TMS Lecture 1. on Q-inv. 1/16, 2019.

rel GW. inv (Y, E) $\qquad \Gamma = (g, n, \beta, \rho, \mu)$

$\mu = (\mu_1, \dots, \mu_\rho) \quad |\mu| = \sum \mu_i = (\beta . E)$

$\langle A \mid \varepsilon, \mu \rangle^{(Y,E)}_\Gamma \qquad A \in H(Y)^{\oplus n}$

$\varepsilon \in H(E)^{\oplus \rho}$ $\qquad Z_E$

base e_i

Deformation to normal cone

$Z \subset X \qquad$ DNC $\qquad W = Bl_{Z \times 0}\, X \times \mathbb{A}^1$

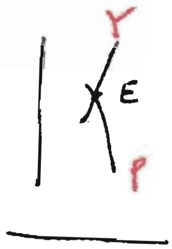
$Y = Bl_Z X$

$$W_0 = Y_1 \cup_E Y_2 = \mathbb{P}_Z(N \oplus \mathcal{O})$$

$$\begin{array}{ccc} Y_1 & & Y_2 \\ \phi \downarrow & & \downarrow p \\ X & & Z \end{array}$$

$\Phi: W \to X \times \mathbb{A}^1$, $\quad W \to \mathbb{A}^1$, $\quad X \times \mathbb{A}^1 \to \mathbb{A}^1$

local model (proj)

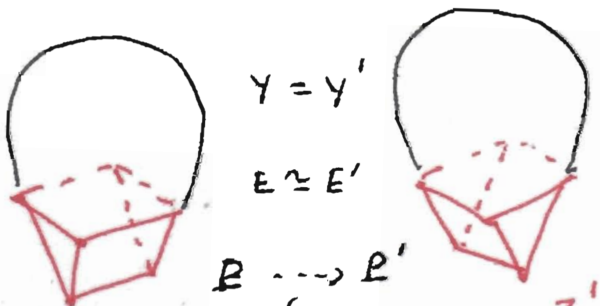


Deg. formula (J. Li)

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_{\eta = (\Gamma_1, \Gamma_2, I_\rho) \in \Omega_\beta} C_\eta \langle \alpha_1 \mid e_I, \mu \rangle^{\bullet (Y_1, E)}_{\Gamma_1} \langle \alpha_2 \mid e^I, \mu \rangle^{\bullet (Y_2, E)}_{\Gamma_2}$$

$\mu = \vec{\mu}$, $\quad \mu = \vec{\mu}^\vee$ weighted partition

$f : X \dashrightarrow X'$ ordinary flop with base S



$Y = Y'$

$E \cong E'$

$E \overset{f}{\dashrightarrow} E'$

$\mathbb{P}_S(F) = Z \xrightarrow{\bar{\psi}} S \qquad Z' = \mathbb{P}_S(F') \xrightarrow{\bar{\psi}'} S$

$N = N_{Z/X} = \mathcal{O}_Z(-1) \otimes \bar{\psi}^* F'$

$N_{E/Y} = \text{“}\mathcal{O}_Z(-1) \otimes \mathcal{O}_{Z'}(-1)\text{”}$

$\mathcal{F} = [\bar{\Gamma}_f]_* \qquad \alpha \sim (\alpha_1, \alpha_2) \qquad \mathcal{F}\alpha \sim (\alpha_1', \alpha_2')$

liftings

then $\alpha_1 = \alpha_1' \iff \mathcal{F}\alpha_2 = \alpha_2'$.

~~Prop 1.~~ To prove $\mathcal{F}\langle \alpha \rangle^X \cong \langle \mathcal{F}\alpha \rangle^{X'}$ $\quad \forall \alpha$

enough to show $\mathcal{F}\langle A \mid \varepsilon, \mu \rangle^{(P,E)} \cong \langle \mathcal{F}A \mid \varepsilon, \mu \rangle^{(P',E)}$

where $\mathcal{F}_* \xi^\beta := \xi^{\mathcal{F}\beta}$, so $\mathcal{F}\xi^\ell = \xi^{-\ell'}$

weighted partition

Relative back to absolute : (local case)

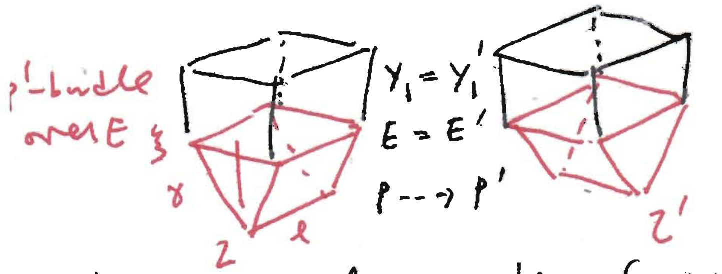
~~Prop 2~~. For local P^r flop $f: P \dashrightarrow P'$

enough to show $\mathcal{F}\langle A, \underline{\tau_{k_1}\varepsilon_1}, \dots, \tau_{k_\ell}\varepsilon_\ell\rangle^P \simeq \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1 \cdots \tau_{k_\ell}\varepsilon_\ell\rangle^{P'}$

means $i_2\varepsilon_1$ $i_2 : E \hookrightarrow P = Y_2$

sketch (for simple case $S = pt$) :

DNC for $Z \hookrightarrow P$



induction on $(|\mu|, n, \rho)$ reverse order

Given $\langle \alpha_1, \dots, \alpha_n \mid \varepsilon, \mu\rangle$

lower order triple $< \infty$

$\beta = d_1\ell + d_2\gamma$

$\Rightarrow d_2 = (E.\beta) = |\mu|$

idea: analyze deg. formula for

$$\langle \alpha_1, \dots, \alpha_n, \tau_{\mu_1 - 1}\varepsilon_1, \dots, \tau_{\mu_\rho - 1}\varepsilon_\rho\rangle^P$$

$$= \sum_{\mu'} m(\mu') \sum_{I'} \langle \tau_{\underline{\mu_1}-1} e_{i_1} \cdots \tau_{\underline{\mu_\rho}-1} e_{i_\rho} \mid e^{I'}, \underline{\mu'}\rangle^{\bullet (Y_1, E)} \langle \alpha_1 \cdots \alpha_n \mid e_{I'}, \mu'\rangle^{(P,E)}$$

$+ R \longrightarrow$ · contact order $< d_2$, or
· # insertion in (P,E) $< n$, or
· inv on (P,E) are disconnected

$\begin{pmatrix} * & * & \cdots \\ 0 & & \ddots \end{pmatrix}$

In fact, virt. dim count for main terms

$\Rightarrow (e_I, \mu)$ occurs $\Rightarrow |\mu'| = |\mu| = d_2$

$\Rightarrow R$ is $\mathcal{F}$-inv by induction.

Moreover, $\deg e_I - \deg e_{I'} = \rho - \rho'$ & must be fiber integral

claim: $\exists!$ term in the main term $= C(\mu)\langle \alpha_1 \cdots \alpha_n \mid e_I, \mu\rangle^{(P,E)}$

Key point ! fiber class ~~inv~~ rel inv on P^1-bundle $/E$

$= (\text{pairing of classes in } E)\cdot(\text{rel-GW on } P^1)$

$\#_0$ $\Rightarrow$ the fiber class is $P_*\gamma = 1$.

+ virt. dim count $\Rightarrow$ claim.

~~Def'n~~: $\langle A, \underline{\tau.\varepsilon.}\rangle$ is called descendent inv of ~~f-special type~~ i.e. $\xi \mid a$

caution: general descendents inv are NOT $\mathcal{F}$-inv !

$+ R \longrightarrow$ · # insertion in (P,E) $< n$ or

$\begin{pmatrix} * & * & \cdots \end{pmatrix}$

Example 1. Simple $\mathbb{P}^1$-flop with $\beta \neq d\ell$.

$$\langle \alpha \rangle^X_{g,n,\beta} = \langle \tilde{\alpha} \rangle^Y_{g,n,\phi^*\beta} = \langle \mathcal{F}\alpha \rangle^{X'}_{g,n,\mathcal{F}\beta}$$

i.e. no deg. terms and no analy cont is needed!

pf: $(K_X . p_*\beta_2) = 0 \Rightarrow$

$$(K_Y . \beta_1) = (\phi^* K_X + E . \beta_1) = (K_X . \underbrace{\phi_* \beta_1}_{\beta - p_*\beta_2}) + \mu = (K_X . \beta) + \mu$$

so $d_{\Gamma_1} = -(K_Y . \beta_1) + \underline{\rho - \mu} = \underbrace{-(K_X . \beta)}_{d^{\exp}} + (\rho - 2\mu)$

- $\rho \neq 0 \Rightarrow d_{\Gamma_1} < d^{\exp}$

May assume $\alpha_i \in H^{2\ell_i}$, $\ell_i \geqslant 2$, hence $|\alpha_i| \cap Z = \phi$

$\Rightarrow \alpha_i$ goes to Y_1 (blow-up) $\Rightarrow$ zero MV.

- hence $\rho = 0$ and result follows #

Example 2. Simple $\mathbb{P}^2$-flop. $n \leq 3$, $\alpha_i \in H^{2\ell_i}(X)$

$\ell_i \geqslant 2$, $\beta \neq d\ell$. Consider $\eta \in \Omega_\beta$ with $\rho \neq 0$. $(g = 0)$

$= (\Gamma_1, \Gamma_2, I)$

Then $v_1 = \rho = \mu$ $\qquad v_1 := |\Gamma_1| = \#$

$v_2 = 1$, $\ell_i = 2 \ \forall i$. $\qquad$ of conn. comp.

i.e.

$(\mu_1, \ldots, \mu_n) = (1, \ldots, 1)$

they are of lowest order with fixed $|\mu|$

since $\rho = \max$

$\Rightarrow$ can be reduced to abs. MV. in one step.

pf: $c_1(P \to Y_2) = 4E$, $d_{\Gamma_2} = 4(E . \beta_2) + (\underbrace{\dim X - 3}_{5}) v_2 + \rho - \mu$

"pure" v-dim $\qquad = 3\mu + \rho + 2v_2$

so $\underline{d_{\Gamma_2} - 4\rho} = 3(\mu - \rho) + 2v_2 \geqslant \underline{2}$.

$n = 1$: $\ell_1 = d + 1 = d_{\Gamma_1} + d_{\Gamma_2} - 4\rho + 1 \geqslant d_{\Gamma_1} + 3$

$\Rightarrow \alpha_1 \curvearrowright Y_2 \ ? \ \ell_1 = 2$ ✗ $\dim E$ (i.e. must be $\rho = 0$)

$n = 2$: $\ell_1 + \ell_2 = d + 2 \geqslant d_{\Gamma_1} + 4$ $\quad$ Similarly $? v_2 = 1$, $4\rho + 2$

a solution is $\ell_1 = \ell_2 = 2$, $d = 2$, $d_{\Gamma_1} = 0$, $\mu = \rho$, d_{Γ_2}

$n = 3$: ...

$v_2 = 1$, $\ell_i = 2 \ \forall i$. $\qquad$ of conn. comp.

TRR and divisor axiom :

<u>Thm</u> : $\mathcal{F}$-inv for descendent inv of f-sp type
$\Leftrightarrow$ $\mathcal{F}$-inv of big quantum ring

Pf : Only need $\Leftarrow$: Notice that

$$\begin{tikzcd} N_1(P) \arrow[rr, "\mathcal{F}"] \arrow[dr] & \circlearrowleft & N_1(P') \arrow[dl] \\ & N_1(S) \oplus \mathbb{Z} & \end{tikzcd}$$

via $N_1(P) = \iota_* N_1(Z) \oplus \mathbb{Z}\gamma = P_* \oplus d_2$

$\downarrow$ $\quad Z = P_S(F)$ Ker

$N_1(S)$ $\quad$ $\langle$ fiber " = extr ray ℓ $\rangle$

$* = \langle \tau_{k_1} a_1, \ldots, \tau_{k_n} a_n \rangle_{\beta_S, d_2}$ f-sp, $(\beta_S, d_2) \neq (0,0)$

induction on $k = \sum k_i$.

$k=0$, may assume $n \geq 3$ by adding div ξ, or $D \in H^2(S)$

in $H(P) = H(S)[h,\xi] / (f_F(h), f_{N \oplus O}(\xi))$

since $(\xi \cdot \ell) = 0 = (D \cdot \ell)$ $\Rightarrow$ affected by a const.

First for $n \geq 3$: TRR $\psi_1 \cdot [D_{1|23}]^{virt}$

$\Rightarrow$ $* = \sum_\mu \underbrace{\langle \tau_{k_1-1} a_1 \cdots T_\mu \rangle}_{①}{}_{\beta_S' d_2'} \langle \underbrace{T^\mu, \tau_{k_2} a_2}_{②}, \tau_{k_3} a_3, \cdots \rangle_{\beta_S'' d_2''}$

set $k_1 \geq k_2 \geq \cdots$ $\quad$ $\neq (0,0)$ since $\xi | a_1$

$\Rightarrow$ $\mathcal{F}$-inv $\Rightarrow$ $\mathcal{F}$-inv of ② since $k \downarrow$ and # ins ≥ 3.

$(n=1) \Leftarrow (n=2)$: Let $b = \xi$ or $D \in H^2(S)$

st $b \cdot (\beta_S + d_2 \gamma) \neq 0$

$\Rightarrow$ $(b \cdot \underline{\beta}) \neq 0$ is ind of $\underline{d_1 \ell}$

over β_S, d_2

$\langle b, \tau_k a \rangle_{\beta_S d_2} = \underline{(b.\beta) \langle \tau_k a \rangle_{\beta_S, d_2}} + \langle \tau_{k-1} ab \rangle_{\beta_2 d_2}$

again ind. on k.

$(n=2) \Leftarrow (n=3)$ by a similar method. #

conclusion : $\mathcal{F}$-inv of $QH(X)$ is reduced to the local model case $X_{loc} = P$.

<u>Problem</u> : How to do this for flips ?