Quantum Cohomology under Birational Maps and Transitions

Chin-Lung Wang
National Taiwan University
(joint with Y.-P. Lee and H.-W. Lin)

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Contents

Morphisms in algebraic geometry

- $f : Y \longrightarrow X$ **a projective morphism, e.g. a blow-up**
- **Factorization,** $P = P_X(\mathscr{E})$

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & P \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

- **If** $\iota(Y) = \sigma^{-1}(0)$ **is a complete intersection,** $\sigma \in \Gamma(P, V)$

LHT = Lefschetz Hyperplane Theorem.

- **Usually need** ***DNC = deformations to the normal cone***
- **For** π

LRT = Leray–Hirsch Theorem.

Quantum cohomology

- **Basis** $T_i \in H = H(X)$**, dual** $\{T^i\}$**,** $t = \sum t^i T_i$**,** $F(t) = \langle\langle\rangle\rangle$

$$\langle\langle a_1, \ldots, a_m \rangle\rangle = \sum_{\beta \in NE(X)} \sum_{n=0}^{\infty} \frac{q^\beta}{n!} \langle a_1, \ldots, a_m, t, \ldots, t \rangle_{g=0,m+n,\beta}$$

- $F_{ijk} = \partial^3_{ijk} F = \langle\langle T_i, T_j, T_k \rangle\rangle$**,** $A^k_{ij} = F_{ijl}\, g^{lk}$

$$T_i *_t T_j = \sum A^k_{ij}(t)\, T_k$$

- **Dubrovin connection on** $TH \otimes \mathbb{C}[\![q^\bullet]\!]\{z\}$ **is flat**

$$\nabla = d - \frac{1}{z} A = d - \frac{1}{z} \sum_i dt^i \otimes A_i$$

Cyclic $\mathscr{D}^z$ modules

- $t = t_0 + t_1 + t_2$, $t_0 \in H^0$, $t_1 \in H^2$

$$J(t, z^{-1}) = 1 + \frac{t}{z} + \sum_{\beta,n,i} \frac{q^\beta}{n!} T_i \left\langle \frac{T^i}{z(z-\psi)}, (t)^n \right\rangle_\beta$$

$$= e^{\frac{t}{z}} + \sum_{\beta\neq 0,n,i} \frac{q^\beta}{n!} e^{\frac{t_0+t_1}{z} + (t_1.\beta)} T_i \left\langle \frac{T^i}{z(z-\psi)}, (t_2)^n \right\rangle_\beta$$

- **TRR $\Longrightarrow$ QDE**

$$z\partial_i z\partial_j J = \sum_k A_{ij}^k z\partial_k J$$

- $QH(X) =$ **cyclic $\mathscr{D}^z$ module $\mathscr{D}^z J$ with base (frame)**

$$z\partial_i J \equiv e^{t/z} T_i \pmod{q^\bullet} = T_i + \cdots$$

QLHT toric base/concavex bundle $c_1 \geq 0$

- $L_i \to P$ **convex,** $\sigma \in \Gamma(P, \bigoplus_{i=1}^r L_i)$**,** $Y = \sigma^{-1}(0) \hookrightarrow P$
- **Factorial trick** $(L)_\beta := \prod_{m=1}^{L.\beta} (L + mz)$

$$I^Y := \sum_\beta q^\beta J_\beta^P \times \prod_{i=1}^r (L_i)_\beta$$

- **Lian–Liu–Yau, Givental (1996):**
  P **toric, semi-Fano** $c_1(P) \geq 0$**,** $\mathbb{C}^\times$ **localization** $\Longrightarrow J^P$**.**
- **(Mirror Theorem)** $c_1(Y) \geq 0$**,** $t \in H^0 \oplus H^2$**,**

$$(I^Y / I_0^Y)(t, z^{-1}) \sim J^Y(\tau, z^{-1})$$

up to mirror map $t \mapsto \tau(t)$ **to match** $1/z$

QLHT general base/split bundle

- **Coates–Givental (2005):** P **general,** L_i **arbitrary.** **Given** $J^P(t)$ **in** $t \in H(X)$**, then** $I^Y \in \mathscr{D}^z J^P$**, i.e.** $\exists\, p, t \mapsto \tau(t)$

$$I^Y(t, z, z^{-1}) = p(\tau, z, q^\bullet, z\partial_\bullet)\, J^Y(\tau, z^{-1})$$

- **Birkhoff factorization**

$$(z\vec{\partial}\, I)(t, z, z^{-1}) = (z\vec{\partial}\, J)(\tau, z^{-1}) B(\tau, z)$$

- $z\partial_i I \equiv e^{t/z} T_i \equiv z\partial_i J \pmod{q^\bullet} \Longrightarrow B \equiv Id \pmod{q^\bullet}$

$$\mathscr{D}^z I(t) \cong \mathscr{D}^z J(\tau)$$

up to a *generalized mirror transform* $\tau(t)$ **on** $H(X)$ **which matches** $1/z$ **in** $p^{-1}(t, z)\, I^Y(t, z, z^{-1}) = J^Y(\tau, z^{-1})$

Quantum Leray–Hirsch

- $\pi : P = P_X(V) \to X$ **projective (toric) bundle**
- **Leray–Hirsch,** $h = c_1(\mathcal{O}_P(1))$

$$H(P) \cong \pi^* H(X)[h]/(f_V(h))$$

- **Factorial trick:** $V = \bigoplus_{i=1}^r L_i$, $\bar{t} \in H(X)$, $D = t^h h$, $\hat{t} = \bar{t} + D$

$$I^P(\hat{t}, z, z^{-1}) = \sum_{\beta \in NE(P)} q^\beta J^X_{\pi_* \beta}(\bar{t}) \times e^{\frac{D}{z} + (D.\beta)} \prod_{i=1}^r \frac{1}{(h + L_i)_\beta}$$

- **J. Brown (2009),** $\exists\, p, \tau(\hat{t}) : H(X) \oplus \mathbb{C}h \to H(P)$,

$$I^P(\hat{t}, z, z^{-1}) = p(\tau, z, q^\bullet, z\partial_\bullet)\, J^P(\tau, z^{-1})$$

- **Lee–Lin–Wang (2011):** $\mathrm{PF}^{P/X} + \nabla^X \Rightarrow \nabla^P$
- **Picard–Fuchs ideal:** $\ell \in NE(P/X)$**,** $\Box_\ell I = 0$

$$\Box_\ell = \prod_{i=1}^{r} z\partial_{h+L_i} - q^\ell e^{t^h}$$

- **Lifting of QDE from** X **to** P**:**

$$z\partial_i\, z\partial_j\, I = \sum_{k,\bar\beta} q^{\bar\beta^*} e^{D.\bar\beta^*}\, \bar A^k_{ij,\bar\beta}(\bar t)\, D_{\beta^*}(z)\, z\partial_k\, I$$

- $\bar\beta^* \in NE(P)$ **is an admissible lift of** $\bar\beta \in NE(X)$

$$D_{\bar\beta^*}(z) := \prod_{i=1}^{r} \prod_{m=0}^{-(h+L_i).\bar\beta^*-1\geq 0} (z\partial_{h+L_i} - mz)$$

- **The admissible lift exists, and the lifting of QDE is independent of the choices of** $\bar\beta^*$ **modulo** $\langle \Box_\ell \rangle$

- $\bar{t} = \sum \bar{t}^i \bar{T}_i \in H(X)$, $e = h^l \bar{T}_i \in H(P)$, **näive quantization**

$$\hat{e} = \partial^{ze} := (z\partial_h)^l z\partial_{\bar{T}_i} = (z\partial_{t^h})^l z\partial_{\bar{t}^i}$$

- $\Longrightarrow$ **first order system of** $\partial^{ze} I$ **over** $t^a = \bar{t}^i, t^h$

$$z\partial_a (\partial^{ze} I) = (\partial^{ze} I) C_a(\hat{t}, z)$$

- **Birkhoff factorization** $B =$ **gauge to remove** z **in** C_a

$$(\partial^{ze} I)(\hat{t}, z, z^{-1}) = (z\vec{\partial} J)(\tau, z^{-1}) B(\tau, z)$$

- $\hat{t} \mapsto \tau(\hat{t})$ **matches** $1/z$ **of first column of** $(\partial^{ze} I)B^{-1}$ **with** J
- **Set** $z = 0$ **in the gauge transform:** $-(z\partial_a B)B^{-1} \mapsto 0$

$$B_0 \, C_{a;0} \, B_0^{-1}(\hat{t}) = \sum_{i=1}^{\dim H(P)} A_i(\tau(\hat{t})) \frac{\partial \tau^i}{\partial t^a}(\hat{t})$$

App-I, (ordinary) flops

- **Let** $f : X \dashrightarrow X'$ **be a** P^r **flop,** $F, F' \to S$ **be v.b. of rank** r

$$\mathrm{Exc} f = Z = P_S(F) \xrightarrow{\bar{\psi}} S, \qquad N_Z = \bar{\psi}^* F' \otimes \mathscr{O}(-1)$$

- $\mathscr{T} = [\Gamma_f]_* : H(X) \cong H(X')$**, but** $\mathscr{T}\ell = -\ell'$
  $\mathscr{T}$ **preserves Poincaré pairing, but not** $\cup$

- **Theorem:** $\mathscr{T} : QH(X) \cong QH(X')$ **as big quantum rings under analytic continuations** $q^\beta \mapsto q^{\mathscr{T}\beta}$

- $\dim X = 3$**, MCF of** $\mathscr{O}_{P^1}(-1)^2 \to P^1$ **gives quantum corrections of** $(\mathscr{T}D)^3 - D^3$ **(Witten 1991)**
- $\dim X = 3$ **global case (Li–Ruan 2000: no degenerations)**
- **Simple flops** $S = \mathrm{pt}$ **(LLW 2006), general** S**, split** F, F' **(LLW I-II, 2011), quantum splitting principle (Qu-LLW III, 2013)**

- **Reduction to local models (J. Li's degeneration formula)**

$$\begin{array}{ccc} X = P_Z(N \oplus \mathscr{O}) & \overset{f}{\dashrightarrow} & X' = P_{Z'}(N' \oplus \mathscr{O}) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

- $X \to Z \to S$, $NE(X/S) = \langle \ell, \gamma \rangle$, $H^2(X/S) \ni D = t^h h + t^\xi \xi$
- $H(X) = \mathbb{Z}[h, \xi]/(f_F, f_{N\oplus\mathscr{O}})$, $F = \bigoplus L_i$, $F' = \bigoplus L'_i$

$$f_F(h) = \prod (h + L_i), \qquad f_{N\oplus\mathscr{O}}(h, \xi) = \xi \prod (\xi - h + L'_i)$$

Not compatible with X' since $\mathscr{T} h = \xi' - h'$, $\mathscr{T} \xi = \xi'$

- **Lemma:** $\mathscr{T} : \langle \Box_\ell, \Box_\gamma \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle$

$$\Box_\ell = \prod z\,\partial_{h+L_i} - q^\ell e^{t^h} \prod z\,\partial_{\xi-h+L'_i}$$

$$\Box_\gamma = z\,\partial_\xi \prod z\,\partial_{\xi-h+L'_i} - q^\gamma e^{t^\xi}$$

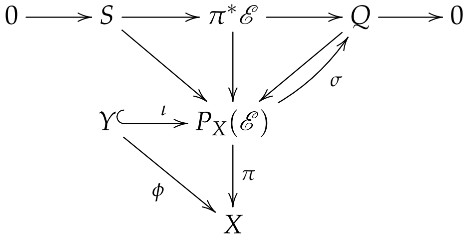
- **QLR** $\Longrightarrow \mathscr{T} : \nabla^X \cong \nabla^{X'}$ **(analytic continuation)**

App-II, blow-ups (along c.i. center)

Let $L_i = \mathscr{O}_X(D_i)$**,** $Z = D_1 \cap \cdots \cap D_r$ **smooth,** $\mathscr{E} = \bigoplus_{i=1}^r L_i$

$$\begin{array}{ccc} E & \hookrightarrow & Y = \mathrm{Bl}_Z X \\ \downarrow & & \downarrow \phi \\ Z & \hookrightarrow & X \end{array}$$

$\mathscr{E}^* \twoheadrightarrow \mathscr{I}_Z \Longrightarrow Y = \sigma^{-1}(0) \subset P := P_X(\mathscr{E})$ **for** $\sigma \in \Gamma(P, Q)$



$\eta = c_1(\mathscr{O}_\pi(1)) \Longrightarrow S = \mathscr{O}_P(-\eta)$ **and** $-\eta|_Y = E$
QLHT (for exact sequence) + QLR $\Longrightarrow$

$$I^Y_\beta = J^P_\beta \frac{\prod_{i=1}^r (D_i)_\beta}{(-\eta)_\beta}$$

$$\sim J^X_{\pi_*\beta}\, e^{\frac{\eta}{z}+\eta.\beta}\, \frac{\prod_{i=1}^r (D_i)_\beta}{(-\eta)_\beta \prod_{i=1}^r (\eta + D_i)_\beta}$$

Relative factor (hypergeometric modification)

$$I^{Y/X}_\beta = e^{\frac{\eta}{z}+\eta.\beta} \left(\prod_{i=1}^r \frac{(D_i)_\beta}{(D_i - E)_\beta (E)_\beta} \right) (E)^{r-1}_\beta$$

As in QLR, (1) näive quantization basis (2) Picard-Fuchs on fiber (3) lifting of QDE on base $X \Longrightarrow \mathscr{D}^z J^Y \Longrightarrow QH(Y)$

App-III, (simple) (r, r') flips

Local model of $(2,1)$ flips, $Z = P^2$, $Z' = P^1$

$$f : X = P_{P^2}(\mathscr{O}(-1)^2 \oplus \mathscr{O}) \dashrightarrow X' = P_{P^1}(\mathscr{O}(-1)^3 \oplus \mathscr{O})$$

$H(X) = \mathbb{Z}[h, \xi]/(h^3, \xi(\xi - h)^2)$, $\dim H(X) = 9$
$H(X') = \mathbb{Z}[h', \xi']/(h'^2, \xi'(\xi' - h')^3)$, $\dim H(X') = 8$
$\mathbf{k}_1 = (\xi - h)^2 = [Z]$ **and** $K = \mathbb{C}\mathbf{k}_1$, $\mathscr{T} = [\Gamma_f]_*$

$$0 \longrightarrow K \longrightarrow H(X) \xrightarrow{\mathscr{T}} H(X') \longrightarrow 0$$

$\mathscr{T}^{-1} : H(X') \hookrightarrow H(X)$ **induces** $H(X) = \mathscr{T}^{-1} H(X') \oplus^{\perp} K$
$\mathscr{T}^{-1}$ **preserves pairing but not** $\cup$, $K^{\perp}$ **is not** $\cup$**-closed**

Use $q_1 = q^{\ell} e^{t^1}$, $q_2 = q^{\gamma} e^{t^2}$, $q'_1 = q^{\ell'} e^{-t^1}$, $q'_2 = q^{\gamma'} e^{-t^1 + t^2}$

$QH(X)$ vs $QH(X')$

- X' **is bad,** $c_1(X') = -h' + 4\xi'$

$$\Box_{\ell'} = (z\partial_{h'})^2 - q_1'(z\partial_{\xi'-h'})^3, \quad \Box_{\gamma'} = z\partial_{\xi'}(z\partial_{\xi'-h'})^3 - q_2'$$

Difficult to compute $\nabla^{X'}$

- X **is Fano,** $c_1(X) = h + 3\xi$

$$\Box_{\ell} = (z\partial_h)^3 - q_1(z\partial_{\xi-h})^2, \quad \Box_{\gamma} = z\partial_{\xi}(z\partial_{\xi-h})^2 - q_2$$

$QH(X)$ **is "easy",** $I = J$**.**

- **Question: Can we get** $QH(X')$ **from** $QH(X)$**?**
- **Kähler moduli** $(q_1, q_2) \bigcup (q_1' = 1/q_1, q_2' = q_1 q_2) = \mathscr{O}_{P^1}(1)$

$$\mathscr{T} : \langle \Box_\ell, \Box_\gamma \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle$$

outside $D_0 = \{q_1 = 0\}$ **and** $D_\infty = \{q_1' = 0\}$

Exact formula for ∇^X

The following frame (recall $I = J$)

$$
\begin{aligned}
v_1 &= \hat{\mathbf{1}}J = J, \\
v_2 &= \hat{h}J, \quad v_3 = (\hat{\xi} - \hat{h})J, \\
v_4 &= \hat{h}^2 J - (\hat{\xi} - \hat{h})^2 J, \quad v_5 = \hat{h}(\hat{\xi} - \hat{h})J + (\hat{\xi} - \hat{h})^2 J, \\
v_6 &= \hat{h}^3 J - \hat{h}(\hat{\xi} - \hat{h})^2 J, \quad v_7 = \hat{h}^2(\hat{\xi} - \hat{h})J + \hat{h}(\hat{\xi} - \hat{h})^2 J, \\
v_8 &= \hat{h}^3(\hat{\xi} - \hat{h})J + \hat{h}^2(\hat{\xi} - \hat{h})^2 J, \\
v_9 &= \hat{\mathbf{k}}_1 = (\hat{\xi} - \hat{h})^2 J,
\end{aligned}
$$

respects $H(X) = \mathscr{T}^{-1}H(X') \oplus^{\perp} K$ **when modulo** q_1, q_2**.**
They are precisely

$$z\partial_i J \quad \textbf{at } t \in H^0 \oplus H^2, \quad 1 \le i \le 9,$$

and we get the Dubrovin connection:

$$A_1 = h*_{small} = \begin{bmatrix} & & & & & q_1q_2 & & & \\ 1 & & & & & & & & \\ & & & & & & & q_1q_2 & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & -1 \\ & & & & 1 & & & & \\ & & & & & -1 & 1 & & \\ & 1 & -1 & & & & & & {\color{blue}q_1} \end{bmatrix}$$

$$A_2 = \xi*_{small} = \begin{bmatrix} & & & -q_2 & q_2 & q_1q_2 & & & q_2 \\ 1 & & & & & -q_2 & q_2 & & \\ 1 & & & & & & & q_1q_2 & \\ & 1 & & & & & & q_2 & \\ & 1 & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & 1 & 1 & & & & \\ & & & & & & 1 & & \\ & & & & & & & q_2 & \end{bmatrix}$$

Irregular singularity in K

Set $x = q_1'$, $y = q_2'$ (and a slight change of basis), the fundamental solution matrix S satisfies

$$z(x\,\partial_x)S = \begin{bmatrix} & & & -\frac{1}{2}xy & xy & & & & xy \\ & & & & & -\frac{1}{2}xy & xy & & \\ 1 & & & & & \frac{1}{4}xy & -\frac{1}{2}xy & & \\ & & & & & & & xy & \\ & 1 & & & & & & -\frac{1}{2}xy & \\ & & & & & & & & 1 \\ & & & 1 & & & & & -\frac{1}{2} \\ & & & & & 1 & & & \\ & -\frac{1}{2} & 1 & & & & & xy & -1/x \end{bmatrix} S$$

which is irregular in the K-block, of Poincaré rank one.

Block diagonalization w.r.t. $H(X) = \mathscr{T}^{-1}H(X') \oplus^{\perp} K$

- **(Wasow 1960's) + flatness of $\nabla^X \Longrightarrow$ there exists a unique formal gauge transformation $S = PZ$**

$$P(x,y,z) = \begin{bmatrix} 1 & & & g_1 \\ & \ddots & & \vdots \\ & & 1 & g_8 \\ f_1 & \cdots & f_8 & 1 \end{bmatrix},$$

such that $z(x\,\partial_x)Z = B_1\,Z$, $z(y\,\partial_y)Z = B_2\,Z$, with B_1, B_2 block diagonalized. ($f_i(x,y,z) = -g_{9-i}(x,y,-z)$)

- **Under the new z-dependent frame $\tilde{v}_1, \ldots, \tilde{v}_8, \tilde{\mathbf{k}}_1$,**

$$QH(X) \cong \langle \tilde{v}_1(0), \ldots, \tilde{v}_8(0) \rangle \times \mathbb{C}$$

- **$\langle \tilde{v}_1(0), \ldots, \tilde{v}_8(0) \rangle \cong QH(X')$ as $\mathscr{D}^z$ modules, but not rings.**

Conifold transiitons $X \nearrow Y$ of CY 3-folds

- $p_i =$ **ODP**, $N_{S_i/X} = T^*S^3$, $N_{C_i/Y} = \mathscr{O}(-1)^2$, $1 \le i \le k$

$$\begin{array}{ccc} & & C_i \subset Y \\ & & \Big\downarrow \psi \\ S_i \subset X & \overset{\pi}{\rightsquigarrow} & p_i \in \bar{X} \end{array}$$

- $\mu := h^{2,1}(X) - h^{2,1}(Y)$, $\rho := h^{1,1}(Y) - h^{1,1}(X)$

$$\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2) \Longrightarrow \mu + \rho = k$$

- ψ**-exceptional curve classes** $NE(Y/\bar{X})$, $\rho = \rho(Y/\bar{X})$

$$A = (a_{ij}) \in M_{k\times\mu} \qquad \sum_{i=1}^{k} a_{ij}[C_i] = 0$$

- π**-vanishing cycles** $V \hookrightarrow H_3(X) \to H^3(\bar{X})$, $\mu = \dim V$

$$B = (b_{ij}) \in M_{k\times\rho} \qquad \sum_{i=1}^{k} b_{ij}[S_i] = 0$$

Basic exact sequence

- **The Hodge realization of $\mu + \rho = k$:**

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} V \to 0$$

  is an exact sequence of weight two Hodge structures.
- **Indeed $V \cong H^{1,1}_\infty H^3(X)$ in limiting Hodge diamond**

$$H^{2,2}_\infty H^3$$

$$\mathbb{C} \cong H^{3,0}_\infty H^3 \qquad H^{2,1}_\infty H^3 \qquad \sim \Big\downarrow N \qquad H^{1,2}_\infty H^3 \qquad H^{0,3}_\infty H^3$$

$$H^{1,1}_\infty H^3$$

- **The invariant subsystem is $Gr^W_3 H^3(X) \cong H^3(Y)$**

B model Yukawa coupling near $[\bar{X}]$

- $V = \mathbb{C}\langle \Gamma_1, \ldots, \Gamma_\mu \rangle$, **the** α**-periods** $r_j = \int_{\Gamma_j} \Omega$ **are the degeneration coordinates around** $[\bar{X}] \in \mathscr{M}_{\bar{X}} \cong \Delta^\mu \times \mathscr{M}_Y$
- **Friedman's (partial) smoothing of ODP's: Let**

$$w_i = a_{i1} r_1 + \cdots + a_{i\mu} r_\mu,$$

then $D_i = \{w_i = 0\} \subset \mathscr{M}_{\bar{X}}$ **is the loci with** $S_i \mapsto p_i$

- $D_B = \bigcup_{i=1}^k D_i$ **is not normal crossing**
- **The transversal** β**-periods are** $u_p = \int_{\beta_p} \Omega$ **for some** u
- **The Bryant–Griffiths–Yukawa couplings satisfy**

$$u_{pmn} = O(1) + \sum_{i=1}^{k} \frac{1}{2\pi\sqrt{-1}} \frac{a_{ip} a_{im} a_{in}}{w_i}$$

- $\{u_{pmn}\} \Longrightarrow \nabla^{GM}$ **(regular singular along** D_B**)**

Local transition between $A(Y)$ and $B(X)$

- $u = \sum_{p=1}^{\rho} u^p T_p \in H^2(Y/X)$, $D^i := \{\sum_{p=1}^{\rho} b_{ip} u^p = 0\}$, $i = 1, \ldots, k$. $QH(Y)$ **is regular singular along** $D^A = \bigcup D^i$
- **Let** $y = \sum_{i=1}^{k} y_i e_i \in \mathbb{C}^k$, $e^1, \cdots, e^k$ **dual basis on** $(\mathbb{C}^k)^\vee$. **The trivial logarithmic connection on** $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee \longrightarrow \mathbb{C}^k$ **is**

$$\nabla^k = d + \frac{1}{z} \sum_{i=1}^{k} \frac{dy_i}{y_i} \otimes (e^i \otimes e_i^*)$$

- $A^t B = 0 \Longrightarrow \mathbb{C}^k = \operatorname{image} A \oplus^{\perp} \operatorname{image} B$
- ∇^k **restricts to the logarithmic part of** ∇^{GM} **on** V
- ∇^k **restricts to the logarithmic part of** ∇ **on** $H^2(Y)/H^2(X)$

"excess A theory" + "excess B theory" = "trivial"

Global aspects (arXiv:1502.03277)

- **Let $[X]$ be a nearby point of $[\bar{X}]$ in $\mathscr{M}_{\bar{X}}$. Then**

(1) **$A(X)$ is a sub-theory of $A(Y)$ (quantum sub-ring).**

(2) **$B(Y)$ is a sub-theory of $B(X)$ (invariant sub-VHS).**

(3) **$A(Y)$ can be reconstructed from a "refined A model" on**

$$X^\circ := X \setminus \bigcup_{i=1}^{k} S_i$$

"linked" by the vanishing spheres in $B(X)$.

(4) **$B(X)$ can be reconstructed from the VMHS on $H^3(Y^\circ)$,**

$$Y^\circ := Y \setminus \bigcup_{i=1}^{k} C_i,$$

"linked" by the exceptional curves in $A(Y)$.

An example

- **(Tsung-Ju Lee, H.-W. Lin 2015) Tautological systems (Lian–Yau) in conifold transitions from toric degenerations**

$$
\begin{array}{ccc}
 & & Y \subset \hat{P} = \hat{P}(2,4) \\
 & & \Big\downarrow \Psi \\
X \subset G = G(2,4) & \rightsquigarrow & P(2,4)
\end{array}
$$

- τ_G**:** $\mathrm{SL}(4,\mathbb{C})$**,** $16 - 1 = 15 = 12$ **roots** $+3$ **torus action**
- $\tau_{\hat{P}} =$ **extended GKZ:** $\mathrm{Aut}^0(\hat{P})$ **generated by** T^4 **and 14 roots:**

$$R(\Sigma, N) = \{\alpha \in M \mid \exists p \in \Sigma_1, (\alpha, p) = -1, \ (\alpha, p') \geq 0 \quad \forall p' \neq p\}$$

(Cox 1995). The 2 roots $\pm(1,1,1,1)$ **are discarded since they do not preserve** Ψ**. Thus** $(\tau_{\hat{P}}, \bigcup C_i)$ **determine** τ_G**.**

- **The definition of the linked GW invariant in (3) is really a reformulation of the discreteness of components appearing in the virtual cycle form of**

$$\langle - \rangle^X_{g,\beta} = \sum_{\gamma \mapsto \beta} \langle - \rangle^Y_{g,\gamma}.$$

- **So far it is not effectively computable.**
- **It requires a blow-up formula of GW along Weil divisors.**

THANK YOU