Quantum Flips

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Contents

1. What is Quantum Cohomology?

A: Deformation of $(H(X), \cup)$ by rational curves.

- Let $X/\mathbb{C}$ be a projective manifold, $\overline{M}_n(X,\beta)$ be the moduli space of stable maps
$$f : (C, p_1, \ldots, p_n) \to X$$
from n-pointed rational nodal curves to X with image class $\beta \in NE(X)$, the Mori cone of effective 1-cycles.
- For $i \in [1, n]$, let $e_i : \overline{M}_n(X,\beta) \to X$ be the evaluation map
$$e_i(f) := f(p_i) \in X.$$
- Let $\mathbf{t} \in H = H(X)$. The $g = 0$ Gromov–Witten potential
$$F(\mathbf{t}) = \langle\langle - \rangle\rangle(\mathbf{t}) := \sum_{n,\,\beta} \frac{q^\beta}{n!} \langle \mathbf{t}^{\otimes n} \rangle^X_{n,\beta}$$
$$= \sum_{n\geq 0,\,\beta\in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_n(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* \mathbf{t}$$
is a formal function in $\mathbf{t}$ and q^β's (Novikov variables).

- We call $\mathscr{R} := \mathbb{C}[\![q^\bullet]\!]$ the (formal) Kähler moduli and denote

$$H_{\mathscr{R}} = H \otimes \mathscr{R}.$$

- Let $\{T_\mu\}$ be a basis of H and $\{T^\mu := \sum g^{\mu\nu} T_\nu\}$ the dual basis with respect to the Poincaré pairing

$$g_{\mu\nu} = (T_\mu . T_\nu), \qquad (g^{\mu\nu}) = (g_{\mu\nu})^{-1}.$$

- Let $\mathbf{t} = \sum t^\mu T_\mu$. The *big quantum ring* $(QH(X), *)$ is a $\mathbf{t}$-family of rings $Q_{\mathbf{t}}H(X) = (T_{\mathbf{t}}H_{\mathscr{R}}, *_{\mathbf{t}})$:

$$\begin{aligned} T_\mu *_{\mathbf{t}} T_\nu &:= \sum_{\epsilon,\kappa} \partial_\mu \partial_\nu \partial_\epsilon F(\mathbf{t})\, g^{\epsilon\kappa} T_\kappa \equiv \sum F_{\mu\nu\epsilon}\, g^{\epsilon\kappa} T_\kappa \\ &= \sum_{\epsilon,\kappa} \langle\!\langle T_\mu, T_\nu, T_\epsilon \rangle\!\rangle(\mathbf{t})\, g^{\epsilon\kappa} T_\kappa \\ &= \sum_{\kappa,\, n \geq 0,\, \beta \in NE(X)} \frac{q^\beta}{n!} \langle T_\mu, T_\nu, T^\kappa, \mathbf{t}^{\otimes n} \rangle^X_{n+3,\,\beta} T_\kappa. \end{aligned}$$

- The WDVV associativity equations equip $(H_{\mathscr{R}}, g_{\mu\nu}, F_{ijk}, T_0 = \mathbf{1})$ a structure of *formal Frobenius manifold* over $\mathscr{R}$.
- It is equivalent to the flatness of the Dubrovin connection

$$\nabla^z = d - \frac{1}{z} A := d - \frac{1}{z} \sum_{\mu} dt^{\mu} \otimes T_{\mu} *_{\mathbf{t}}$$

on the formal relative tangent bundle $TH_{\mathscr{R}}$ for all $z \in \mathbb{C}^{\times}$:

$$\partial_{\mu} A_{\nu} = \partial_{\nu} A_{\mu}, \qquad [A_{\mu}, A_{\nu}] = 0,$$

- where the (connection) matrix A_{μ} for $z\nabla^z_{\mu}$ is z-free:

$$A_{\mu}(\mathbf{t}) = T_{\mu} *_{\mathbf{t}} .$$

- This *z-free property* uniquely characterizes the constant frame $\{T_{\mu}\}$ among all frames $\{\tilde{T}_{\mu}\}$ with

$$\tilde{T}_{\mu}(q^{\bullet}, \mathbf{t}, z) \equiv T_{\mu} \pmod{\mathscr{R}}.$$

- Let $\psi = c_1(\mathbf{p}_1^* \omega_{\mathscr{C}/\overline{M}_n})$ be the class of cotangent line at the first marked section $\mathbf{p}_1 : \overline{M}_n \to \mathscr{C}$ of $\mathscr{C} \to \overline{M}_n$, then

$$J(\mathbf{t}, z^{-1}) := 1 + \frac{\mathbf{t}}{z} + \sum_{\beta,n,\mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \mathbf{t}^{\otimes n} \right\rangle_{n+1,\beta}^X$$

encodes invariants with one descendent insertion.

- The topological recursion relation (TRR):

$$\langle\langle \tau_{d+1} T_i, T_j, T_k \rangle\rangle = \sum\nolimits_\mu \langle\langle \tau_d T_i, T_\mu \rangle\rangle \langle\langle T^\mu, T_j, T_k \rangle\rangle$$

implies the quantum differential equation (QDE):

$$z\partial_\mu \, z\partial_\nu J = \sum\nolimits_\kappa A_{\mu\nu}^\kappa \, z\partial_\kappa J.$$

- Let $\mathscr{D}^z$ be the ring of differential operators generated by $z\partial_i$ with coefficients in $\mathscr{O} = \mathbb{C}[z][\![q^\bullet, \mathbf{t}]\!]$. The $\mathscr{D}^z$-module $\mathscr{O}^{\dim H}$ associated to $z\partial_i \mapsto z\nabla_i^z$ is isomorphic to the *cyclic* $\mathscr{D}^z$*-module* $\mathscr{D}^z J$.

- In practice, one might be able to find element

$$I(\hat{\mathbf{t}}, z, z^{-1}) \in \mathscr{D}^z J(\mathbf{t}, z^{-1})$$

but only along some restricted variables $\hat{\mathbf{t}} \in H_1 \subset H$.

- If H_1 generates H (either in classical product or quantum product), then often one may compute $J(\mathbf{t}, z^{-1})$ and ∇^z.
- For a toric manifold X, such an I function can be found through the $\mathbb{C}^\times$-localization data with $\hat{\mathbf{t}} \in H^{\leq 2}(X)$.
- [Lian–Liu–Yau 1996, Givental 1996] For $c_1(X) \geq 0$, $I(\hat{\mathbf{t}}, z^{-1})$ can be found and $J(\hat{\mathbf{t}}, z^{-1})$ is obtained by a *mirror transform*.
- [Coates–Givental 2005, Iritani 2008, Brown 2010] $I(\hat{\mathbf{t}}, z, z^{-1})$ is found for all toric manifolds. However, the structures and computations are *far more complicated*. Need BF/GMT:

Birkhoff Fatcorizations + Generalized Mirror Transform.

Example: a Fano toric bundle

$$X = P_{P^1}(\mathscr{O}(-1) \oplus \mathscr{O}) \xrightarrow{\pi} P^1,$$

$$c_1(X) = h + 2\xi > 0,$$

$$H(X) = \mathbb{C}[h, \xi]/(h^2, \xi(\xi - h)).$$

Let ℓ be the zero section, γ the fiber line, then

$$NE(X) = \mathbb{Z}\ell + \mathbb{Z}\gamma.$$

$$QH(X) = ?$$

- $\{T_0, T_1, T_2, T_3\} = \{1, h, \xi, \xi^2\}$,

$$\hat{\mathbf{t}} = t^0 T_0 + D, \qquad D = t^1 h + t^2 \xi \in H^2.$$

- Let $q_1 = q^\ell e^{t^1}$ and $q_2 := q^\gamma e^{t^2}$ (small parameters), then

$$I(\hat{\mathbf{t}}, z^{-1}) := e^{\frac{t^0 T_0}{z}} \sum_{\beta = d_1 \ell + d_2 \gamma} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta = e^{\frac{\hat{\mathbf{t}}}{z}} \sum_{d_1, d_2 = 0}^{\infty} q_1^{d_1} q_2^{d_2} I_{d_1, d_2},$$

$$I_{d_1, d_2} := \frac{1}{\prod_{m=1}^{d_1} (h + mz)^2 \prod_{m=1}^{d_2 - d_1} (\xi - h + mz) \prod_{m=1}^{d_2} (\xi + mz)} = O(z^{-2}).$$

- [LLY, Givental] $\Longrightarrow I(\hat{\mathbf{t}}, z^{-1}) = J(\hat{\mathbf{t}}, z^{-1})$. However, t^3 is missing.
- In general, if $c_1(X).\beta < 0$ for some β, then the z power $\to +\infty$.

- Technique: use *Naive Quantization* to replace $z\partial_3 J$: e.g.

$$\widehat{T}_i I = z\partial_i I, \quad i = 0,1,2, \qquad \widehat{T}_3 I = \widehat{\xi^2} I := (z\partial_2)^2 I.$$

- In general, since $I \in \mathscr{D}^z J$, we get $\widehat{T}_i I \in \mathscr{D}^z J$ too. Hence

$$\left(\widehat{T}_i I\right)(\hat{\mathbf{t}}, z, z^{-1}) = z\nabla J(\sigma(\hat{\mathbf{t}}), z^{-1}) B(\hat{\mathbf{t}}, z).$$

- The unique gauge transform is called the BF. It implies

$$J(\sigma(\hat{\mathbf{t}}), z^{-1}) = z\partial_0 J = \sum_i \widehat{T}_i I \cdot (B^{-1})_i^0 =: P(\hat{\mathbf{t}}, z\partial_1, z\partial_2) I(\hat{\mathbf{t}}, z, z^{-1}).$$

- The z^{-1} coefficient of PI gives the GMT: $\hat{\mathbf{t}} \mapsto \sigma(\hat{\mathbf{t}}) \in H_{\mathscr{R}}$.
- In practice, we study $B, \sigma(\hat{\mathbf{t}})$ via the Picard–Fuchs equations of I:

$$\begin{aligned} \Box_\ell &= (z\partial_1)^2 - q_1(z\partial_2 - z\partial_1), \\ \Box_\gamma &= (z\partial_2 - z\partial_1) z\partial_2 - q_2. \end{aligned}$$

- This leads to the connection matrix in the frame $\widehat{T}_i I$:

$$z\partial_a(\widehat{T}_i I) = (\widehat{T}_i I) C_a(\hat{\mathbf{t}}, z), \qquad a = 1, 2.$$

- In this example the choice of $\{\widehat{T}_i I\}$ leads to

$$C_1 = \begin{bmatrix} & & -q_2 & q_1 q_2 \\ 1 & -q_1 & & q_2 \\ & q_1 & & \\ & & 1 & \end{bmatrix}, \quad C_2 = \begin{bmatrix} & -q_2 & & q_1 q_2 + z q_2 \\ & & & q_2 \\ 1 & & & q_2 \\ & 1 & 1 & \end{bmatrix}.$$

- $B(\hat{\mathbf{t}}, z) = I_4 + q_2 e_{03}$ ($\widehat{\xi^2} \mapsto \widehat{h\xi}$) removes the z-dependence:

$$\tilde{C}_2(\hat{\mathbf{t}}) = -(z\partial_2 B)B^{-1} + BC_2(\hat{\mathbf{t}}, z)B^{-1} = \begin{bmatrix} & & q_2 & q_1 q_2 \\ & & & q_2 \\ 1 & & & \\ & 1 & 1 & \end{bmatrix}.$$

- The first column $\Longrightarrow \sigma(\hat{\mathbf{t}}) = \hat{\mathbf{t}}$. In general $\tilde{C} = \sigma^* A$: i.e.

$$\tilde{C}_a(\hat{\mathbf{t}}) = \sum_\mu A_\mu(\sigma(\hat{\mathbf{t}})) \frac{\partial \sigma^\mu}{\partial t^a}.$$

2. Quantum Motives? The Functoriality Problem

Q: Which part of the structure on $QH(X)$ is functorial?

- $\mathcal{M}_k$: the category of Chow motives, k the ground field.
- Objects: $\hat{X}$, where X a smooth variety over k.
- Morphisms are correspondences

$$\Gamma \in \mathrm{Mor}(\hat{X}, \hat{X}') := A(X \times X').$$

- Induced map on Chow groups: $[\Gamma]_* : A(X) \to A(X')$:

$$\alpha \mapsto \pi'_*(\Gamma.\pi^*\alpha).$$

- Linear structures: if $\hat{X} \cong \hat{X}'$ then $A^i(X) \cong A^i(X')$ for all i. If k is a number field, X and X' have the same L functions for each i.
- However, the ring structures are different: $A(X) \ncong A(X')$!
- [Wang 2002] Is there a *universal product structure* defined on Chow motives? Namely a universal family $(\mathscr{A}, *) \to T$ such that all geometric realizations $(A(X), \bullet)$ correspond to *special points*.

- Typical examples come from ordinary (r, r')-flops/flips:

$$
\begin{array}{ccccc}
 & & E = Z \times_S Z' \subset Y & & \\
 & \swarrow^{\phi} & & \searrow^{\phi'} & \\
P^1 \cong \ell \subset Z \subset X & & \overset{f}{\dashrightarrow} & & X' \supset Z' \supset \ell' \cong P^1 \\
 & \searrow_{\psi} & & \swarrow_{\psi'} & \\
 & & S \subset \overline{X} & &
\end{array}
$$

- $\bar{\psi} : Z = P_S(F) \to S$, $\operatorname{rk} F = r + 1$, ψ-extremal ray $\ell = [C]$.
- $N_{Z/X}|_{\bar{\psi}^{-1}(s)} \cong \mathscr{O}_{P^r}(-1)^{\oplus(r'+1)}$ for all $s \in S$.
- $Y = \mathrm{Bl}_Z X = \mathrm{Bl}_{Z'} X'$, $K_Y = \phi^* K_X + r'E = \phi'^* K_{X'} + rE$. Hence

$$\phi^* K_X = \phi'^* K_{X'} + (r - r')E.$$

- For flops $r = r'$, we have K-equivalence and $\hat{X} \cong \hat{X}'$ via

$$\Phi := [\bar{\Gamma}_f]_* = \phi'_* \circ \phi^* : H(X) \xrightarrow{\sim} H(X').$$

- It preserves the Poincaré pairing

$$(\Phi a.\Phi b)^{X'} = (\phi'^* \Phi a.\phi^* b)^Y = ((\phi^* a + \xi).\phi^* b)^Y = (a.b)^X,$$

but NOT the cup product!

- For the simple case ($S = \mathrm{pt}$), let $\alpha_i \in H^{2l_i}(X)$, $\sum_{i=1}^3 l_i = \dim X$,

$$(\Phi\alpha_1.\Phi\alpha_2.\Phi\alpha_3)^{X'} = (\alpha_1.\alpha_2.\alpha_3)^X - \prod_{i=1}^3 (\alpha_i.h^{r-l_i})^Z,$$

where $h = c_1(\mathscr{O}_Z(1)) \in H^2(Z)$.

- Solution: use quantum product $(QH_{\mathbf{t}}, *_{\mathbf{t}})$ instead.

- The effectivity of extremal curve is not preserved:

$$\Phi\ell = -\ell' \notin NE(X').$$

- It is necessary to consider analytic continuations $\overline{QH(X)}$ of $QH(X)$ along the Kähler moduli via the *partial compactification*

$$\Phi q^{\beta} = q^{\Phi\beta} \quad \text{toward} \quad \text{“}q^{\ell} = \infty\text{”}.$$

- For flops, the functoriality is simply the canonical isomorphism

$$\Phi : \overline{QH(X)} \xrightarrow{\sim} \overline{QH(X')}.$$

- In terms of Gromov–Witten invariants: for $\mathbf{t} \in H(X)$,

$$\Phi\langle\langle T_i, T_j, T_k\rangle\rangle^X(\mathbf{t}) = \langle\langle \Phi T_i, \Phi T_j, \Phi T_k\rangle\rangle^{X'}(\Phi\mathbf{t}).$$

- [Li–Ruan] for 3-folds, [LLW, LLQW] for general ordinary flops.

- The simplest non K-equivalent birational maps *preserving the dimension of Kähler moduli* are smooth ordinary flips.
- *Pseudo-abelian completion of Chow motives* $\widetilde{\mathcal{M}}$: objects $(\hat{X}, p)$, where $p \in \mathrm{End}(\hat{X}) = A(X \times X)$ is a projector: $p^2 = p$. Then

$$\hat{X} \equiv (\hat{X}, 1) = (\hat{X}, p) \oplus (\hat{X}, 1 - p).$$

- For flips with $r > r'$, $\Psi := [\bar{\Gamma}_{f^{-1}}]$ induces a sub-motive

$$\Psi : \hat{X}' \xrightarrow{\sim} (\hat{X}, p), \qquad p := \Psi \circ \Phi.$$

- On cohomology

$$\Psi : H(X') \hookrightarrow H(X),$$

the Poincaré pairing is still preserved $(\Psi a . \Psi b)^X = (a.b)^{X'}$, but not the cup product. **Not even the quantum product!**

- Solutions?

3. Statements of Results for Simple Flips

$$f : X \dashrightarrow X'$$

- We would like to show that $QH(X')$ can still be regarded as a sub-theory of $QH(X)$ in a canonical, though *non-linear*, manner.
- First of all, there is a basic *split* exact sequence

$$0 \longrightarrow K \longrightarrow H(X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} H(X') \longrightarrow 0 .$$

- The kernel space (vanishing cycles) K has dimension $d := r - r'$ and is orthogonal to $\Psi H(X')$:

$$K = \bigoplus_{j=r'+1}^{r} \mathbb{C}[P^j].$$

- Secondly, the Dubrovin connection ∇ can be analytically continued *along the Kähler moduli* to a connection $\Phi\nabla$ under the rule

$$\Phi q^{\beta} = q^{\Phi\beta}, \qquad \beta \in NE(X).$$

- As before $\Phi\ell = -\ell'$ and analytic continuations are required.

- We use identification of *divisorial coordinates* t^i and Novikov variables q^{β_i} (divisor axiom): let $D = \sum t^i D_i$, $(D_i.\beta_j) = \delta_{ij}$,

$$q_i := q^{\beta_i} e^{t^i}, \qquad \partial_i = \frac{\partial}{\partial t^i} = q_i \frac{\partial}{\partial q_i}.$$

- Hence

$$\nabla_\mu = \partial_\mu - \frac{1}{z} T_\mu *$$

  has only (formal) *regular singularities* at $q_i = 0$.

- The resulting connection $\Phi\nabla$ turns out to be analytic in the extremal ray variable q^ℓ and contains *irregular singularities in the K directions* along $q^\ell = \infty$, that is $q^{\ell'} = 0$.

- This suggests to extract the Dubrovin connection ∇' on $TH'_{\mathscr{R}'}$, where $H' = H(X')$ and $\mathscr{R}' = \mathbb{C}[\![NE(X')]\!]$, from $\Phi\nabla$

  by removing the K directions

  — since ∇' is (formlly) regular.

- We will show that there is a *bundle-decomposition*

$$TH \otimes \mathscr{R}'[1/q^{\ell'}] = \mathscr{T} \oplus \mathscr{K} \qquad (*)$$

  into irregular eigenbundle $\mathscr{K}$ which extends K over $\mathscr{R}'[1/q^{\ell'}]$ and the regular eigenbundle $\mathscr{T} = \mathscr{K}^{\perp}$.
- From WDVV equations, both $\mathscr{T}$ and $\mathscr{K}$ are shown to be integrable distributions.
- The integrable submanifold passing through the section

$$\mathcal{M}_{q'} \supset \{(q' \neq 0, \mathbf{t} = 0)\}$$

  is then the proposed manifold corresponding to $QH(X')$.
- However, to relate $\mathscr{T}$, and hence $\mathcal{M}_{q'}$, to $QH(X')$, we need to work on the connection (z-dependent) version of $(*)$.
- Hence there are non-trivial BF/GMT involved, and it is unclear what kind of functoriality should exist.

- The end result turns out to be quite satisfactory — the product structure is preserved but not the metric (Poincaré pairing)!

Theorem (Lee–Lin–Wang, 2017)

For the local model $f : X \dashrightarrow X'$ of simple (r, r') flips, there is a unique $\mathscr{R}'$-point $\sigma_0(q') \in H'_{\mathscr{R}'}$ and a unique embedding $\hat{\Psi}(q', \mathbf{s})$ over $\mathscr{R}'$:

$$\begin{aligned} \hat{\Psi} : H(X')_{\mathscr{R}'} &\longrightarrow \mathcal{M} \hookrightarrow H(X)_{\mathscr{R}'}, \\ \sigma_0(q') + \mathbf{s} &\longmapsto \hat{\Psi}(q', \mathbf{s}). \end{aligned}$$

where $\mathbf{s} \in H(X')$, such that

(1) $(\hat{\Psi}, \sigma_0)$ *restricts to* $(\Psi : H' \hookrightarrow H, 0)$ *when modulo* $q^{\ell'}$,

(2) $\hat{\Psi}$ *induces an F-embedding over* $\mathscr{R}'[1/q^{\ell'}]$:

$$(TH'_{\mathscr{R}'[1/q^{\ell'}]}, \nabla') \xhookrightarrow{d\hat{\Psi}} (TH_{\mathscr{R}'[1/q^{\ell'}]}, \nabla)|_{\mathcal{M}} \longrightarrow \mathscr{K} \cong N_{\hat{\Psi}} .$$

- In particular, outside the divisor $q^{\ell'} = 0$, the big quantum products on the corresponding tangent spaces are preserved.
- Denote the tangent frame by $\widehat{\Psi}_i = \partial_i \widehat{\Psi}$ and the induced metric by

$$\mathbf{g}_{ij} = (\widehat{\Psi}_i, \widehat{\Psi}_j), \qquad \widehat{\Psi}^i := \sum \mathbf{g}^{ij} \widehat{\Psi}_j.$$

- Then $\widehat{\Psi}$ is an F-embedding:

$$\langle\langle \widehat{\Psi}_\mu, \widehat{\Psi}^i, \widehat{\Psi}_j \rangle\rangle^X (\widehat{\Psi}(q', \mathbf{s})) = \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'} (\sigma_0(q') + \mathbf{s}).$$

- Hence there is a family of ring isomorphisms/decompositions:

$$Q_{\widehat{\Psi}(q', \mathbf{s})} H(X) \cong Q_{\sigma_0(q') + \mathbf{s}} H(X') \times \mathbb{C}^{r - r'},$$

which depend on the points $(q', \mathbf{s})$.

4. STEP (i)

Irregular Singularity of $\overline{QH(X)}$ along Vanishing Cycles

- Small parameters $\hat{\mathbf{t}} = t^0 T_0 + D \in H^{\leq 2}(X)$, $\hat{\mathbf{s}} = s^0 T_0' + D'$.

$$D = t^1 h + t^2 \xi = \Psi D' = \Psi(s^1 h' + s^2 \xi') = s^1(\xi - h) + s^2 \xi.$$

$$s^1 = -t^1, \qquad s^2 = t^2 + t^1.$$

- Kähler moduli: $NE(X) = \mathbb{Z}\ell \oplus \mathbb{Z}\gamma$, $NE(X') = \mathbb{Z}\ell' \oplus \mathbb{Z}\gamma'$.

$$\Phi\ell = -\ell', \qquad \Phi\gamma = \gamma' + \ell',$$

$$q_1 = q^\ell e^{t^1}, \qquad q_2 = q^\gamma e^{t^2},$$

$$x = q_1' = q^{\ell'} e^{s^1} = 1/q_1, \qquad y = q_2' = q^{\gamma'} e^{s^2} = q_1 q_2.$$

- Naive quantization, for $i \in [0, r]$, $j \in [0, r' + 1]$, $a = h^i \xi^j$,

$$\hat{a} \equiv \partial^{za} := \hat{h}^i \hat{\xi}^j = (z\partial_h)^i (z\partial_\xi)^j = (z\partial_1)^i (z\partial_2)^j.$$

- X is Fano, $c_1(X) = (r - r')h + (r' + 2)\xi$ is ample,
- X' is bad, $c_1(X') = (r' - r)h' + (r + 2)\xi'$ has no fixed sign.

- For $\beta = d_1 \ell + d_2 \gamma \in NE(X)$,

$$I_\beta = \frac{1}{\prod_{m=1}^{d_1} (h + mz)^{r+1} \prod_{m=1}^{d_2 - d_1} (\xi - h + mz)^{r'+1} \prod_{m=1}^{d_2} (\xi + mz)}$$

- $I = e^{\hat{\mathbf{t}}/z} \sum_\beta e^{D.\beta} q^\beta I_\beta$ is annihilated by Picard–Fuchs equations:

$$\Box_\ell = (z\partial_h)^{r+1} - q_1 (z\partial_{\xi - h})^{r'+1},$$
$$\Box_\gamma = z\partial_\xi (z\partial_{\xi - h})^{r'+1} - q_2.$$

- $I = I(z^{-1}) \implies I = J_{small}$ and $Q_0 H(X)$ is "easy". It is still non-trivial to write down the Dubrovin connection ∇^X.
- The naive frame, for $\mathbf{e} = h^i \xi^j$ (or even $h^i (\xi - h)^j$),

$$\partial^{z\mathbf{e}} I \equiv \hat{h}^i \hat{\xi}^j I := (z\partial_h)^i (z\partial_\xi)^j I$$

does not lead to z-free connection matrices for $z\partial_1, z\partial_2$!

Example: the case of $(2,1)$ flips.

- For the naive frame respecting $H(X) = \Psi H(X') \oplus^{\perp} K$, with $v_6 = \hat{\kappa}_0 = (\hat{\xi} - \hat{h})^2$, we have

$$z\partial_1(\partial^{z\mathbf{e}} I) = z\, q_1 \frac{\partial}{\partial q_1}(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I)\, C_1(q,z),$$

$$C_1(q,z) = \begin{bmatrix} & & & & & & q_1q_2 & & -zq_1q_2 \\ 1 & & & & & & & & -q_1q_2 \\ & & & & & & & & q_1q_2 \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & 1 & -1 & q_1 & & & -zq_1 & & z^2q_1 \\ & & & & 1 & & & & \\ & & & -1 & 1 & 1 & -q_1 & & 2zq_1 \\ & & & & & & & 1 & q_1 \end{bmatrix}.$$

- It is even unclear where the irregular singularities at $q_1 = \infty$ are located. (Not just in the K directions?)

The Ψ-corrected quantum frame

- The quantized basis corresponding to $\ker\Phi$ is chosen to be

$$\hat{\kappa}_i I = \hat{h}^i(\hat{\xi} - \hat{h})^{r'+1} I, \qquad i \in [0, r - r' - 1].$$

- For $e_1 \in [0, r+1]$, $e_2 \in [0, r']$, we define

$$v_{\mathbf{e}} := \hat{h}^{e_1}(\hat{\xi} - \hat{h})^{e_2} I + \delta_{(e_1,e_2)}(-1)^{r'-e_2}\hat{\kappa}_{e_1+e_2-(r'+1)},$$

  where

$$\begin{cases} \delta_{(e_1,e_2)} = 0 & \text{if } e_1 + e_2 \in [0, r'], \text{ and} \\ \delta_{(e_1,e_2)} = 1 & \text{otherwise.} \end{cases}$$

- The added term comes from $\ker\Phi \iff e_1 + e_2 \in [r'+1, r]$.
- But $H^{2j}(X')$ with $j \geq r+1$ are also corrected accordingly.
- The frame reduces to a classical basis when modulo $NE(X)$.

The connection matrices for $z\partial_1$ and $z\partial_2$.

- For $i = 1, 2$, the connection matrix $C_i(q_1, q_2)$ in the Ψ corrected frame is independent of z. Moreover, $A_i(\hat{\mathbf{t}}) = C_i$.
- Write $C_i = \begin{bmatrix} C_i^{11} & C_i^{12} \\ C_i^{21} & C_i^{22} \end{bmatrix}$ wrt. $H(X) = \Psi H(X') \oplus^{\perp} K$.
- Let $d = \dim K = r - r'$.
- For C_1, the $d \times d$ block corresponding to $\ker \Phi$ is given by

$$C_1^{22} = \begin{bmatrix} & & & (-1)^{r'+1} q_1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}.$$

- Other entries in C_1 and C_2 have "good properties"!

- **Corollary 1.** The Ψ-corrected frame corresponds to the constant frame for ∇^X.

- **Corollary 2.** Under the analytic continuation in the Kähler moduli over $NE(X')$, ∇^X is irregular in the divisor $(x = 0)$ precisely in the kernel block.

- To proceed, we denote

$$R = \dim H(X) = (r+1)(r'+2),$$
$$R' = \dim H(X') = (r+2)(r'+1).$$

And then $d = R - R' = r - r' = \dim K$.

5. STEP (ii)

Block Diagonalizations and BF/GMT over $NE(X')$

- We have $A_j(\hat{\mathbf{t}}) = C_j, j = 1, 2$:

$$C_1^{22} = \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} q_1 \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} = \frac{1}{x} \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} \\ x & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & x & 0 \end{bmatrix}.$$

- We will now work on the irregular system of PDE in variables (x, y) with a parameter z.
- The irregularity comes only from x, and it is thus necessary to keep track of the lowest order entries in x in C_j's.
- A transformation is needed to bring C_1^{22} into its "semisimple" form: let $u = x^{1/d}$, we modify the constant frame to $\{T_i\}$ with

$$\{T_i\}_{i=0}^{R'-1} = \{T_{\mathbf{e}}\}, \qquad \{T_{R'+i}\}_{i=0}^{d-1} = \{u^i k_i\}_{i=0}^{d-1}.$$

Lemma on shearing (= base change in $\mathscr{D}$-modules).

- Let $Y(x) = \mathrm{diag}(1^{R'}, u^0, u^1, \cdots, u^{d-1})$. After substitutions $S = YW$ and $x = u^d$, the equation $zx\frac{\partial}{\partial x}S = C_1 S$ becomes

$$zu\frac{\partial}{\partial u}W = D_1(u,z)W, \qquad (**)$$

$$D_1^{11} = d \cdot C_1^{11},$$
$$D_1^{12} = d \cdot C_1^{12} \cdot \mathrm{diag}(u^0, u^1, \cdots, u^{d-1}),$$
$$D_1^{21} = d \cdot \mathrm{diag}(u^0, u^{-1}, \ldots, u^{-d+1}) \cdot C_1^{21},$$
$$D_1^{22} = \frac{d}{u} \cdot \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} \\ 1 & -z\frac{1}{d}u & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & -z\frac{d-1}{d}u \end{bmatrix}.$$

- D_1^{21} is polynomial in u. Thus, $(**)$ is irregular of Poincaré rank 1 in u, and the irregular part only appears in the $(2,2)$ block D_1^{22}.

- Therefore, $D_1(z=0)$ has R eigenvalues, including $0^{R'}$ and d *distinct nonzero* eigenvalues from $D_1^{22}(0)$ as solutions to

$$\omega^d = (-1)^{r'+1}.$$

- By the classical procedure due to Wasow/Shibuya, together with the flatness of the Dubrovin connection, we conclude that

(i) The connection matrices C_1, C_2 can be *simultaneously block diagonalized* to $\tilde{C}_1, \tilde{C}_2$, such that the $(2,2)$ blocks are *diagonalized*.

(ii) Furthermore, the block-diagonalization frame (gauge matrix)

$$P = [\tilde{T}_0, \ldots, \tilde{T}_{R'-1}, \tilde{T}_{R'}, \ldots, \tilde{T}_{R-1}] = \begin{bmatrix} I_{R'} & * \\ * & I_d \end{bmatrix}$$

can be chosen so that $\tilde{T}_i$ has the initial term T_i in u.

(iii) $\mathscr{T}$ spanned by $\tilde{T}_0, \ldots, \tilde{T}_{R'-1}$ and $\mathscr{K}$ spanned by $\tilde{T}_{R'}, \ldots, \tilde{T}_{R-1}$ lead to *reduction of connection* and are *orthogonal* to each other.

- **Extract $QH(X')$ from $QH(X)$:** On X', let $\beta' = d_1'\ell' + d_2'\gamma'$, then

$$I_{\beta'}^{X'} = \frac{1}{\prod_1^{d_1'}(h'+mz)^{r'+1}\prod_1^{d_2'-d_1'}(\xi'-h'+mz)^{r+1}\prod_1^{d_2'}(\xi'+mz)}.$$

- It has Picard–Fuchs equations

$$\Box_{\ell'} := (z\partial_2 - z\partial_1)^{r'+1} - q_1'(z\partial_1)^{r+1},$$
$$\Box_{\gamma'} := (z\partial_2)(z\partial_1)^{r+1} - q_2'.$$

- Since $\Box_{\ell'} = q_1^{-1}\Box_\ell$ and $\Box_{\gamma'} = z\partial_2\Box_\ell - q_1\Box_\gamma$, we get the
- **Key Lemma.** Over $\mathbb{C}[q_1, q_1^{-1}, q_2] \cong \mathbb{C}[q_1', q_1'^{-1}, q_2']$, we have

$$\langle \Box_\ell, \Box_\gamma \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle.$$

- **Corollary.** The matrices $\tilde{C}_1^{11}, \tilde{C}_2^{11}$ can be used to compute $\nabla^{X'}$.

- For $a, b \in H(X)$ we have $ab = a * b + \sum_\beta q^\beta c_\beta$ for some $c_\beta \in H(X)$. By induction on the Mori cone we conclude that

$$T_\mu * = \sum_{\beta \in NE(X)} q^\beta P_\beta(h*, \xi*)$$

where P_β is a polynomial. Since X is Fano, the sum is finite.

- So the block diagonalization in $u = x^{1/d}, y, z$ extends to all $T_\mu *$.
- In fact $\tilde{C}_1^{11}$ and $\tilde{C}_2^{11}$, hence all $\tilde{C}_\mu^{11}$, are expressible in x, y, z.
- Two technical problems:

(i) Remove the NEW z-dependence in $\tilde{C}_\mu^{11}(x, y, z)$ *introduced in the block-diagonalization*. (Sol. BF/GMT.)

(ii) Since $T_\mu *$ is generated by $h*$ and $\xi*$ over $NE(X)$ instead of over $NE(X')$, *will* $\tilde{C}_\mu^{11}(x, y, z)$ *contain negative powers in x?* (Sol. No!)

(i) Let $B_1 = B_1(x,y,z)$ be the BF matrix and $B_1(0) := B_1(x,y,0)$.

$$[\mathbf{T}_0,\ldots,\mathbf{T}_{R'-1}] := \left([\tilde{T}_0,\ldots,\tilde{T}_{R'-1}]B_1^{-1}\right)(z=0).$$

- Under $x = q^{\ell'} e^{s^1}$, $y = q^{\gamma'} e^{s^2}$, $a = 0,1,2$, the "z-free" matrix

$$C'_a(\hat{\mathbf{s}}) = -(z\partial_a B_1)B_1^{-1} + B_1 \tilde{C}_a^{11} B_1^{-1} = B_1(0)\tilde{C}_{a;0}^{11} B_1(0)^{-1}(x,y)$$

is related to $A'_\mu(\sigma)$ for $T'_\mu *'$ at $\sigma = \sigma(\hat{\mathbf{s}}) \in H(X')[\![x,y]\!]$ via

$$C'_a(\hat{\mathbf{s}}) = \sum_\mu A'_\mu(\sigma(\hat{\mathbf{s}}))\frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}), \qquad a = 0,1,2,$$

$$\langle\langle T_a, \mathbf{T}_j, \mathbf{T}^i \rangle\rangle^X(\hat{\mathbf{s}}) = \sum_\mu \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}) \langle\langle T'_\mu, T'_j, T'^i \rangle\rangle^{X'}(\sigma(\hat{\mathbf{s}})).$$

- Since $(A'_\mu)_0^i = \delta_\mu^i$, $\sigma(\hat{\mathbf{s}})$ is determined by the first column:

$$(C'_a)_0^\mu(\hat{\mathbf{s}}) = \langle\langle T_a, \mathbf{T}_0, \mathbf{T}^\mu \rangle\rangle^X(\hat{\mathbf{s}}) = \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}).$$

6. STEP (iii)

The Non-Linear F-Embedding $QH(X') \hookrightarrow \overline{QH(X)}$

(ii) The next step is to transform $\mathbf{T}_0$ to the identity element (section) $e \in \mathscr{T}$ and normalized $\mathbf{T}_i$'s to $\tilde{\mathbf{T}}_i$'s accordingly.

- **Lemma.** There is a unique element $\mathbf{S}_0 \in \mathscr{T}$ such that

$$\mathbf{S}_0 * \mathbf{T}_0 = e,$$

and so e acts as zero on $\mathscr{K}$. (This requires delicate calculations!)

- Define the *normalized frame* on $\mathscr{T}$ by

$$\widetilde{\mathbf{T}}_\mu := \mathbf{T}_\mu * \mathbf{S}_0.$$

- Theorem (Initial quantum invariance up to a shifting)

Let $\mathbb{T}_i(q') = \widetilde{\mathbf{T}}_i(q', \hat{\mathbf{s}} = 0, z = 0)$ *and* $\sigma_0(q') = \sigma(q', \hat{\mathbf{s}} = 0)$. *Then we have*

$$\langle \mathbb{T}_\mu, \mathbb{T}^i, \mathbb{T}_j \rangle^X = \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'}(\sigma_0(q')).$$

- An F-manifold M is a complex manifold with a commutative product structure on each T_pM, such that a WDVV-type integrability condition is forced when $p \in M$ varies.
- In $QH(X)$, this is the structure which remembers $*_p$ but forgets the metric g_{ij}. Hertling and Manin showed that the WDVV equations can be rewritten as

$$L_{X*Y} * = X * L_Y * + Y * L_X *$$

  for any local vector fields X and Y.
- I.e., for any local vector fields X, Y, Z, W:

$$\begin{aligned}[X * Y, Z * W] - [X * Y, Z] * W - [X * Y, W] * Z \\ = X * [Y, Z * W] - X * [Y, Z] * W - X * [Y, W] * Z \\ + Y * [X, Z * W] - Y * [X, Z] * W - Y * [X, W] * Z.\end{aligned}$$

- Denote by $\mathcal{K}$ the irregular eigenbundle and $\mathcal{T} := \mathcal{K}^{\perp}$ the regular eigenbundle, which extend $\mathscr{K}$ and $\mathscr{T}$ from $\mathbf{s} = 0$ to big $\mathbf{s}$.

- **Lemma**

  $\mathcal{T}$ is an integrable distribution of the relative tangent bundle $TH_{\mathscr{R}'}$.

  In particular, $\operatorname{Im}\widehat{\Psi}$ is the integral submanifold $\mathcal{M}$ (over $\mathscr{R}'$) containing the slice $(q^{\ell'} \neq 0, \mathbf{t} = 0)$ which contains $\operatorname{Im}\Psi$ when modulo $\mathscr{R}'$.

- **Proof.**

  Let X, Z be any local vector fields in $\mathcal{T} = \mathcal{K}^{\perp}$. Let $Y = e_i$ and $W = e_j$ be idempotents in $\mathcal{K}$. Since $a * b = 0$ for $a \in \mathcal{K}, b \in \mathcal{K}^{\perp}$,

$$0 = -X * Z * [e_i, e_j] - \delta_{ij} e_j * [X, Z].$$

  Let $i = j$ we get $e_j * [X, Z] = 0$ for all j. Hence $[X, Z] \in \mathcal{K}^{\perp}$. □

- The quantum product on the Frobenius manifold $H(X') \otimes \mathscr{R}'$ is semi-simple. Let $v'_0, \ldots, v'_{R'-1}$ be the idempotent vector fields.
- **Dubrovin 1996:** $[v'_i, v'_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the corresponding *canonical coordinates* $u'^0, \ldots, u'^{R'-1}$ satisfying

$$(u'^i(q', \mathbf{s} = 0)) = \sigma_0(q')$$

  and $v'_i = \partial / \partial u'^i$ exist.
- This was extended to *F*-manifolds by Hertling. The *F*-manifold $\mathcal{M}$ is semi-simple in the sense that $*_p$ on $T_p\mathcal{M}$ for $p \in \mathcal{M}$ is semi-simple. Denote the idempotent vector fields by $v_1 \ldots, v_{R'}$.
- **Hertling 2002:** $[v_i, v_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the canonical coordinates $u^0, \ldots, u^{R'-1}$ near each $p \in \mathcal{M}$ exist in the sense that $v_i = \partial / \partial u^i$.

- **Fixing the initial correspondence of frames:**
- We have constructed an analytic family of coordinate systems $(u^0(q',p),\ldots,u^{R'-1}(q',p))$ parametrized by $q' \in \mathscr{R}'$. Write

$$\mathbb{T}_i(q') = \sum_{j=0}^{R'-1} a_i^j(q')\, v_j(q', \mathbf{s} = 0)$$

for an invertible $R' \times R'$ matrix $(a_i^j(q'))$.

-

$$\langle \mathbb{T}_\mu, \mathbb{T}^i, \mathbb{T}_j \rangle^X = \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'}(\sigma_0(q')). \tag{1}$$

From this relation, we see easily that:

- ## Lemma

After a possible reordering of $\{v'_j\}$, we have for all $i = 0, \ldots, R'-1$:

$$T'_i = \sum_{j=0}^{R'-1} a_i^j(q')\, v'_j(\sigma_0(q')).$$

- Now we define the map $\hat{\Psi}$ by *matching the canonical coordinates*. Namely, $\hat{\Psi}(q', \mathbf{s}) \in \mathcal{M}$ is the unique point on $\mathcal{M}$ so that

$$u^i(\hat{\Psi}(q', \mathbf{s})) = u'^i(q', \mathbf{s}) = u'^i(\sigma_0(q') + \mathbf{s})$$

  for $i = 0, \ldots, R' - 1$.

- Since the tangent map $\hat{\Psi}_*$ matches the idempotents

$$\hat{\Psi}_* \partial / \partial u'^i = \partial / \partial u^i,$$

  it induces a product structure isomorphism, and hence an F-structure isomorphism by "coordinates-free WDVV".

- Also along $\mathbf{s} = 0$, by Lemma we have

$$\hat{\Psi}_* T'_i = \mathbb{T}_i$$

  which matches the initial condition along the $\mathscr{R}'$-axis.

- $H(X')$ is contractible $\Longrightarrow$ $\hat{\Psi}$ exists globally. **QED**

Ending Remarks

- Work in progress by LLW:

(1) Globalization to simple (r, r') flips.

(2) Generalizations to ordinary flips with non-trivial base.

(3) Reconstruction of $QH(X)$ from $QH(X')$ and "the K-block".

- Other approaches to quantum flips:

(4) [Woodward et. al.] studying wall crossing of GW invariants in different GIT quotients.

(5) [Shoemaker et. al] studying asymptotic of I functions in the toric setup.

- Would be interesting to compare their approaches with ours.

Example: $(2,1)$ **flip**

$R = 9$, $R' = 8$. The following frame (recall $I = J_{small}$)

$$
\begin{aligned}
v_1 &= \hat{\mathbf{1}}J = J, \\
v_2 &= \hat{h}J, \quad v_3 = (\hat{\xi} - \hat{h})J, \\
v_4 &= \hat{h}^2 J - (\hat{\xi} - \hat{h})^2 J, \quad v_5 = \hat{h}(\hat{\xi} - \hat{h})J + (\hat{\xi} - \hat{h})^2 J, \\
v_6 &= \hat{h}^3 J - \hat{h}(\hat{\xi} - \hat{h})^2 J, \quad v_7 = \hat{h}^2(\hat{\xi} - \hat{h})J + \hat{h}(\hat{\xi} - \hat{h})^2 J, \\
v_8 &= \hat{h}^3(\hat{\xi} - \hat{h})J + \hat{h}^2(\hat{\xi} - \hat{h})^2 J, \\
v_9 &= \hat{\kappa}_0 J = (\hat{\xi} - \hat{h})^2 J,
\end{aligned}
$$

respects $H(X) = \Phi^{-1}H(X') \oplus^{\perp} K$ when modulo q_1, q_2.
They are precisely

$$z\partial_i J \quad \text{at } t \in H^0 \oplus H^2, \quad 1 \le i \le 9,$$

and we get the Dubrovin connection:

$$A_1 = h*_{small} = \begin{bmatrix} & & & & & q_1q_2 & & & \\ 1 & & & & & & & & \\ & & & & & & & q_1q_2 & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & -1 \\ & & & & 1 & & & & \\ & & & & & -1 & 1 & & \\ & 1 & -1 & & & & & & {\color{blue}q_1} \end{bmatrix},$$

$$A_2 = \xi*_{small} = \begin{bmatrix} & & & -q_2 & q_2 & q_1q_2 & & & q_2 \\ 1 & & & & & -q_2 & q_2 & & \\ 1 & & & & & & & q_1q_2 & \\ & 1 & & & & & & q_2 & \\ & 1 & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & 1 & 1 & & & & \\ & & & & & & 1 & & \\ & & & & & & & q_2 & \end{bmatrix}.$$

$$x := q_1' = 1/q_1, \qquad y := q_2' = q_1 q_2.$$

Chain rule: $y\,\partial_y = xy\,\partial_{q_2} = \partial_2$, and

$$x\,\partial_x = x(-x^{-2}\,\partial_{q_1} + y\,\partial_{q_2}) = -\partial_1 + \partial_2 = \partial_{\xi - h}.$$

Further simplification: Let $w_i = \sum_j v_j T_{ji}$

$$T := \left[\begin{array}{cccccccc|c}
1 &&&&&&&& \\
& 1 &&&&&&& \\
& \frac{1}{2} & 1 &&&&&& \\
&&& 1 &&&&& \\
&&& \frac{1}{2} & 1 &&&& \\
&&&&& 1 &&& \\
&&&&& \frac{1}{2} & 1 && \\
&&&&&&& 1 & \\
\hline
&&&&&&&& 1
\end{array}\right].$$

$$g_{ij} := (w_i, w_i)^X = \delta_{9,i+j}, \qquad 1 \le i,j \le 8,$$

and $w_9 = v_9 = \kappa_0$ satisfies $(w_9, w_i)^X = \delta_{9,i}$.

$$
A_1 = \left[\begin{array}{cccccccc|c}
 & & & -\frac{1}{2}xy & xy & & & & xy \\
 & & & & & -\frac{1}{2}xy & xy & & \\
1 & & & & & \frac{1}{4}xy & -\frac{1}{2}xy & & \\
 & & & & & & & xy & \\
 & 1 & & & & & & -\frac{1}{2}xy & \\
 & & & & & & & & 1 \\
 & & & 1 & & & & & -\frac{1}{2} \\
 & & & & & 1 & & & \\
\hline
 & -\frac{1}{2} & 1 & & & & & xy & {\color{blue}-1/x}
\end{array}\right],
$$

$$
A_2 = \left[\begin{array}{cccccccc|c}
 & & & -\frac{1}{2}xy & xy & y & & & xy \\
1 & & & & & -\frac{1}{2}xy & xy & & \\
\frac{1}{2} & & & & & \frac{1}{4}xy & -\frac{1}{2}xy & y & \\
 & 1 & & & & & & xy & \\
 & 1 & 1 & & & & & -\frac{1}{2}xy & \\
 & & & 1 & & & & & \\
 & & & 1 & 1 & & & & \\
 & & & & & \frac{1}{2} & 1 & & \\
\hline
 & & & & & & & xy &
\end{array}\right].
$$

Irregular in the K-block, of Poincaré rank one.

Block diagonalization w.r.t. $H(X) = \Phi^{-1}H(X') \oplus^{\perp} K$

(Wasow 1960's) + flatness of $\nabla^X \Longrightarrow$
$\exists!$ formal gauge transformation $S = PZ$

$$P(x,y,z) = I + \begin{bmatrix} 0 & g^{\bullet} \\ f_{\bullet} & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & g_1 \\ & \ddots & & \vdots \\ & & 1 & g_8 \\ f_1 & \cdots & f_8 & 1 \end{bmatrix},$$

such that

$$z(x\,\partial_x)Z = E_1\,Z, \qquad z(y\,\partial_y)Z = E_2\,Z$$

with E_1, E_2 being block diagonalized. Also, for $i' := 9 - i$,

$$f_i(x,y,z) = -\bar{g}_{i'} := -g_{9-i}(x,y,-z).$$

Get the deformed, (x,y,z)-dependent, frame

$$\widetilde{w}_i = w_i + f_i\hat{\kappa}_0, \quad 1 \le i \le 8, \qquad \widetilde{\hat{\kappa}}_0 = \hat{\kappa}_0 + \sum_{i=1}^{8} g_i w_i.$$

From

$$-z\partial_k P + A_k P = P E_k,$$

the block decomposition is equivalent to

$$\begin{bmatrix} A_k^{11} + A_k^{12} f_\bullet & -z\partial_k g^\bullet + A_k^{11} g^\bullet + A_k^{12} \\ -z\partial_k f_\bullet + A_k^{21} + A_k^{22} f_\bullet & A_k^{21} g^\bullet + A_k^{22} \end{bmatrix} = \begin{bmatrix} E_k^{11} & g^\bullet E_k^{22} \\ f_\bullet E_k^{11} & E_k^{22} \end{bmatrix}.$$

In particular we get the equation for f_i:

$$\begin{aligned} z\partial_k f_i &= A_k^{22} f_i + (A_k^{21})_i - \textstyle\sum_{j=1}^{8} f_j (E_k^{11})_{ji} \\ &= -\frac{\delta_{k1}}{x} f_i + (A_k)_{9i} - \textstyle\sum_{j=1}^{8} \Big(f_j \, (A_k)_{ji} + f_j \, (A_k)_{j9} f_i \Big). \end{aligned}$$

$k = 1$: system of inhomogeneous non-linear perturbation of

$$zx\, \partial_x\, h = -\frac{1}{x}\, h.$$

Formality in $s = zx$

$$f_1 = -x^2(1 - 3zx + 11z^2x^2 - 50z^3x^3 + (274z^4 + 6y)x^4 - (1764z^5 + 87yz)x^5 + (13068z^6 + 986yz^2)x^6 - (109584z^7 + 10803yz^3)x^7 + \cdots),$$

$$f_2 = -\frac{1}{2}x(1 - zx + 2z^2x^2 - 6z^3x^3 + (24z^4 + 5y)x^4 - (120z^5 + 54yz)x^5 + (720z^6 + 489yz^2)x^6 - (5040z^7 + 4472yz^3)x^7 + \cdots),$$

$$f_3 = x(1 - zx + 2z^2x^2 - 6z^3x^3 + (24z^4 + 3y)x^4 - (120z^5 + 30yz)x^5 + (720z^6 + 253yz^2)x^6 - (5040z^7 + 2168yz^3)x^7 + \cdots),$$

- Formal part: $f_2, f_3 \sim$ factorial series in zx.
- $f_1 \sim$ *Stirling numbers of first kind*, which counts the number of $\sigma \in S_{n+1}$ with exactly two cycles. It satisfies $a_0 = 1$,

$$a_n = (n+1)a_{n-1} + n!, \qquad n \geq 2.$$

Its closed form is $a_n = (n+1)!H_{n+1}$.

$$
\begin{aligned}
f_4 &= -\tfrac{1}{2}x^4 y(3 - 23zx + 162z^2x^2 - 1214z^3x^3 + (9972z^4 + 29y)x^4 + \cdots),\\
f_5 &= x^4 y(1 - 7zx + 46z^2x^2 - 326z^3x^3 + (2556z^4 + 9y)x^4 + \cdots),\\
f_6 &= -\tfrac{1}{2}x^3 y(3 - 14zx + 70z^2x^2 - 404z^3x^3 + (2688z^4 + 23y)x^4\\
&\qquad - (20376z^5 + 407yz)x^5 + (173808z^6 + 5454yz^2)x^6 + \cdots),\\
f_7 &= x^3 y(1 - 4zx + 18z^2x^2 - 96z^3x^3 + (600z^4 + 7y)x^4\\
&\qquad - (4230z^5 + 115yz)x^5 + (35280z^6 + 1448yz^2)x^6 + \cdots),\\
f_8 &= x^2 y(1 - 2zx + 6x^2z^2 - 24z^3x^3 + (120z^4 + 5y)x^4\\
&\qquad - (720z^5 + 63yz)x^5 + (5040z^6 + 642yz^2)x^6 + \cdots).
\end{aligned}
$$

- f_5: $a_n = n!(n - H_n)$. f_7: $a_n = n \cdot n!$
- $f_4 + \frac{1}{2}f_5 = -x^4y(1 - 8zx + 58z^2x^2 + 444z^3x^3 + 3708z^4x^4 + \cdots)$ with coefficients $a_n = (n+2)(H_{n+2} - 2) + (n+1)!$.
- $f_6 + \frac{1}{2}f_7 = -x^3y(1 - 5zx + 26z^2x^2 - 154z^3x^3 + \cdots)$ with coefficients $a_n = (n+1)!(H_{n+1} - 1)$.

Analyticity/Algebracity in $t = yx^4$

Consider the generalized hypergeometric series

$$b = F(\tfrac{1}{9}, \cdots, \tfrac{8}{9}; \tfrac{2}{8}, \cdots, \widehat{\tfrac{8}{8}}, \tfrac{9}{8}; \tfrac{9^9}{8^8} t)$$
$$= \sum_{n \geq 0} \binom{9n+1}{n} \frac{1}{9n+1} t^n,$$

which solves the algebraic equation

$$tb^9 = b - 1.$$

It is easy to see that

$$b^l = F(\tfrac{l}{9}, \cdots; \tfrac{l+1}{8}, \cdots; \tfrac{9^9}{8^8} t) = \sum_{n \geq 0} \binom{9n+l}{n} \frac{l}{9n+l} t^n$$

is the $(l-1)$-th shift with $\frac{9}{9}$ and $\frac{8}{8}$ skipped.

By solving the quadratic system on h_i's arising from $k = 2$:

Theorem (Algebraicity in the CY class $t = yx^4$)

Denote $f_1(x, y, 0), \ldots, f_8(x, y, 0)$ by

$$x^2h_1,\ xh_2,\ xh_3,\ h_4,\ h_5,\ x^{-1}h_6,\ x^{-1}h_7,\ x^{-2}h_8.$$

Then $h_i(t)$ depends on t only and we have

$$\begin{aligned}
h_1 &= -b^6, \\
h_2 &= \tfrac{1}{2}b^3 - b^4, \qquad h_3 = b^3, \\
h_4 &= \tfrac{1}{2}(1+b) - b^2, \qquad h_5 = -1 + b, \\
h_6 &= -\tfrac{1}{2}b^7t - b^8t, \qquad h_7 = b^7t, \\
h_8 &= b^5t.
\end{aligned}$$

Remark: For (r, r') flips, the CY direction is $(y^{r-r'}x^{r+2})^{1/D}$ where $D = \gcd(r - r', r + 2)$.

BF/GMT along extremal rays $x = q^{\ell'} e^{s_1}$ on X'

Denote $\delta = zx\,\partial_x$ and its (pseudo) inverse $\mathscr{I}$ by

$$\mathscr{I}\phi = \mathscr{I}(\phi - \phi(z=0)) = \int \frac{\phi - \phi(z=0)}{zx}\,dx.$$

For example, $\mathscr{I}(f_1/x) = \frac{3}{2}x^2 - \frac{11}{3}zx^3 + \frac{50}{4}z^2x^4 + \cdots \pmod{y}$.

Lemma (The Birkhoff factorization matrix B modulo y)

By writing $B = I + N$ we have $N^2 = 0$ and $B^{-1} = I - N$. In fact

$$B = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ -\mathscr{I}^2(f_1/x) & \mathscr{I}f_2 & \mathscr{I}f_3 & & & 1 & & \\ \frac{1}{2}\mathscr{I}^2(f_1/x) & -\frac{1}{2}\mathscr{I}f_2 & -\frac{1}{2}\mathscr{I}f_3 & & & & 1 & \\ \mathscr{I}^3(f_1/x - f_3) & -\mathscr{I}^2f_2 & -\mathscr{I}^2f_3 & & & & & 1 \end{bmatrix} \pmod{y}.$$

Corollary

For local $(2,1)$ *flips, the Dubrovin connection matrices modulo* y *and up to GMT are given by*

$$\bar{C}_1' = \begin{bmatrix} 0 & & & & & & & \\ 0 & & & & & & & \\ 1 & & & & & & & \\ & 0 & & & & & & \\ & 1 & & & & & & \\ -3x^2/2 & -x/2 & x & & & & & \\ 3x^2/4 & x/4 & -x/2 & 1 & 0 & & & \\ -13x^3/9 & -x^2/4 & x^2/2 & & & 1 & 0 & 0 \end{bmatrix}$$

and $\bar{C}_2' = A_2^{11} \pmod{y}$*. The GMT in the extremal ray variable is*

$$\begin{aligned} &\sigma(s^1 h' + s^2 \xi') \\ &= s^1 h' + s^2 \xi' + \tfrac{3}{4} e^{2s^1} q^{2\ell'} \xi'^2 h' - \tfrac{13}{27} e^{3s^1} q^{3\ell'} \xi'^3 h' \pmod{q^{\gamma'}}. \end{aligned}$$

Example of quantum invariance without BF/GMT

For local $(2,1)$ flip, the final frame $\mathbb{T}_1 \pmod{y}$ is

$$[\xi - h] := (\tilde{\xi} - \tilde{h})(y = 0, z = 0) = (\xi - h) + x\kappa_0.$$

Theorem (Invariance along extremal rays)

For extremal primary Gromov–Witten invariants of $n \geq 1$ insertions,

$$\langle [\xi - h]^{\otimes n} \rangle^X = \langle (h')^{\otimes n} \rangle^{X'} = q^{\ell'}.$$

This is equivalent to the quantum interpretation of Cayley's formula

$$a_d := \langle \kappa_0^{\otimes (d+1)} \rangle^X_{d\ell} = d^{d-2}, \qquad d \geq 1,$$

which is the number of spanning trees in the complete graph on d vertexes (and hence with $d-1$ edges).

Degenerate case I: Flops, $r = r'$**,** $\ker \Phi = 0$

E.g. Atiyah flops $r = 1$. The Ψ-corrected frame is

$$
\begin{aligned}
v_1 &= I, \\
v_2 &= \hat{h} I, \quad v_3 = (\hat{\xi} - \hat{h}) I, \\
v_4 &= \hat{h}^2 I - (\hat{\xi} - \hat{h})^2 I, \quad v_5 = \hat{h}(\hat{\xi} - \hat{h}) I + (\hat{\xi} - \hat{h})^2 I, \\
v_6 &= \hat{h}^2 (\hat{\xi} - \hat{h}) I + \hat{h}(\hat{\xi} - \hat{h})^2 I.
\end{aligned}
$$

Let

$$
\mathbf{f} = \mathbf{f}(q_1) = \frac{q_1}{1 - q_1}.
$$

Then Picard–Fuchs $\Rightarrow$

$$
v_4 = -\mathbf{f}^{-1} (z\partial_1)^2 I = -q_1 \mathbf{f}^{-1} (z\partial_2 - z\partial_1)^2 I = (q_1 - 1)\hat{\kappa}_0.
$$

Then we absorb $\hat{\kappa}_0$ into v_4 to get A_1, A_2 as

$$A_1 = \begin{bmatrix} & & & q_1q_2 & & \\ 1 & & & & & \\ & & & & & q_1q_2 \\ & -\mathbf{f} & q_1^{-1}\mathbf{f} & & & \\ & & 1 & & & \\ & & & -1 & 1 & \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & & & q_2(1-q_1) & q_2 & \\ 1 & & & & & q_2 \\ 1 & & & & & q_1q_2 \\ & 1 & & & & \\ & 1 & 1 & & & \\ & & & & 1 & \end{bmatrix}.$$

Now $v_6 = \hat{h}\hat{\xi}(\hat{\xi} - \hat{h})I = \hat{h}\hat{\xi}^2 I - q_1q_2 I$ does not come from a naive quantization. The z-independence fails if v_6 is not Ψ-corrected.

Degenerate case II: $(r, 0)$ **flips, i.e blow-ups**

Example: For $(1, 0)$ flips,

$$f : X = \Sigma_{-1} = P_{P^1}(\mathscr{O}(-1) \oplus \mathscr{O}) \to X' = P^2.$$

$$A_x = \left[\begin{array}{ccc|c} & xy & & xy \\ & & xy & \\ & & & -1 \\ \hline 1 & & -xy & -1/x \end{array}\right],$$

$$A_y = \left[\begin{array}{ccc|c} & xy & y & xy \\ 1 & & xy & \\ & 1 & & \\ \hline & & -xy & \end{array}\right].$$

- In the diagonalization process all the formal series $f_\bullet$ and $g^\bullet$ in x do not have constant terms.
- For the resulting 3×3 matrices E_x^{11} and E_y^{11}, the BF matrix $B \equiv I_3 \pmod{x}$.
- Thus after substituting $x = 0$ the resulting matrices for A_x, A_y go to $\mathbf{0}_3$ and

$$A_{\xi'} = \begin{bmatrix} & & y \\ 1 & & \\ & 1 & \end{bmatrix},$$

which recovers the Dubrovin connection on P^2 with $y = q^{\gamma'} e^{t'}$.

THANK YOU