpha_i \in A^{l_i}(X)$ with $1 \le l_i \le r$ and $\sum_{i=1}^n l_i = 2r + 1 + (n-3)$, there are recursively determined universal constants $N_{l_1,\ldots,l_n}$, such that for $n \le 3$, $N_* \equiv 1$ and

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d} = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3} (\alpha_1.h^{r-l_1}) \cdots (\alpha_n.h^{r-l_n}).$$

Consider the basic geometric series $\mathbf{f}(q) := \dfrac{q}{1-(-1)^{r+1}q}$. Then

$$\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r.$$

3-Point functions (small quantum product):

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle := \sum\nolimits_{\beta \in NE(X)} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta}\, q^\beta$$
$$= (\alpha_1.\alpha_2.\alpha_3) + \sum\nolimits_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d\ell}\, q^{d\ell} + \sum\nolimits_{\beta \notin \mathbb{Z}\ell} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta\, q^\beta.$$

Since $(\mathcal{F}\alpha_i.h'^{(r-l_i)}) = (-1)^{l_i}(\mathcal{F}\alpha_i.\mathcal{F}h^{r-l_i}) = (-1)^{l_i}(\alpha_i.h^{r-l_i})$,

$$\langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3 \rangle - \langle \alpha_1, \alpha_2, \alpha_3 \rangle = (-1)^r(\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})$$
$$+ (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3})((-1)^{2r+1}\mathbf{f}(q^{\ell'}) - \mathbf{f}(q^\ell)) = 0,$$

modulo the 3rd (non-extremal) terms.

Unlike the $r = 1$ case, analytic continuations for the 3rd terms are needed!

Big quantum product: For $n = 3 + k$ point extremal invariants with $k \geq 1$, we get

$$\langle \alpha_1, \ldots, \alpha_n \rangle = N_{l_1,\ldots,l_n}(\alpha_1.h^{r-l_1}) \cdots (\alpha_n.h^{r-l_n}) \left(q^\ell \frac{d}{dq^\ell} \right)^k \mathbf{f}(q^\ell)$$

Since $(-1)^{\sum l_i} = (-1)^{k+1}$, taking into account of

$$q^{-\ell} \frac{d}{dq^{-\ell}} = -q^\ell \frac{d}{dq^\ell}$$

we get $\langle \mathfrak{F}\alpha_1, \ldots, \mathfrak{F}\alpha_n \rangle = \langle \alpha_1, \ldots, \alpha_n \rangle$ for all $k \geq 1$ $(n \geq 4)$.

Sketch of proof:

The virtual fundamental class: $[\bar{M}_{0,n}(X, d\ell)]^{virt}$ is represented by the Euler class of $U_d = R^1 ft_* e_{n+1}^* N$, where $N = N_{Z/X}$:

$$\begin{array}{ccc} \overline{M}_{0,n+1}(\mathbb{P}^r, d) & \xrightarrow{e_{n+1}} & \mathbb{P}^r \\ \Big\downarrow ft & & \\ \overline{M}_{0,n}(\mathbb{P}^r, d) & & \end{array}.$$

That is, $[\overline{M}_{0,n}(X, d\ell)]^{virt} = e(U_d) \cap [\overline{M}_{0,n}(\mathbb{P}^r, d\ell)]$ and

$$\int_{[\bar{M}_{0,n}(X,d\ell)]^{virt}} ev^*\alpha = \int_{\bar{M}_{0,n}(\mathbb{P}^r,d)} ev^*(\alpha|_{\mathbb{P}^r}).e(U_d).$$

The theorem is equivalent to

$$\int_{\bar{M}_{0,n}(\mathbb{P}^r,d)} e_1^* h^{l_1} \cdots e_n^* h^{l_n} . e(U_d) = (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}.$$

Descendent invariants: Let L_i be the line bundle on $\bar{M}_{g,n}(X,\beta)$ whose fiber at $(f; C, (x_1, \ldots, x_n))$ is $T^*_{x_i} C$. Let $\psi_i = c_1(L_i)$.

$$\left\langle \tau_{k_1}(h^{l_1}), \cdots, \tau_{k_n}(h^{l_n}) \right\rangle_d = \int_{\bar{M}_{0,n}(\mathbb{P}^r,d)} \left(\prod_{i=1}^n \psi_i^{k_i} e_i^* h^{l_i} \right) . e(U_d).$$

Step 1. One point invariants. *For* $l + k = 2r - 1$, $1 \le l \le r$,

$$\left\langle \tau_k h^l \right\rangle_d = \frac{(-1)^{d(r+1)+k}}{d^{k+2}} C_r^{k+1}.$$

The invariant is zero if $l + k \neq 2r - 1$.

Consider a $\mathbb{C}^\times$ action on $\mathbb{P}^1$ with weight z. By the localization theorem and the work of Lian-Liu-Yau (1996, Mirror Principle I),

$$J(d\ell, z^{-1}) \equiv e_{1*}\frac{e(U_d)}{z(z-\psi)} = P_d \equiv (-1)^{(d-1)(r+1)}\frac{1}{(h+dz)^{r+1}}.$$

No mirror transformations are needed since $r+1 \geq 2$.

Step 2. Divisor relation for $g = 0$. [Lee-Pandharipande 2003]
For $L \in \mathrm{Pic}(X)$ and $i \neq j$,

$$e_i^* L = e_j^* L + (\beta, L)\psi_j - \sum_{\beta_1+\beta_2=\beta}(\beta_1, L)[D_{i,\beta_1|j,\beta_2}]^{vir}.$$

Also $\psi_i + \psi_j = [D_{i|j}]^{vir}$ and for $n \geq 3$, $\psi_j = [D_{j|ik}]^{vir}$.
For toric varieties, $H^* = A^*$ is generated by divisors.

Deformations to the Normal Cone

$\mathcal{X} = X \times \mathbb{A}^1$

$\Phi : W \to \mathcal{X}$ is the blowing-up along $Z \times \{0\}$

$W_t \cong X$ for all $t \neq 0$

$W_0 = Y \cup \tilde{E}$ with $\tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$

$\phi = \Phi|_Y : Y \to X$ is the blowing-up along Z

$p = \Phi|_{\tilde{E}} : \tilde{E} \to Z \subset X$ is the compactified normal bundle.

$Y \cap \tilde{E} = E = \mathbb{P}_Z(N_{Z/X})$ is the $\phi -$ exceptional divisor

By similar constructions we also have $\Phi' : W' \to \mathcal{X}' = X' \times \mathbb{A}^1$ and $W_0' = Y' \cup \tilde{E}'$. By definition of ordinary flips we have $Y = Y'$ and $E = E'$.

Representatives of Classes on W_0

All cohomology classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ are global and the restriction $\alpha(t)$ on W_t is defined for all t.

Let $j_1 : Y \hookrightarrow W_0$, $j_2 : \tilde{E} \hookrightarrow W_0$, $j : E \hookrightarrow Y$ and $j^+ : E \hookrightarrow \tilde{E}$. The class $\alpha(0)$ can be represented by explicit data

$$(j_1^*\alpha(0), j_2^*\alpha(0)) = (\alpha_1, \alpha_2)$$

such that

$$j^*\alpha_1 = j^{+*}\alpha_2 \quad \text{and} \quad \phi_*\alpha_1 + p_*\alpha_2 = \alpha.$$

Such representatives are not unique. For e being a class in E,

$$(\phi^*\alpha, p^*\alpha) \sim (\phi^*\alpha - j_*e, p^*\alpha + j_*^+e).$$

Cohomology Reduction to Local Models

For a simple flop $f : X \dashrightarrow X'$, let $\alpha \cap Z \neq \emptyset$ with representatives $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathcal{F}\alpha(0) = (\alpha'_1, \alpha'_2)$.

$$\text{If} \quad \alpha_1 = \alpha'_1 \quad \text{then} \quad \mathcal{F}\alpha_2 = \alpha'_2.$$

Degeneration formula: $\triangle(E) = \sum_i S_i \otimes S^i$.

$$\langle\alpha\rangle^X_\beta = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \langle\alpha_1; S_I\rangle^{(Y,E)}_{\Gamma_1} \langle\alpha_2; S^I\rangle^{(\tilde{E},E)}_{\Gamma_2}.$$

Let $\langle\alpha\rangle^X = \sum_\beta \langle\alpha\rangle^X_\beta q^\beta$ and $\mathcal{F}f(q^\beta) = f(q^{\mathcal{F}\beta})$ be the change of variables.

To prove the functional equation $\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}$, it is enough to show that

$$\mathcal{F}\langle\alpha_2; S^I\rangle^{(\tilde{E},E)}_\mu \cong \langle\mathcal{F}\alpha_2; S^I\rangle^{(\tilde{E}',E)}_\mu.$$

Apply deformation to normal cone to $\tilde{E}$, $W_0 = \tilde{Y} \cup \tilde{E}$, . the degeneration formula (with descendent) implies that

$$\begin{aligned}&\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1} S_{i_1}, \ldots, \tau_{\mu_\rho - 1} S_{i_\rho} \rangle^{\tilde{E}}_{\beta} \\ &= \langle \alpha_1, \ldots, \alpha_n; S_{i_1}, \ldots, S_{i_\rho} \rangle^{(\tilde{E}, E)}_{\mu, \beta} + \sum \langle \cdots \rangle^{(\tilde{Y}, E_0)} \langle * * * \rangle^{(\tilde{E}, E)}\end{aligned}$$

where $* * *$ is of lower order in cohomology degree and contact order. May apply induction.

So in order to prove $\mathfrak{F}\langle \alpha \rangle^X \cong \langle \mathfrak{F}\alpha \rangle^{X'}$, it is enough to show that

$$\begin{aligned}&\mathfrak{F}\langle \alpha_1, \ldots, \alpha_n, \tau_{k_1} S_{i_1}, \ldots, \tau_{k_\rho} S_{i_\rho} \rangle^{\tilde{E}} \\ &\qquad \cong \langle \mathfrak{F}\alpha_1, \ldots, \mathfrak{F}\alpha_n, \tau_{k_1} \mathfrak{F} S_{i_1}, \ldots, \tau_{k_\rho} \mathfrak{F} S_{i_\rho} \rangle^{\tilde{E}'}\end{aligned}$$

for projective bundles $\tilde{E}$ and $\tilde{E}'$. Descendent of special type!

Setup on Local Models

The ordinary cohomology ring of $\tilde{E} = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{\oplus(r+1)} \oplus \mathcal{O})$ is given by

$$H^*(\tilde{E}) = \mathbb{Z}[h, \xi]/(h^{r+1}, (\xi - h)^{r+1}\xi).$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^r}(1))$ and $\xi = c_1(\mathcal{O}_{\tilde{E}}(1))$.

Since $c_1(\tilde{E}) = (r+2)\xi$ is semi-positive, $\tilde{E}$ is a semi-Fano toric variety.

$NE(\tilde{E}) = \mathbb{R}_+\ell \oplus \mathbb{R}_+\gamma$ with ℓ the line class in $Z(= \mathbb{P}^r)$ and γ the fiber line class of $\tilde{E} \to Z$. Denote

$$\beta = d_1\ell + d_2\gamma.$$

The virtual dimension $= c_1(\tilde{E}).\beta + \cdots = (r+2)d_2 + \cdot$. So every $\langle\alpha\rangle = \sum_\beta \langle\alpha\rangle_\beta q^\beta$ is a sum over β with a fixed d_2.

Two Important Special Cases

CASE I: $d_2 = 0$. Then $\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}$ has been proved before by the generalized multiple cover formula.

CASE II: One-point descendent invariant for any $d_2 \in \mathbb{N}$.

By the theory of Euler data of Lian-Liu-Yau on semi-Fano smooth toric varieties we get (here no mirror transform is needed)

$$e_*^v \frac{1}{z(z-\psi)} = P_\beta = \frac{\prod_{m=-\infty}^{0} (\xi - h + mz)^{r+1}}{\prod_{m=1}^{d_1} (h+mz)^{r+1} \prod_{m=-\infty}^{d_2-d_1} (\xi - h + mz)^{r+1} \prod_{m=1}^{d_2} (\xi + mz)}.$$

One-point Descendent Invariant of special type

Notation: Denote by $X = \tilde{E}$, $X' = \tilde{E}'$. It is convenient to consider the generating series (Givental's J function)

$$J_X := \sum_{\beta \in NE(X)} q^\beta e_*^v \frac{1}{z(z-\psi)} = \frac{1}{z^2} \sum_{\beta \in NE(X)} q^\beta \sum_{k \geq 0} e_*^v \frac{\psi^k}{z^k}.$$

Theorem

For any $\alpha \in H^*(X)$, the one point function $\langle \tau_k \xi \alpha \rangle^X$ satisfies the functional equation (without analytic continuation):

$$\mathcal{F} \langle \tau_k \xi . \alpha \rangle^X = \langle \tau_k \mathcal{F}(\xi . \alpha) \rangle^{X'} = \left\langle \tau_k \xi' . \mathcal{F} \alpha \right\rangle^{X'}.$$

Equivalently, $\mathcal{F}$ is linear in $J\xi$:

$$\mathcal{F}(J_X \xi . \alpha) = J_{X'} \mathcal{F}(\xi . \alpha) = J_{X'} \xi' . \mathcal{F} \alpha.$$

Corrections for higher genus: Let $\dim X \geq 3$, $\ell \in NE(X)$ with $(K_X.\ell) = 0$, the virtual dimension of $\overline{M}_{g,n}(X, d\ell)$ is given by

$$D_{g,n} = (\dim X - 3)(1 - g) + n.$$

If ℓ is of flopping type, $\langle \alpha \rangle_{g,n,d\ell}$ depends only on $(Z, N_{Z/X})$ for $d \geq 1$. (But not for $d = 0$.). If $D_{g,n} < 0$, all GW invariants vanish.

Genus one: If $g = 1$ then $D_{1,n} = n$ and each insertion is a divisor. Hence if $d \geq 1$ the n-point invariants are determined by

$$\langle - \rangle_{1,d} = \int_{[\overline{M}_{1,0}(X,d\ell)]^{\text{vir}}} 1.$$

For $d = 0$ and $n \geq 2$, the divisor axiom shows that $\langle \alpha \rangle_{1,n,0} = 0$. $n = 1$ case requires different consideration.

Indeed $\overline{M}_{g,n}(X,0) \cong X \times \overline{M}_{g,n}$ and

$$[\overline{M}_{g,n}(X,0)]^{vir} = e(\mathcal{E}) \cap [X \times \overline{M}_{g,n}]$$

where $\mathcal{E} = \pi_1^* T_X \otimes \pi_2^* \mathcal{H}_g^\vee$ with $\mathcal{H}_g$ the Hodge bundle. Let $\lambda_i = c_i(\mathcal{H}_g)$. For $(g,n) = (1,1)$, $e(\mathcal{E}) = c_{\rm top}(X) - c_{\rm top-1}(X).\lambda_1$,

$$\langle \alpha \rangle_{1,0}^X = -(c_{\rm top-1}(X).\alpha)_X \cdot \int_{\overline{M}_{1,1}} \lambda_1 = -\frac{1}{24}(c_{\rm top-1}(X).\alpha)_X,$$

For simple $\mathbb{P}^r$ flops, we verify that $\mathcal{F}\langle \alpha \rangle_1^X = \langle \mathcal{F}\alpha \rangle_1^{X'}$ by proving

$$\langle - \rangle_{1,d} = (-1)^{d(r+1)} \frac{r+1}{24d}$$

and by calculating $(c_{2r}(X).h) - (c_{2r}(X').\mathcal{F}h)$ in local models. (For $r = 1$, BCOV 1993, Graber-Pandharipande 1999.)

For $\dim X = 3$ and $g \geq 2$, $D_{g,n} = n$. It is reduced to $n = 0$. For simple $\mathbb{P}^1$ flop with $d \geq 1$, Faber-Pandharipande 2000 showed

$$\langle - \rangle_{g,d} := \int_{[\overline{M}_{g,0}(X, d\ell)]^{\rm vir}} 1 = C_g \, d^{2g-3}$$

where $C_g = |\chi(M_g)|/(2g-3)!$.

The generating function

$$\langle - \rangle_g := \sum_{d=0}^{\infty} \langle - \rangle_{g,d} \, q^d = \langle - \rangle_{g,0} + C_g \, \delta^{2g-3} \mathbf{f},$$

is invariant under $\mathcal{F}$ since $2g - 3 \geq 1$. For $\langle - \rangle^X_{g,0} = \langle - \rangle^{X'}_{g,0}$, the degeneration analysis reduces the proof to local models, which are both isomorphic to $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-1)^2 \oplus \mathcal{O})$.

Formal loop space: $\mathcal{H}_+ := \bigoplus_{k=0}^{\infty} H\, z^k = H[z]$. $F_g(t)$ is a function on $\mathcal{H}_t$, $t = \sum_{\mu,k} t_k^{\mu} T_{\mu} z^k$. The formal loop space over H is

$$\mathcal{H} := T^*\mathcal{H}_+ = H[z, z^{-1}]].$$

$(\mathcal{H}, \Omega)$ is symplectic. Let $\widehat{\cdot}$ be the Heisenberg quantization.

Ancestor potential: In the stable range $2g + m \geq 3$, let $\pi = ft \circ st : \overline{M}_{g,m+l}(X, \beta) \to \overline{M}_{g,m}$. $\bar{\psi}_i := \pi^* c_1(L_i)$.

$$\bar{F}_g(\bar{t}, s) := \sum_{\beta,m,l} \frac{q^{\beta}}{m!\,l!} \langle \bar{t}^m, s^l \rangle_{g,m+l,\beta}$$

is a function on $\mathcal{H}_+ \times H$ where $\bar{t} = \sum \bar{t}_k^{\mu} T_{\mu} \bar{\psi}^k$, $s = \sum s^{\mu} T_{\mu}$.

$$\mathcal{A}_X(\bar{t}, s) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} \overline{F}_g^X(\bar{t}, s).$$

Frobenius formalism: The Dubrovin connection on TH is

$$\nabla_z = d - \frac{1}{z}\sum\nolimits_\mu ds^\mu \circ T_\mu \, * \, .$$

Recall that $\nabla_z^2 = 0 \iff$ WDVV. The fundamental solution $N \times N$ matrix S ($N = \dim H$) is found at ∞ by $S = J(s, 1/z)$.

Semi-simple Frobenius manifolds: If $(QH, *)$ is semi-simple, i.e. there exist eigen-vector fields ϵ_i with $\epsilon_i * \epsilon_j = \delta_{ij}\,\epsilon_i$, let u^i be the dual (canonical) coordinates and $U = \mathrm{diag}(u^1, \cdots, n^N)$. Let Ψ^{-1} be the transition matrix from $\{\epsilon_i\}$ to $\{T_\mu\}$. Then Givental shows that $\nabla_z S = 0$ near $z = 0$ for

$$S = \Psi^{-1}(s)R(s,z)e^{U/z}$$

where R is a formal series in z, c.f. Lee-Pandharipande's notes.

Ancestor potentials via quantization, the s.s. case: Let $\mathcal{D}_N(\mathbf{t}) = \prod_{i=1}^N \mathcal{D}_{pt}(t^i)$ be the descendent potential of N points.
C. Teleman 2007 classified all semi-simple TFT's. In particular the following formula (conjectured by Givental) holds:

$$\mathcal{A}_X(\bar{t}, s) = e^{\bar{c}(s)} \widehat{\Psi}^{-1}(s) \widehat{R}_X(s, z) e^{\widehat{U/z}}(s) \mathcal{D}_N(\mathbf{t}),$$

where $\bar{c}(s) = \frac{1}{48} \ln \det(\epsilon_i, \epsilon_j)$.
Semi-simplicity for local models: For $X = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{r+1} \oplus \mathcal{O})$, $QH^*(X)$ is semi-simple. Indeed, for $q_1 = q^\ell$ and $q_2 = q^\gamma$,

$$QH^*_{small}(X) \cong \mathbb{C}[h, \xi][q_1, q_2]/(h^{r+1} - q_1(\xi - h)^{r+1}, (\xi - h)^{r+1}\xi - q_2).$$

The eigenvalues of $h*$ and $\xi*$ are all different, hence $(QH^*, *)$ is semi-simple at the origin $s = 0$. Since semi-simplicity is an open condition, $QH^*(X)$ is also semi-simple.

From descendent to ancestors: Let D_j be the (virtual) divisor on $\overline{M}_{g,m+l}(X,\beta)$ as the image of the gluing morphism

$$\sum_{\beta'+\beta''=\beta}\sum_{l'+l''=l}\overline{M}_{0,\{j\}+l'+\bullet}(X,\beta')\times_X\overline{M}_{g,(m-1)+l''+\bullet}(X,\beta'')\to\overline{M}_{g,m+l}(X,\beta),$$

Then $\psi_j-\bar\psi_j=[D_j]$. The j-th point is in the $g=0$ component. In the stable range $2g+n\geq 3$,

$$\begin{aligned}&\langle\tau_{k+1,\bar l}\,\alpha_1,\cdots\rangle_g(\bar t,s)\\&=\langle\tau_{k,\overline{l+1}}\,\alpha_1,\cdots\rangle_g(\bar t,s)+\sum_\nu\langle\tau_k\,\alpha_1,T_\nu\rangle_0(s)\,\langle\tau_{\bar l}\,T^\nu,\cdots\rangle_g(\bar t,s).\end{aligned}$$

This reduces all descendent of special type to ancestors. The proof for higher genus is complete by the degeneration analysis.

Simple extremal transition in dimension k:

Let $\bar{X}$ be a k dimensional variety contain only a hypersurface canonical singularity $(p, \bar{X})$ defined by $x_0^k + \cdots + x_k^k = 0$.

A crepant resolution can be obtained by a standard blow-up $\phi : Y = \mathrm{Bl}_p \bar{X} \to \bar{X}$. $\bar{X}$ can be smoothed into a flat family $\mathfrak{X} \to \Delta$ with general smooth fiber $X = \mathfrak{X}_t$ with $t \neq 0$ and $\mathfrak{X}_0 = \bar{X}$. We call $Y \mapsto X$ a simple extremal transition in dimension k.

Semi-stable degenerations $W \to \Delta$ attached to $\mathfrak{X} \to \Delta$:

Notice that the total space $\mathfrak{X}$ is a smooth variety and W can be achieved by taking a degree k base change

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{t \mapsto t^k} & \Delta \end{array}$$

and then set $W = \mathrm{Bl}_{p'}\mathfrak{X}'$. Here $p' \in \mathfrak{X}'$ is now a $k+1$ dimensional simple hypersurface singularity of order k in $\mathbb{C}^{k+2}$. Thus $W_0 = Y \cup \tilde{E}$ with $\tilde{E} \subset \mathbb{P}^{k+1}$ being a degree k Fano hypersurface. The intersection $E = Y \cap \tilde{E}$, which is the ϕ exceptional divisor, can be regarded as a degree k hypersurface in $\mathbb{P}^k$, which is still Fano.

$N_{E/\tilde{E}} = \mathcal{O}(1)$ and $N_{E/Y} = \mathcal{O}(-1)$. (E, Y) is equivalent to $\mathbb{P}^k$ "cut out" by a rank 2 split bundle $V_k = \mathcal{O}(k) \oplus \mathcal{O}(-1)$.

The A model on X can be compared with the one on Y through the degeneration analysis on the semi-stable family

$$\begin{array}{c} W \\ \downarrow \pi \\ \Delta \end{array}$$

thanks to the description of $\tilde{E}$ as a toric Fano hypersurface.

$$\langle a \rangle^X = \sum_{\mu} \langle a_1 \mid \mu \rangle^{(Y,E)} * \langle a_2 \mid \mu \rangle^{(\tilde{E},E)}$$

with μ being the splitting/gluing data of curves.

Let $\ell \in NE(Y)$ be the ϕ extremal ray, which is of flopping type. The genus 0 extremal function is defined by

$$f(a) = \langle a \rangle^Y_{extr} := \sum_{d \in \mathbb{Z}_+} \langle a \rangle^Y_{0,d\ell}\, q^{d\ell}.$$

By the localization calculation in local mirror symmetry or rather the quantum Serre duality principle, the calculation of $f(a)$ may be transformed into a calculation on

$$\begin{array}{c} V_k^+ = \mathcal{O}(k) \oplus \mathcal{O}(1) \\ \downarrow \\ \mathbb{P}^k \end{array}$$

which in turn reduced to $\mathcal{O}(k)$ over $\mathbb{P}^{k-1}$, that is the case of Calabi-Yau hypersurface CY_k.

From A model of Y to B model of X:

The key observation is to “observe” the appearance of CY_k in the degeneration family $\pi : W \to \Delta$. In fact,

Theorem

There is a sub-degeneration of VHS which corresponds the the vanishing cycle along π, whose Picard-Fuchs equation turns out have $f(a)$ as its solution, up to a mirror change of variable!