Lecture III
A Quantum Splitting Principle

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Contents

Quantum cohomology

- Let X be smooth projective variety over $\mathbb{C}$.
- Basis $T_i \in H = H(X)$, dual $\{T^i\}$, $t = \sum t^i T_i$, $g_{ij} := \langle T_i, T_j \rangle$.
- Genus zero GW formal prepotential $F(t) = \langle\langle \rangle\rangle$:

$$\langle\langle a_1, \dots, a_m \rangle\rangle = \sum_{\beta \in NE(X)} \sum_{n=0}^{\infty} \frac{q^\beta}{n!} \langle a_1, \dots, a_m, t^{\otimes n} \rangle_{g=0, m+n, \beta}.$$

- 3-pt function $F_{ijk} = \partial^3_{ijk} F = \langle\langle T_i, T_j, T_k \rangle\rangle$, $A_{ij}^k := F_{ijl} \, g^{lk}$, then

$$T_i *_t T_j = \sum A_{ij}^k(t) \, T_k.$$

- The Dubrovin connection ∇ on $T_0 \widehat{H} \otimes \mathbb{C}[\![q^\bullet]\!] \times \mathbb{A}^1_z$ is flat:

$$\nabla = d - \frac{1}{z} \sum_i dt^i \otimes A_i = d - \frac{1}{z} \sum_i dt^i \otimes T_i *_t .$$

Gromov–Witten invariants

Let $\overline{\mathscr{M}}_{g,n}(X,\beta)$ be the moduli stack of n-pointed genus g stable maps $f:(C;x_1,\ldots,x_n)\to X$ with $f_*[C]=\beta\in H_2(X)$. We have

$$\mathrm{ev}_j:\overline{\mathscr{M}}_{g,n}(X,\beta)\to X,\qquad f\mapsto f(x_j),\qquad 1\le j\le n.$$

For $\alpha_j\in H^*(X)$, $\psi_j=c_1(x_j^*\omega_{\mathscr{C}/\overline{\mathscr{M}}_{g,n}(X,\beta)})$, the *descendant invariant* is

$$\Big\langle \prod_{j=1}^n \tau_{k_j}(\alpha_j)\Big\rangle_{g,\beta}^X=\int_{[\overline{\mathscr{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_j \mathrm{ev}_j^*(\alpha_j)\prod_j \psi_j^{k_j}.$$

When $2g+n\ge 3$, there is a stabilization map

$$\mathrm{st}:\overline{\mathscr{M}}_{g,n}(X,\beta)\to\overline{\mathscr{M}}_{g,n}.$$

Now let $\bar\psi_j\in H^2(\overline{\mathscr{M}}_{g,n})$ instead. Then the *ancestor invariant* is

$$\Big\langle \prod_{j=1}^n \bar\tau_{k_j}(\alpha_j)\Big\rangle_{g,\beta}^X=\int_{[\overline{\mathscr{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}}\prod_j \mathrm{ev}_j^*(\alpha_j)\,\mathrm{st}^*\Big(\prod_j \bar\psi_j^{k_j}\Big).$$

Cyclic $\mathscr{D}^z$-modules

- $t = t_0 + t_1 + t_2$, $t_0 \in H^0$, $t_1 \in H^2$:

$$J(t, z^{-1}) = 1 + \frac{t}{z} + \sum_{\beta,n,i} \frac{q^\beta}{n!} T_i \left\langle \frac{T^i}{z(z-\psi_1)}, (t)^n \right\rangle_\beta$$
$$= e^{\frac{t}{z}} + \sum_{\beta \neq 0,n,i} \frac{q^\beta}{n!} e^{\frac{t_0+t_1}{z} + (t_1.\beta)} T_i \left\langle \frac{T^i}{z(z-\psi_1)}, (t_2)^n \right\rangle_\beta .$$

- TRR $\Longrightarrow$ QDE (denote by $z\partial_i = z\partial_{t^i} = z\partial_{T_i}$):

$$z\partial_i \, z\partial_j \, J = \sum_k A_{ij}^k(t) \, z\partial_k \, J.$$

- $QH(X) \equiv$ cyclic $\mathscr{D}^z$-module $\mathscr{D}^z J$ with basis (frame)

$$z\partial_i J \equiv e^{t/z} T_i \pmod{q^\bullet} = T_i + \cdots .$$

Given an ordinary P^r-flop with extremal ray ℓ, ℓ' resp.

$$f : X \dashrightarrow X',$$

the graph $\Gamma_f \subset X \times X'$ induces an isomorphism of motives

$$\mathscr{F} = [\bar{\Gamma}_f]_* : H(X) \xrightarrow{\sim} H(X'),$$

which preserves the Poincaré pairing. We set

$$\mathscr{F}(q^\beta) = q^{\mathscr{F}(\beta)}.$$

Theorem (Analytic continuation in $q^\ell = 1/q^{\ell'}$)

$\mathscr{F}$ induces an isomorphism of big quantum rings

$$QH(X) \cong QH(X').$$

The results also hold for relative invariants and (relative) ancestors.

Step 1 [LLW 2008]

- Degeneration + Reconstruction reduce the proof to the case of *local models*.
- Let (S, F, F') consist of two v.b.'s F and F' of rank $r+1$ over a smooth S. The f-exc loci $Z \subset X$ and $Z' \subset X'$ are

$$\bar{\psi} : Z = P_S(F) \to S, \qquad \bar{\psi}' : Z' = P_S(F') \to S,$$

  and the (projective) local model of f is

$$X = P_Z(N \oplus \mathscr{O}) \overset{f}{\dashrightarrow} X' = P_{Z'}(N' \oplus \mathscr{O}),$$

  where $N = N_{Z/X} \cong \mathscr{O}_Z(-1) \otimes \bar{\psi}^* F'$ and similarly for N'.
- The flop f is the blowup of X along Z followed by contracting the exc-divisor $E = Z \times_S Z'$ along the $\bar{\psi}$-ruling.
- The local model of f is a functor over the triples (S, F, F')'s.

Step 2 [LLW 2011]

- For $F = \bigoplus_{i=0}^{r} L_i$, $F' = \bigoplus_{i=0}^{r} L_i'$ being split bundles, based on [Brown 2009], a quantum Leray–Hirsch theorem is proved:

$$QH(X) \cong_{\mathscr{D}^z} p^* QH(S)[\hat{h}, \hat{\xi}] / (\hat{f}_F, \hat{f}_{N \oplus \mathscr{O}}).$$

- Here $\hat{h} = z\partial_h$, $\hat{\xi} = z\partial_\xi$, and

$$\hat{f}_F = \Box_\ell = \prod z\,\partial_{h+L_i} - q^\ell e^{t^h} \prod z\,\partial_{\xi - h + L_i'},$$
$$\hat{f}_{N\oplus\mathscr{O}} = \Box_\gamma = z\,\partial_\xi \prod z\,\partial_{\xi - h + L_i'} - q^\gamma e^{t^\xi},$$

are the Picard–Fuchs operators which are the "quantized version" of the Chern polynomials.
- The pullback $p^*QH(S)$ is an admissible lifting of the Dubrovin connection on $H(S)$ to $H(X)$:

Let $D = t^h h + t^\xi \xi$ be the relative divisor class, $\bar{t} \in H(S)$, then

$$z\partial_i \, z\partial_j = \sum_{\bar{\beta} \in NE(S),k} q^{\beta} e^{D.\bar{\beta}^*} \, [A_S]^k_{ij,\bar{\beta}}(\bar{t}) \, z\partial_k \, \mathbf{D}_\beta(z)$$

for some *admissible lifting* $\beta \in NE(X)$ and differential operator

$$\mathbf{D}_\beta(z) := \prod_{m=0}^{-\xi.\beta-1} (z\partial_\xi - mz) \times$$
$$\prod_{i=0}^{r} \left(\prod_{m=0}^{-(h+L_i).\beta-1} (z\partial_{h+L_i} - mz) \prod_{m=0}^{-(\xi-h+L_i').\beta-1} (z\partial_{\xi-h+L_i'} - mz) \right).$$

such that $-(h+L_i).\beta \geq 0$, $-(\xi - h + L_i').\beta \geq 0$ and $-\xi.\beta \geq 0$.

β exists, but might not be unique. Nevertheless, $\mathbf{D}_\beta(z)$ is well-defined modulo the Picard–Fuchs ideal $\langle \Box_\ell, \Box_\gamma \rangle$.

- Now we may compute the first order system

$$z\partial_{t^a}(\hat{T}_i) = (\hat{T}_i)C_a(z, q^\bullet), \qquad t^a = t^h, t^\xi, \bar{t}^i.$$

  under the naive frame $\hat{T}_i = z\partial_{\bar{t}^i}(z\partial_{t^h})^j(z\partial_{t^\xi})^k$'s.
- This is "equivalent" to $\mathscr{D}^z J^X$ as $\mathscr{D}^z$-modules.
- The analytic continuation of $\mathscr{D}^z$-modules in q^ℓ follows easily from the above presentation of C_a and

$$\mathscr{F} : \langle \square_\ell, \square_\gamma \rangle \cong \langle \square_{\ell'}, \square_{\gamma'} \rangle.$$

- To get $QH(X)$ from the $\mathscr{D}^z$-module, we need BF/GMT: Birkhoff factorization/generalized mirror transform.
- A technical induction was performed so that B, τ are compatible with analytic continuations.

Example

Let $f : X \dashrightarrow X'$ be a P^1-flop, $(S, F, F') = (P^1, \mathscr{O} \oplus \mathscr{O}, \mathscr{O} \oplus \mathscr{O}(1))$.
Write $H(S) = \mathbb{C}[p]/(p^2)$ with Chern polynomials

$$f_F(h) = h^2, \qquad f_{N\oplus\mathscr{O}}(\xi) = \xi(\xi - h)(\xi - h + p).$$

Then $H = H(X) = H(S)[h, \xi]/(f_F, f_{N\oplus\mathscr{O}})$ has dimension $N = 12$ with basis $\{T_i \mid 0 \leq i \leq 11\}$ being

$$1,\ h,\ \xi,\ p,\ h\xi,\ hp,\ \xi^2,\ \xi p,\ h\xi^2,\ h\xi p,\ \xi^2 p,\ h\xi^2 p.$$

Denote by $q_1 = q^\ell e^{t^1}$, $q_2 = q^\gamma e^{t^2}$, $\bar{q} = q^b e^{t^3}$, where $b = [S] \cong [P^1]$.
The Picard-Fuchs operators are

$$\Box_\ell = (z\partial_h)^2 - q_1 z\partial_{\xi-h}\, z\partial_{\xi-h+p},$$
$$\Box_\gamma = z\partial_\xi\, z\partial_{\xi-h}\, z\partial_{\xi-h+p} - q_2.$$

They lead to a Grobner basis:

$$(z\partial_h)^2 = \mathbf{f}(q_1)\big((z\partial_\xi)^2 - z\partial_p\, z\partial_h + z\partial_p\, z\partial_\xi - 2z\partial_h\, z\partial_\xi\big),$$
$$(z\partial_\xi)^3 = q_2(1-q_1) - z\partial_p(z\partial_\xi)^2 + 2z\partial_h(z\partial_\xi)^2 + z\partial_p\, z\partial_h\, z\partial_\xi.$$

Here $\mathbf{f}(q) := q/(1-(-1)^{r+1}q)$ which satisfies

$$\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r.$$

$H(S) = \mathbb{C}\mathbf{1} \oplus \mathbb{C}p$ has only small parameter $\bar{q}$ with QDE

$$z\partial_p(z\partial_{\mathbf{1}}, z\partial_p) = (z\partial_{\mathbf{1}}, z\partial_p)\begin{pmatrix} 0 & \bar{q} \\ 1 & 0 \end{pmatrix}.$$

We have admissible lifting $b^I = b - \gamma$ and $\mathbf{D}_b = z\partial_\xi\, z\partial_{\xi-h}$, hence the lifted QDE:

$$(z\partial_p)^2 = \bar{q}q_2^{-1}\, z\partial_\xi\, z\partial_{\xi-h}.$$

We calculate C_a in $z\partial_a \hat{T}_j = \sum_k C^k_{aj}(z)\hat{T}_k$.
Let $q^* = \bar{q}q_2^{-1}$ be the chosen admissible lift.
Set $\mathbf{g} = \mathbf{f}(q^*)$, $A = q_2 - q_1q_2$, $S = q_2 + q_1q_2$. Then

$$C_h = \begin{bmatrix}
 & & & & q_1q_2 & \mathbf{f}\,q_2q^* & & & zq_1q_2 & & & \\
1 & & & & & & & & & & & \\
 & & & & & & & & q_1q_2 & & & \\
 & & & & & & & & & q_1q_2 & & zq_1q_2 \\
 & -2\mathbf{f} & 1 & & & z\mathbf{f}\,q^* & & & & & & \\
 & -\mathbf{f} & & 1 & & & & & & & & \\
 & \mathbf{f} & & & & -z\mathbf{f}\,q^* & & & & & & \\
 & \mathbf{f} & & & & & & & & & & q_1q_2 \\
 & & & & & & 1 & & & & & \\
 & & & & & \mathbf{f}(q^*-2) & & 1 & & & & \\
 & & & & & \mathbf{f}(1-q^*) & & & & & & \\
 & & & & & & & & & & 1 &
\end{bmatrix},$$

$$
C_\xi = \begin{bmatrix}
 & & & & & & A & & zq_1q_2 & & zA\mathbf{g} & z^2q_1q_2\mathbf{g} \\
 & & & & & & & & A & & & zA\mathbf{g} \\
1 & & & & & & & & 2q_1q_2 & & -q_2\mathbf{g} & zq_1q_2\mathbf{g} \\
 & & & & & & & & q_1q_2 & & A(1+\mathbf{g}) & zq_1q_2(1+2\mathbf{g}) \\
 & 1 & & & & & & & & & z^2\mathbf{g} & -q_2q^*(1+\mathbf{g}) \\
 & & & & & & & & & & & A(1+\mathbf{g}) \\
 & & 1 & & & & & & & & -z^2\mathbf{g} & \\
 & & & 1 & & & & & & & & q_1q_2(2+\mathbf{g}) \\
 & & & & 1 & & 2 & & & & z\mathbf{g} & -z^2\mathbf{g} \\
 & & & & & 1 & 1 & & & & 2z\mathbf{g} & \\
 & & & & & & -1 & 1 & & & -2z\mathbf{g} & \\
 & & & & & & & & -1 & 1 & 2+\mathbf{g} & -2z\mathbf{g}
\end{bmatrix},
$$

and $C_p =$

$$\begin{bmatrix}
 & & & & & -q_1q_2q^* & & Aq^* & & zq_1q_2q^* & z(q_1q_2q^* - A\mathbf{g}) & -z^2q_1q_2\mathbf{g} \\
 & & & & & & & & & Aq^* & Aq^* & -zA\mathbf{g} \\
 & & & & & & & & & q_1q_2q^* & (S - q_1q_2q^*)\mathbf{g} & -zq_1q_2\mathbf{g} \\
1 & & & & & & & & & q_1q_2q^* & q_1q_2q^* - A\mathbf{g} & -2zq_1q_2\mathbf{g} \\
 & & & -q^* & & & & zq^* & & & -z^2\mathbf{g} & (A + q_1q_2q^*)\mathbf{g} \\
 & 1 & & & & & & & & & & -A\mathbf{g} \\
 & & & q^* & & & & -zq^* & & & z^2\mathbf{g} & q_1q_2q^* \\
 & & 1 & & & & & & & & & -q_1q_2\mathbf{g} \\
 & & & & & q^* & & q^* & & -zq^* & z(q^* - 2)\mathbf{g} & z^2\mathbf{g} \\
 & & & & 1 & & & q^* & & & -2z\mathbf{g} & \\
 & & & & & & 1 & -q^* & & & 2z\mathbf{g} & \\
 & & & & & & & & 1 & -q^* & (q^* - 2)\mathbf{g} & 2z\mathbf{g}
\end{bmatrix}.$$

A gauge transform is needed to remove all appearances of z. In this example GMT is not needed since the first column vectors in C_a's are correct: $\hat{T}_i * \hat{\mathbf{1}} = \hat{T}_i$.

Step 3: splitting principle [L-L-Qu-W 2014]

Proposition

Given a $\mathbb{C}^k$-bundle $F \to S$, there exists a sequence of blow-ups on smooth centers $\phi : \tilde{S} \to S$ such that there is a filtration of subbundles

$$0 = F_0 \subset F_1 \subset \ldots \subset F_k = \phi^* F$$

with rk $F_{i+1}/F_i = 1$ *for all i; $\phi^* F$ can be deformed to a split bundle.*

Proof.

Consider the *complete flag bundle* over S and a *rational section* s:

$$\mathcal{F}_S(F) \underset{s}{\overset{p}{\rightleftarrows}} S.$$

Let $\phi : \tilde{S} \to S$ resolves s. Then $\phi^* F$ admits a complete flag, and there is a deformation of sending all extension classes to 0. □

- In the classical setting

$$p^* : H(S) \hookrightarrow H(\mathcal{F}_S(F)), \qquad \phi^* : H(S) \hookrightarrow H(\tilde{S})$$

  are both ring monomorphisms.
- They lead to the *classical splitting principle*.
- Such functorialties are not yet available for QH.
- Instead, we develop a *quantum splitting principle* to study

$$QH(S) \dashrightarrow QH(\mathcal{F}_S(F)), \qquad QH(S) \dashrightarrow QH(\tilde{S}).$$

- In particular, $\mathscr{F}$-invariance (analytic continuations)

$$\mathscr{F} : QH(X_{(S,F,F')}) \cong QH(X'_{(S,F,F')})$$

  with $\mathscr{F} q^\ell = (q^{\ell'})^{-1}$ is reduced to the split case.

Starting with $(S_0, F_0, F_0') = (S, F, F')$, we construct $(S_i, F_i, F_i')_{i\geq 0}$:

$$\phi_i : S_{i+1} = \mathrm{Bl}_{T_i} S_i \to S_i$$

for some smooth $T_i \subset S_i$, $F_{i+1} = \phi_i^* F_i$ and $F_{i+1}' = \phi_i^* F_i'$.

- *Will show the $\mathscr{F}$-invariance for (S_i, F_i, F_i') can be reduced to the $\mathscr{F}$-invariance for $(S_{i+1}, F_{i+1}, F_{i+1}')$.*
- *The problem is then solved for $S_{i+1} = \tilde{S}$, the split case, since GW theory is invariant under smooth deformations.*

We consider the deformation to the normal cone for $T_i \hookrightarrow S_i$:

$$\Phi_i : \mathbf{S} = \mathrm{Bl}_{T_i \times \{0\}}(S_i \times \mathbb{A}^1) \to \mathbb{A}^1,$$

$$\mathbf{S}_t = S_i \sim S_{i+1} \cup_{E_i} P_i = \mathbf{S}_0,$$

$E_i = \mathrm{Exc}\, \phi_i = P_{T_i}(N_{T_i/S_i})$, and $P_i = \mathrm{Exc}\, \Phi_i = P_{T_i}(N_{T_i/S_i} \oplus \mathscr{O})$.

- For simplicity, we write

$$X_{S_i} \equiv X_{(S_i, F_i, F_i')}$$

etc. when the bundles are from pullbacks (restrictions).

- The degeneration formula in GW theory says that

$$\langle \alpha \rangle^{X_{S_i}} = \sum_{\vec{\mu}} \langle \alpha_1 \mid \vec{\mu} \rangle^{\bullet (X_{S_{i+1}}, X_{E_i})} \langle \alpha_2 \mid \vec{\mu}^{\vee} \rangle^{\bullet (X_{P_i}, X_{E_i})}$$

where $\vec{\mu} = \{(\mu_i, e_i)\}$ is a $H(X_{E_i})$-weighted partition.

- Thus, for both factors, we need to control

relative invariants for a smooth divisor pair (X_S, X_D)

by the *absolute invariants* of X_S and X_D.

- A trivial degeneration (to the normal cone)

$$S \sim S \cup_D P, \qquad P = P_D(N \oplus \mathscr{O}) \xrightarrow{\pi} D$$

leads to

$$\langle \alpha \rangle^{X_S} = \sum_{\vec{\mu}} \langle \alpha_1 \mid \vec{\mu} \rangle^{\bullet(X_S, X_D)} \langle \alpha_2 \mid \vec{\mu}^\vee \rangle^{\bullet(X_P, X_D)}.$$

- The problem becomes "inversion of this linear system", with coefficients being relative invariants of (X_P, X_D).
- Here $X_P \to X_D$ is a split P^1-bundle arising from $\pi : P \to D$.
- Since $D = P_T(N_{T/S}) \to T$ has

$$\dim T < \dim S.$$

$\Longrightarrow$ the absolute invariants for X_P are handled inductively.

- To handle (X_P, X_D), *fiberwise localization* was used in [Maulik–Pandharipande 2006].
- Among other technical issues, localizations create *descendants* which breaks $\mathscr{F}$-invariance.

- Now, to treat general $P = P_D(N \oplus \mathscr{O})$, localizations are replaced by *more complex degeneration argument* and
- the *strong virtual pushforward property*, which extends earlier works of [Hsin-Hong Lai 2008, Manolache 2012] from absolute GW to relative GW.

Review of relative obstruction theory

The universal curve $\mathscr{C} = \overline{\mathscr{M}}_{g,n+1}(X, \beta)$ with $f = \mathrm{ev}_{n+1} : \mathscr{C} \to X$:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ f\ } & X \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathscr{M}}_{g,n}(X, \beta), & & \end{array}$$

leads to a perfect obstruction theory and virtual cycle

$$E^\bullet := (R\pi_* f^* T_X)^\vee \to \mathbb{L}_{\overline{\mathscr{M}}}, \quad \text{and} \quad [\overline{\mathscr{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$$

[Li–Tian 1998, Behrend–Fantachi 1997].

Also a relative theory for

$$i : X \hookrightarrow X' \quad \text{with} \quad i_* : A_1(X) \hookrightarrow A_1(X')$$

Then

$$\bar{i} : \overline{\mathscr{M}}_{g,n}(X, \beta) \to \overline{\mathscr{M}}_{g,n}(X', i_*\beta),$$

with

$$E^\bullet_{\bar{i}} := (R\pi_* f^* \mathbb{L}_i^\vee)^\vee \to \mathbb{L}_{\bar{i}}.$$

It is perfect if $g = 0$ and X' is convex (e.g. homogeneous).

Then the *virtual pull-back formula* holda:

$$[\overline{\mathscr{M}}_{0,n}(X, \beta)]^{\text{vir}} = \bar{i}^! [\overline{\mathscr{M}}_{0,n}(X', i_*\beta)]^{\text{vir}}.$$

[Manolache 2012]:

Strong Virtual Pushforward for Relative GW

- Consider the split P^1-bundle

$$\pi : Y = P_X(L \oplus \mathscr{O}) \to X,$$

which has two sections

$$i_0 : Y_0 \hookrightarrow Y, \qquad i_\infty : Y_\infty \hookrightarrow Y.$$

- Relative/Log GW invariants on (Y, Y_0) and (Y, Y_∞) are called *type I*;
- those on $(Y, Y_0 \sqcup Y_\infty)$ are called *type II*.
- Relative/Log GW are equivalent for $g = 0$ [Abramovich et. al 2014].

- Let $(Y, Y_0 \sqcup Y_\infty)$, (Y, Y_0) and (Y, Y_∞) denote the log schemes, which are log smooth and integral. And

$$\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu) := \overline{\mathscr{M}}_{0,n}((Y, Y_0 \sqcup Y_\infty), \beta; \mu, \nu) \quad \text{etc.}$$

  be the log stack of stable log maps with curve class β.
- μ, ν are partitions of

$$d_0 = \int_\beta Y_0, \qquad d_\infty = \int_\beta Y_\infty,$$

  (contact orders of marked points in Y_0 and Y_∞).
- When $\theta := \pi_*\beta \neq 0$ or $n \geq 3$, we have induced maps:

$$p : \overline{\mathscr{M}}_{0,n}(Y;\mu,\nu) \to \overline{\mathscr{M}}_{0,n}(X,\theta),$$
$$q : \overline{\mathscr{M}}_{0,n}(Y;\nu) \to \overline{\mathscr{M}}_{0,n}(X,\theta).$$

Lemma (Virtual dimension count)

1. $\dim\,[\overline{\mathscr{M}}_{g,n}(Y;\mu,\nu)]^{\mathrm{vir}} = \dim\,[\overline{\mathscr{M}}_{g,n}(X,\theta)]^{\mathrm{vir}} + 1 - g.$
2. $\dim\,[\overline{\mathscr{M}}_{g,n}(Y;\nu)]^{\mathrm{vir}} = \dim\,[\overline{\mathscr{M}}_{g,n}(X,\theta)]^{\mathrm{vir}} + 1 - g + \int_\beta Y_0.$

Proof. For log moduli stack we need to impose conditions by the contact orders:

In (1) it is $d_0 + d_\infty$ and in (2) it is d_∞.

Now

$$c_1(Y).\beta = (\pi^* c_1(X) + Y_0 + Y_\infty).\beta = c_1(X).\theta + d_0 + d_\infty.$$

Also

$$(\dim Y - \dim X)(1-g) = 1 - g.$$

□

Proposition (Strong virtual pushforward for $g = 0$)

1. *In* $A_*(\overline{\mathscr{M}}_{0,n}(X,\theta))$, *there exists* $N(\mu,\nu) \in \mathbb{Q}$ *such that*

$$p_*[\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu)]^{\mathrm{vir}} = 0,$$
$$p_*([\overline{\mathscr{M}}_{0,n}(Y;\mu,\nu)]^{\mathrm{vir}} \cap \mathrm{ev}_1^*[Y_0]) = N(\mu,\nu)[\overline{\mathscr{M}}_{0,n}(X,\theta)]^{\mathrm{vir}}.$$

2. *Assume* $\int_\beta Y_0 \geq 0$, *then* $q_*[\overline{\mathscr{M}}_{0,n}(Y;\nu)]^{\mathrm{vir}} = 0$.

Proof. Choose $M \in \operatorname{Pic} X$ such that M and $L \otimes M$ are both very ample. Then we have a cartesian diagram of embeddings

$$\begin{array}{ccc} Y & \xrightarrow{\;j\;} & P(\mathscr{O}(-1,1) \oplus \mathscr{O}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ X & \xrightarrow{\;i\;} & P^{|M|} \times P^{|L \otimes M|}, \end{array}$$

with $L = i^*\mathscr{O}(-1,1)$. The proposition holds for $\tilde{\pi}$.

It induces a cartesian diagram between (log) stacks

$$\begin{array}{ccc} \overline{\mathscr{M}}_{0,n}(Y;\mu,\nu) & \xrightarrow{\bar{j}} & \overline{\mathscr{M}}_{0,n}(P(\mathscr{O}(-1,1)\oplus\mathscr{O});\mu,\nu) \\ \downarrow p & & \downarrow \tilde{p} \\ \overline{\mathscr{M}}_{0,n}(X,\theta) & \xrightarrow{\bar{i}} & \overline{\mathscr{M}}_{0,n}(P^{|M|}\times P^{|L\otimes M|},(\int_\theta M,\int_\theta L\otimes M)). \end{array}$$

The *relative perfect obstruction theories* $E^\bullet_{\bar{i}} \to \mathbb{L}_{\bar{i}}$ and $E^\bullet_{\bar{j}} \to \mathbb{L}_{\bar{j}}$ fit in

$$\begin{array}{ccc} E^\bullet_{\bar{j}} & \longrightarrow & \mathbb{L}_{\bar{j}} \\ \uparrow \approx & & \uparrow \\ p^*E^\bullet_{\bar{i}} & \longrightarrow & p^*\mathbb{L}_{\bar{i}} \end{array}$$

since $p^*\mathbb{L}_i \cong \mathbb{L}_j$ (cf. Manolache). Now $\bar{i}^!$ and $\bar{j}^!$ pullback virtual cycles. The results for (π, p) follow from that for $(\tilde{\pi}, \tilde{p})$. $\square$

Back to (X_P, X_D), i.e. type I invariants

- Recall $\pi : P = P_D(N \oplus \mathscr{O}) \to D$, with $P_0, P_\infty \cong D$, induces

$$\pi : X_P = P_{X_D}(L \oplus \mathscr{O}) \to X_D,$$

  with $L = N_D|_{X_D}$ and sections $X_{P_0}, X_{P_\infty} \cong X_D$.
- A *non-vanishing theorem* modelled on $(P^1, \{0\}) \times X_{\mathrm{pt}}$ is proved to show the invertibility of the linear system.
- **Claim:** under the trivial degenerations

$$P \sim P \cup_{P_\infty} P, \qquad (P, P_0) \sim (P, P_0) \cup_{P_\infty} P,$$

  the "*strong virtual pushforward*" and "*TRR for ancestors*" $\Longrightarrow$ *type I invariants* are determined by *absolute, type II,* and *rubber invariants* modulo lower degree ones.

- For $X_P \to Z_P \to P$, a class $\beta \in NE(X_P)$ has $(\beta_P, d) \in NE(P) \times \mathbb{Z}$ with $d = \int_\beta \xi$. The *generating series*

$$\langle \alpha \rangle^{X_P}_{(\beta_P,d)} := \sum_{\beta \in (\beta_P, d)} \langle \alpha \rangle^{X_P}_{\beta} q^\beta$$

is a sum over the extremal ray. Similarly for type I, II, etc.

- Then for ω being pullback insertions from X_D, we have

$$\left\langle \vec{v} \mid \omega \cdot \prod_{i=1}^k i_{\infty *}(\alpha_i) \right\rangle^{(X_P, X_{P_\infty})}_{(\beta_P, d)} =$$
$$\sum_{I, \eta = (\Gamma_1, \Gamma_2)} C_\eta \left\langle \vec{v} \mid \omega_1 \cdot \prod_{i=1}^k i_{\infty *}(\alpha_i) \mid \mu, e^I \right\rangle^{\bullet}_{\Gamma_1} \cdot \langle \mu, e_I \mid \omega_2 \rangle^{\bullet}_{\Gamma_2}$$

spanned by type II and *type I series with pullback insertions*.

- Moreover, if $\int_{\beta_P} P_0 \geq 0$ then $\langle \omega \mid \vec{v} \rangle^{(X_P, X_{P_\infty})}_{\beta_S, d} = 0$.

Proposition (Type I reduction)

Assume $\int_{\beta_P} P_0 < 0$. *An ordering is introduced on* $\{\vec{v}\}$ *such that*

1. *If* $\vec{v} = \{(v_j, B_j)\} \neq \emptyset$ *then there exists* $C(\vec{v}) > 0, k(\vec{v}) \in \mathbb{Z}_{\geq 0}$,

$$C(\vec{v}) \langle\, \omega \mid \vec{v}\, \rangle^{(X_P, X_{P_\infty})}_{(\beta_P, d)} - \left\langle\, \omega \cdot [X_{P_\infty}]^{k(\vec{v})} \cdot \prod_j \bar{\tau}_{v_j - 1}(i_{\infty *}(B_j))\, \right\rangle^{X_P}_{(\beta_P, d)}$$

is generated by "relative and rubber series" on X_P *of class at most* (β_P, d), *and those of* (X_P, X_{P_∞}) *involving class* (β_P, d) *whose orders are lower than* $\langle\, \omega \mid \vec{v}\, \rangle_{(\beta_P, d)}$.

2. *If* $\vec{v} = \emptyset$ *then*

$$\langle\, \omega \mid \vec{v}\, \rangle^{(X_P, X_{P_\infty})}_{(\beta_P, d)} - \langle\, \omega\, \rangle^{X_P}_{(\beta_P, d)}$$

is generated by series of relative invariants on X_P *with curve classes lower than* (β_P, d).

Theorem (Type II invariance)

$\mathscr{F}$-invariance for X_D implies $\mathscr{F}$-invariance for $(X_P, X_{P_0} \sqcup X_{P_\infty})$.

- For fiber class inv., i.e. $\beta \in NE(X_P/D)$, they are reduced to the cup product on a birational $D' \to D$ and the case

$$(P^1, \{0, \infty\}) \times X_{\text{pt}}.$$

  Thus we consider non-fiber class type II-inv.
- Let $k \geq 0$ be the number of non-pullback insertions in $\pi : X_P \to X_D$. If $k \leq 1$, the strong pushforward (1) applies.
- If $k \geq 2$, since

$$[X_{P_0}] - [X_{P_\infty}] = \pi^* c_1(N_{X_D/X_P}),$$

  modulo type II-inv with $k-1$ non-pullback insertions, we may assume one is $i_{0*}(\alpha)$ and the others are $i_{\infty *}(\alpha_i)$.

The family $W = \mathrm{Bl}_{X_{P_\infty} \times \{0\}} X_P \times \mathbb{A}^1 \to X_P \times \mathbb{A}^1$ gives

$$\left\langle \vec{\mu} \mid \omega \cdot i_{0*}(\alpha) \prod_{i=1}^{k-1} i_{\infty *}(\alpha_i) \mid \vec{\nu} \right\rangle^{(X_P, X_{P_0}, X_{P_\infty})}_{(\beta_P, d)} =$$

$$\sum_{I,\eta} C_\eta \left\langle \vec{\mu} \mid \omega_1 \cdot i_{0*}(\alpha) \mid \lambda, e^I \right\rangle^{\bullet}_{\Gamma_1} \left\langle \lambda, e_I \mid \omega_2 \cdot \prod_{i=1}^{k-1} i_{\infty *}(\alpha_i) \mid \vec{\nu} \right\rangle^{\bullet}_{\Gamma_2},$$

where $\eta = (\Gamma_1, \Gamma_2)$ is the splitting type.

- Here $\omega, \omega_1, \omega_2$ are pullbacks insertions from X_D.
- The RHS is determined by type II generating functions with at most $k-1$ non-pullback insertions.
- This relation is compatible with $\mathscr{F}$-invariance, and the theorem follows by induction on $k \in \mathbb{N}$.

We omit the discussion on rubber calculus. **QED**