Mean field equations, Lame equations, and modular forms

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- This is a joint project with Chang-Shou Lin and Ching-Li Chai.
- The Green function $G(z, w)$ on a flat torus $T = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the unique function on $T \times T$ which satisfies

$$-\triangle_z G(z, w) = \delta_w(z) - \frac{1}{|T|}$$

  and $\int_T G(z, w)\, dA = 0$, where δ_w is the Dirac measure with singularity at $z = w$.

- Because of the translation invariance of $\triangle_z$, we have $G(z, w) = G(z - w, 0)$ and it is enough to consider *the Green function* $G(z) := G(z, 0)$. Asymptotically

$$G(z) = -\frac{1}{2\pi} \log |z| + O(|z|^2).$$

- Not surprisingly, G can be explicitly solved in terms of elliptic functions.
- Let $z = x + iy$, $\tau := \omega_2/\omega_1 = a + ib \in \mathbb{H}$ and $q = e^{\pi i \tau}$ with $|q| = e^{-\pi b} < 1$. Then

$$\vartheta_1(z;\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}.$$

- (Neron):

$$G(z) = -\frac{1}{2\pi} \log \left| \frac{\vartheta_1(z)}{\vartheta_1'(0)} \right| + \frac{1}{2b} y^2.$$

- The structure of G, especially its critical points and critical values, will be the fundamental objects that interest us. $\nabla G(z) = 0 \iff$

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi} \left((\log \vartheta_1)_z + 2\pi i \frac{y}{b} \right) = 0.$$

- Recall $\wp(z) = 1/z^2 + \cdots$, $\zeta(z) = -\int^z \wp = 1/z + \cdots$. and $\sigma(z) = \exp \int^z \zeta(w)\, dw = z + \cdots$ is entire, odd with a simple zero on lattice points and

$$\sigma(z + \omega_i) = -e^{\eta_i (z + \frac{1}{2}\omega_i)} \sigma(z)$$

with $\eta_i = \zeta(z + \omega_i) - \zeta(z) = 2\zeta(\frac{1}{2}\omega_i)$ the quasi-periods.

- Indeed

$$\sigma(z) = e^{\eta_1 z^2/2} \frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Hence $\zeta(z) - \eta_1 z = (\log \vartheta_1(z))_z$.

- Let $z = t\omega_1 + s\omega_2$. By Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, $\nabla G(z) = 0$ if and only if

$$G_z = -\frac{1}{4\pi}\Big(\zeta(t\omega_1 + s\omega_2) - (t\eta_1 + s\eta_2)\Big) = 0.$$

- **Question:** How many critical points can G have in T?

- The 3 half periods are trivial critical points. Indeed,

$$G(z) = G(-z) \Rightarrow \nabla G(z) = -\nabla G(-z).$$

  Let $p = \frac{1}{2}\omega_i$ then $p = -p$ in T and so $\nabla G(p) = -\nabla G(p) = 0$.
- Other critical points must appear in pair $\pm z \in T$.

- Example (Maximal principle)

  For rectangular tori T: $(\omega_1, \omega_2) = (1, \tau = bi)$, $\frac{1}{2}\omega_i$, $i = 1, 2, 3$ are precisely all the critical points.

- Example ($\mathbb{Z}_3$ symmetry)

  For the torus T with $\tau = e^{\pi i/3}$, there are at least 5 critical points: 3 half periods $\frac{1}{2}\omega_i$ plus $\frac{1}{3}\omega_3$, $\frac{2}{3}\omega_3$.

  - However, it is very difficult to study the critical points from the "simple equation" $\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2$ directly.

- **In PDE**, the geometry of $G(z,w)$ plays fundamental role in the non-linear mean field equations (= Liouville equation with singular RHS): On a flat torus T it takes the form ($\rho \in \mathbb{R}_+$)

$$\triangle u + \rho e^u = \rho \delta_0.$$

- It is originated from the prescribed curvature problem (Nirenberg problem, constant K with cone metrics etc.).
- It is the mean field limit of Euler flow in statistic physics.
- It is related to the self-dual condensation of abelian Chern-Simons-Higgs model (Nolasco and Tarantello 1999).
- **In Arithmetic Geometry**, $G(z,w)$ also appears in the Arakelov geometry as the intersection number of two sections z and w of the arithmetic surface $\mathcal{T} \to \operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ at the ∞ fiber $\mathcal{T}_\infty =$ Riemann surface T.

- When $\rho \notin 8\pi\mathbb{N}$, it has been proved by C.-C. Chen and C.-S. Lin that the Leray-Schauder degree is

$$d_\rho = n + 1 \quad \text{for} \quad \rho \in (8n\pi, 8(n+1)\pi),$$

so the equation has solutions, regardless on the shape of T.

- The first interesting case is when $\rho = 8\pi$ where the degree theory fails completely.

Theorem (Existence criterion via ∇G for $n = 1$)

For $\rho = 8\pi$, the mean field equation on a flat torus $T = \mathbb{C}/\Lambda$:

$$\triangle u + \rho e^u = \rho \delta_0$$

has solutions if and only if the G has more than 3 critical points. Moreover, each extra pair of critical points $\pm p$ corresponds to an one parameter family of solutions u_λ, where $\lim_{\lambda\to\infty} u_\lambda(z)$ blows up precisely at $z \equiv \pm p$.

- **Structure of solutions.**
- Liouville's theorem says that any solution u of $\triangle u + e^u = 0$ in a simply connected domain $\Omega \subset \mathbb{C}$ must be of the form

$$u = c_1 + \log \frac{|f'|^2}{(1+|f|^2)^2},$$

where f, called a developing map of u, is meromorphic in Ω.
- It is straightforward to show that for $\rho = 8\pi\mu$,

$$S(f) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2}u_z^2 = -2\mu(\mu+1)\frac{1}{z^2} + O(1).$$

I.e., any developing map f of u has the same Schwartz derivative $S(f)$, which is elliptic on T.

- By the theory of ODE, locally $f = w_1/w_2$ for two solutions w_i of the Lamé equation $L_{\eta,B}\,y = 0$:

$$y'' + \frac{1}{2}S(f)y = y'' - (\eta(\eta+1)\wp(z) + B)y = 0$$

  for some $B \in \mathbb{C}$.
- Even more, for any two developing maps f and $\tilde{f}$ of u, there exists $S = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in PSU(2)$ such that $\tilde{f} = Sf := \dfrac{pf - \bar{q}}{qf + \bar{p}}$.

Lemma (Existence of developing map for $\mu \in \frac{1}{2}\mathbb{Z}$)

Given Λ, for $\rho = 4\pi\ell$, $\ell \in \mathbb{N}$, by analytic continuation across Λ, f is glued into a meromorphic function on $\mathbb{C}$. (Instead of on $T = \mathbb{C}/\Lambda$.)

- **First constraint from the double periodicity:**

$$f(z+\omega_1) = S_1 f, \quad f(z+\omega_2) = S_2 f$$

  with $S_1 S_2 = \pm S_2 S_1$.

- **Second constraint from the Dirac singularity:**
  (1) If $f(z)$ has a zero/pole at $z_0 \notin \Lambda$ then order $r = 1$.
  (2) $f(z) = a_0 + a_{\ell+1}(z - z_0)^{\ell+1} + \cdots$ be regular at $z_0 \in \Lambda$.

- **Type I (Topological) Solutions $\Longleftrightarrow \ell = 2n+1$:**

$$f(z+\omega_1) = -f(z), \qquad f(z+\omega_2) = \frac{1}{f(z)}.$$

  Then

$$g = (\log f)' = \frac{f'}{f}$$

  is elliptic on $T' = \mathbb{C}/\Lambda'$, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ with the only (highest order) zeros at $z_0 \equiv 0 \pmod{\Lambda}$ of order $\ell = 2n+1$.

- The equations $0 = g(0) = g''(0) = g^{(4)}(0) = \cdots$ implies that f is an even function. So f has simple zeros at $\pm p_1, \ldots, \pm p_n$ and $\omega_1/2$.

- The remaining equations $0 = g'(0) = g'''(0) = g^{(5)}(0) = \cdots$ leads to the polynomial system for $\wp(p_i)$'s:

Theorem (Type I evenness and algebraic integrability)

(1) *For* $\rho = 4\pi\ell$, $\ell = 2n+1$. *All type I solutions* u *are even.* f *has simple zeros at* $\omega_1/2$ *and* $\pm p_i$ *for* $i = 1, \dots, n$, *and poles* $q_i = p_i + \omega_2$.

(2) *For* $x_i := \wp(p_i)$, $\tilde{x}_i := \wp(q_i)$, *and* $m = 1, \dots, n$,

$$\sum_{i=1}^{n} x_i^m - \sum_{i=1}^{n} \tilde{x}_i^m = c_m, \quad (x_m - e_2)(\tilde{x}_m - e_2) = \mu,$$

for some constants c_m *and* $\mu = (e_2 - e_1)(e_2 - e_3)$. *This is a* $2n$ *affine polynomial system in* $\mathbb{C}^{2n}$ *of degree* $2^n n!$.

(3) *The corresponding Lamé equation* $L_{\eta = n+1/2, B}\, y = 0$ *has finite monodromy group* M *(in fact* $PM = V_4$*) hence there is a polynomial* p_n *of degree* $n+1$ *such that* $p_n(B) = 0$. *(Brioschi-Halphen 1894.)*

- **Type II (Scaling Family) Solutions $\Longleftrightarrow \eta = n$ ($\ell = 2n$):**

$$f(z+\omega_1) = e^{2i\theta_1} f(z), \qquad f(z+\omega_2) = e^{2i\theta_2} f(z).$$

- If f satisfies this, $e^{\lambda} f$ also satisfies this for any $\lambda \in \mathbb{R}$. Thus

$$u_\lambda(z) = c_1 + \log \frac{e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}$$

  is a scaling family of solutions with developing maps $\{e^{\lambda} f\}$.

- The blow-up points for $\lambda \to \infty$ (resp. $-\infty$) are precisely zeros (resp. poles) of $f(z)$.
- $g = (\log f)'$ is elliptic on $T = \mathbb{C}/\Lambda$, with highest order zero at $z = 0$ of order $\ell = 2n$.

- $0 = g'(0) = g'''(0) = \cdots = g^{(2n-1)}(0)$ implies that g is even.
- We may write

$$g(z) = \frac{\wp'(p_1)}{\wp(z) - \wp(p_1)} + \cdots + \frac{\wp'(p_n)}{\wp(z) - \wp(p_n)}$$

  constraint by $0 = g(0) = g''(0) = \cdots = g^{(2n-2)}(0)$. These give rise to $n-1$ equations on $p_1, \ldots, p_n$.
- And then

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

  which should satisfies (the n-th equation)

$$\int_{L_i} g \in \sqrt{-1}\mathbb{R}, \qquad i = 1, 2.$$

- **Periods integrals.** Let L_1, L_2 be the fundamental 1-cycles. Then

$$F_i(p) := \int_{L_i} \Omega(\xi, p)\, d\xi,$$

where $p \not\equiv \frac{1}{2}\omega_i \pmod{\Lambda}$ and

$$\Omega(\xi,p) = A\,\frac{\sigma^2(\xi)}{\sigma(\xi-p)\sigma(\xi+p)} = \frac{\wp'(p)}{\wp(\xi)-\wp(p)}$$
$$= 2\zeta(p) - \zeta(p+\xi) - \zeta(p-\xi).$$

- Lemma (Periods integrals and critical points)

Let $p = t\omega_1 + s\omega_2$, then up to $4\pi i\mathbb{N}$,

$$F_1(p) = 2(\omega_1\zeta(p) - \eta_1 p) = 2(\zeta(p) - t\eta_1 - s\eta_2)\omega_1 - 4\pi i s,$$
$$F_2(p) = 2(\omega_2\zeta(p) - \eta_2 p) = 2(\zeta(p) - t\eta_1 - s\eta_2)\omega_2 + 4\pi i t.$$

- E.g. when $\rho = 8\pi$ ($\ell = 2$), $p_1 = p$, $p_2 = -p$, $g(z) = \Omega(z, p)$ and

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

gives rise to a type II solution $\iff F_i(p) \in i\,\mathbb{R} \iff \nabla G(p) = 0$.

- **Theorem (Uniqueness, Lin-W 2006, Annals 2010)**

  For $\rho = 8\pi$, the mean field equation $\triangle u + \rho e^u = \rho\delta_0$ on a flat torus has at most one solution **up to scaling**.

- **Theorem (Number of critical points)**

  The Green function has either 3 or 5 critical points.

  - We were unable to prove it from the critical point equation.

- It remains to study the geometry of critical points over $\mathcal{M}_1$, which relies on methods of deformations and the degeneracy analysis of half periods.

Theorem (Moduli dependence, Lin-W 2011)

(1) *Let $\Omega_3 \subset \mathcal{M}_1 \cup \{\infty\} \cong S^2$ (resp. Ω_5) be the set of tori with 3 (resp. 5) critical points, then $\Omega_3 \cup \{\infty\}$ is closed containing $i\mathbb{R}$, Ω_5 is open containing the vertical line $[e^{\pi i/3}, i\infty)$.*

(2) *Both Ω_3 and Ω_5 are simply connected with $C := \partial\Omega_3 = \partial\Omega_5$ homeomorphic to S^1 containing ∞.*

(3) *Moreover, the extra critical points are split out from some half period point when the tori move from Ω_3 to Ω_5 across C.*

(4) *(Strong uniqueness) The map $\Omega_5 \to [0,1]^2$ by $\tau \mapsto (t,s)$ for $p(\tau) = t\omega_1 + s\omega_2$ is a bijection onto $\triangle = [(\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})]$.*

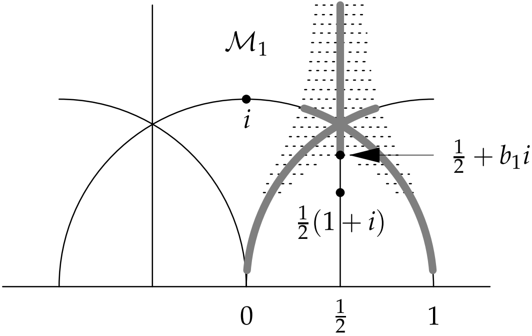

Figure: Ω_5 contains a neighborhood of $e^{\pi i/3}$.

- On the line $\operatorname{Re} \tau = 1/2$ which are equivalent to the rhombuses tori, the proof relies on *functional equations* of ϑ_1.
- The general case uses *modular forms of weight one*.

- **Idea of proof: Hecke (1926):**

$$\Psi(N) := \#\{\,(k_1,k_2)\mid (N,k_1,k_2)=1, 0\le k_i\le N-1\,\}.$$

Consider the weight one modular function for $\Gamma(N)$:

$$Z_{N,k_1,k_2}(\tau) := \zeta\Big(\frac{k_1\omega_1+k_2\omega_2}{N};\tau\Big) - \frac{k_1\omega_1+k_2\omega_2}{N} = -Z_{N,N-k_1,N-k_2}(\tau);$$

- and the weight $\Psi(N)$ one for full modular group:

$$Z(\tau)\equiv Z_N(\tau) := \prod_{(N,k_1,k_2)=1} Z_{N,k_1,k_2}(\tau)\in M_{\Psi(N)}(\mathrm{SL}(2,\mathbb{Z})).$$

- Each $\tau\in\mathbb{H}$ with $Z(\tau)=0$ is (at least) a double zero.

- For odd $N \geq 5$, $\nu_i(Z) = \nu_\rho(Z) = 0$,
- At ∞, Hecke calculated the asymptotic expansion: $\nu_\infty = \phi(N/2) = 0$,
- Then the RR:

$$(Z)_{\text{red}} = \frac{1}{2} \deg Z = \frac{1}{2} \sum_p \nu_p(Z) = \frac{\Psi(N)}{24}.$$

- Take N prime, this suggests a 1-1 correspondence between Ω_5 and

$$\triangle = [(\tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{2})]$$

under the map $\Omega_5 \to [0,1]^2$:

$$\tau \mapsto (t,s), \quad \text{where} \quad p(\tau) = t\omega_1 + s\omega_2.$$

Theorem (Periods integrals and type II evenness)

- *If solutions exist for $\rho = 8n\pi$, then there is a unique even solution within each type II scaling family. ($\ell = 2n$, $a_{n+i} = -a_i$.)*
- *The solution u is determined by the zeros $a_1, \dots, a_n$ of f. In fact*

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \sum_{i=1}^{n} \Omega(z, a_i),$$

$$f(z) = f(0) \exp \int^{z} g(\xi)\, d\xi = f(0) \prod_{i=1}^{n} \exp \int^{z} \Omega(\xi, a_i)\, d\xi.$$

The condition $\mathrm{ord}_{z=0}\, g(z) = 2n$ leads to equations for $a_1, \dots, a_n$:

Theorem (Green/polynomial system)

For $\rho = 8n\pi$, $n \in \mathbb{N}$, the n equations for $a_1, \ldots, a_n$ are precisely

$$\wp'(a_1)\wp^r(a_1) + \cdots + \wp'(a_n)\wp^r(a_n) = 0,$$

where $r = 0, \ldots, n-2$, and

$$\nabla G(a_1) + \cdots + \nabla G(a_n) = 0.$$

Theorem (Hyperelliptic geometry/Lamé curve)

For $x_i := \wp(a_i)$, $y_i := \wp'(a_i)$, the first $n-1$ algebraic equations

$$\sum y_i x_i^r = 0, \quad r = 0, \ldots, n-2,$$

defines a hyperelliptic curve under the 2 to 1 map $a \mapsto \sum \wp(a_i)$:

$$X_n := \{(x_i, y_i)\} \subset \mathrm{Sym}^n T \longrightarrow (x_1 + \cdots + x_n) \in \mathbb{P}^1.$$

- **The proof relies on its relation to Lamé equations:**

$$f = \exp \int g\, dz = \exp \int \sum_{i=1}^{n} (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\, dz$$

$$= e^{2\sum_{i=1}^{n} \zeta(a_i) z} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z + a_i)} = \frac{w_a}{w_{-a}},$$

where $w(z) = w_a(z) := e^{z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z)}$.

- Theorem (Explicit map $a \mapsto B_a$)

$a \in X_n$ if and only if w_a and w_{-a} are solutions of the Lamé equation

$$\frac{d^2 w}{dz^2} - \Big(n(n+1)\wp(z) + (2n-1) \textstyle\sum_{i=1}^{n} \wp(a_i) \Big) w = 0.$$

- **Idea of the Analytic Proof.** Consider $y^2 = p(x) = 4x^3 - g_2x - g_3$, where $(x, y) = (\wp(z), \wp'(z))$, and we set $(x_i, y_i) = (\wp(a_i), \wp'(a_i))$. Consider a basis of solutions to the Lamé equation by $\Lambda_a(z)$, $\Lambda_{-a}(z)$, where

$$\Lambda_a(z) := \frac{w_a(z)}{\prod_{i=1}^n \sigma(a_i)} = e^{z\sum\zeta(a_i)} \prod_{i=1}^n \frac{\sigma(z - a_i)}{\sigma(z)\sigma(a_i)}. \tag{1}$$

- Let $X = \Lambda_a\Lambda_{-a}$. By the addition theorem,

$$X(z) = (-1)^n \prod_{i=1}^n \frac{\sigma(z + a_i)\sigma(z - a_i)}{\sigma(z)^2\sigma(a_i)^2} = (-1)^n \prod_{i=1}^n (\wp(z) - \wp(a_i)).$$

  That is, $X(x) = (-1)^n \prod_{i=1}^n (x - x_i)$ is a polynomial in x.

- **Key:** $X(z)$ satisfies the second symmetric power:

$$\frac{d^3X}{dz^3} - 4(n(n+1)\wp + B)\frac{dX}{dz} - 2n(n+1)\wp' X = 0,$$

hence a polynomial solution, in variable x, to

$$p(x)X''' + \tfrac{3}{2}p'(x)X'' - 4((n^2+n-3)x + B)X' - 2n(n+1)X = 0. \tag{2}$$

X is determined by B and certain initial conditions.

- Write $X(x) = (-1)^n(x^n - s_1x^{n-1} + \cdots + (-1)^n s_n)$, (2) translates to a linear recursive relation for $\mu = 0, \cdots, n-1$ (we set $s_0 = 1$):

$$0 = 2(n-\mu)(2\mu+1)(n+\mu+1)s_{n-\mu} - 4(\mu+1)Bs_{n-\mu-1}$$
$$+ \tfrac{1}{2}g_2(\mu+1)(\mu+2)(2\mu+3)s_{n-\mu-2} - g_3(\mu+1)(\mu+2)(\mu+3)s_{n-\mu-3}.$$

- Since $B = (2n-1)s_1$, the initial relation for $\mu = n-1$ is automatic. Thus all s_i's, X, and $\pm a$, are determined by B alone.

C.-L. Chai offered a purely algebraic proof without Lamé equations:

Theorem (Chai-Lin-W 2012)

- *There is a natural projective compactification $\bar{X}_n$ as a, possibly singular, hyperelliptic curve defined by*

$$\begin{aligned} C^2 &= \ell_n(B, g_2, g_3) \\ &= 4Bs_n^2 + 4g_3 s_{n-2} s_n - g_2 s_{n-1} s_n - g_3 s_{n-1}^2, \end{aligned} \tag{3}$$

  in (B, C), where $s_k = s_k(B, g_2, g_3) = r_k B^k + \cdots \in \mathbb{Q}[B, g_2, g_3]$, is an universal polynomial of homogeneous degree k with $\deg g_2 = 2$, $\deg g_3 = 3$, and $B = (2n-1)s_1$.
- *Thus $\deg \ell_n = 2n + 1$ and $\bar{X}_n$ has arithmetic genus $g = n$.*
- *The curve $\bar{X}_n$ is smooth except for a finite number of τ, namely the discriminant loci of $\ell_n(B, g_2, g_3)$, so that $\ell_n(B)$ has multiple roots.*

- Now we study the last equation on $\bar{X}_n$:

$$0 = -4\pi \sum_{i=1}^{n} \nabla G(a_i) = \sum_{i=1}^{n} \zeta(a_i) - \sum_{i=1}^{n} (t_i\eta_1 + s_i\eta_2), \tag{4}$$

  where $a_i = t_i\omega_1 + s_i\omega_2$.

- Then for the rational function on T^n:

$$E_n(a_1, \ldots, a_n) := \zeta(a_1 + \cdots + a_n) - \sum_{i=1}^{n} \zeta(a_i),$$

  we get, by assuming (4),

$$\begin{aligned} E_n(a) &= \zeta(\textstyle\sum a_i) - (\sum t_i)\eta_1 - (\sum s_i)\eta_2 \\ &= Z(\textstyle\sum a_i) \\ &= -4\pi\nabla G(\textstyle\sum a_i). \end{aligned}$$

- It is thus crucial to study the branched covering map

$$\sigma : \bar{X}_n \to T, \qquad a \mapsto \sigma(a) := \sum_{i=1}^{n} a_i.$$

Theorem (New modular functions)

(1) *The map σ has degree equals $\frac{1}{2}n(n+1)$.*

(2) *There is a universal (weighted homogeneous) polynomial $W_n(x) \in \mathbb{C}[g_2, g_3, \wp(\sum a_i), \wp'(\sum a_i)][x]$ of degree $\frac{1}{2}n(n+1)$ such that*

$$W_n(E_n) = 0.$$

(3) *The function $Z_n := W_n(Z)$ is modular of weight $\frac{1}{2}n(n+1)$.*

- **Idea of proof for (1):** Apply *Theorem of the Cube*: For any three morphisms $f, g, h : V_n \longrightarrow T$ and $L \in \operatorname{Pic} T$,

$$(f+g+h)^*L \cong (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

- Apply to the case $V_n \subset T^n$ which is the ordered n-tuples so that $V_n / S_n = \bar{X}_n$. We prove inductively that the map

$$f_k(a) := a_1 + \cdots + a_k$$

  has degree $\frac{1}{2}k(k+1)n!$. It is easy to check for $k = 1, 2$. From k to $k+1$, we let $f = f_{k-1}$, $g(a) = a_k$, and $h(a) = a_{k+1}$.
- Then f_{k+1} has degree $n!$ times

$$\tfrac{1}{2}k(k+1) + 3 + \tfrac{1}{2}k(k+1) - \tfrac{1}{2}(k-1)k - 1 - 1 = \tfrac{1}{2}(k+1)(k+2).$$

Example ($n = 2$)

For $E_2(a_1, a_2) = \zeta(a_1 + a_2) - \zeta(a_1) - \zeta(a_2)$,

$$E_2^3(a) - 3\wp(a_1 + a_2)E_2(a) - \wp'(a_1 + a_2) = 0$$

on X_n. The equation on T^n has one more term $-\frac{1}{2}(\wp'(a_1) + \wp'(a_2))$.