K-equivalence in Birational Geometry

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Minimal Models for Algebraic Surfaces

Theorem 1 (Castelnuovo) *Let X be a smooth projective surface and $C \subset X$ be an irreducible curve. Then there exists a birational morphism $\phi : X \to \bar{X}$ contracting exactly the curve C down to a smooth surface $\bar{X}$ if and only if C is a (-1) curve. That is, $C \cong \mathbf{P}^1$ and $C^2 = -1$.*

Definition 2 *A smooth surface is called minimal if it contains no (-1) curves.*

Definition 3 *The Kodaira dimension $\kappa(X)$ of a proper smooth variety X is defined to be*

$$\kappa(X) = \lim_{\ell \to \infty} \dim \operatorname{Im} \left[\Phi_\ell : X \cdots\to \mathbf{P}\Big(H^0(X, K^\ell)\Big) \right].$$

Theorem 4 (Enrique) *Any surface X admits birational minimal models and it is unique if $\kappa(X) \neq -\infty$. Moreover, $\kappa(X) = -\infty$ if and only if X is birationally ruled, i.e. $C \times \mathbf{P}^1$.*

Mori Theory: Minimal Model Program

Definition 5 (Reid) *A normal variety X is terminal if K_X is $\mathbf{Q}$-Cartier and for some (hence any) resolution of singularities $\phi : Y \to X$, one has $K_Y =_{\mathbf{Q}} \phi^* K_X + \sum a_i E_i$ with $a_i \geq 0$.*

Theorem 6 (Mori, Kawamata, Shokurov) *Let X be a terminal variety. If K_X is not nef, each extremal ray $R \in \overline{NE}_{K<0}$ is spanned by a rational curve C. The extremal contraction $\psi_R : X \to \bar{X}$ defined by a supporting divisor D of R is a morphism such that $\psi_R(C') = pt \Leftrightarrow [C'] \in R$.*

One end up with 3 possibilities on $\bar{X}$:
1. $\dim \bar{X} < \dim X$: $X \to \bar{X}$ is a fiber space.
2. ψ_R is **divisorial**: $\dim \mathrm{Exc}(\phi_R) = n - 1$, OK.
3. ψ_R is **small**: $\dim \mathrm{Exc}(\phi_R) < n - 1$. In this case, $\bar{X}$ is very singular, it is not $\mathbf{Q}$-Gorenstein!

Definition 7 *X is a* **minimal model** *if it is terminal and K_X is nef.*

Three Dimensional Flips/Flops

Definition 8 *A* $(K_X + D)$ **log-flip** *of a log-extremal contraction* ψ *is a diagram*

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X^+ \\ & \psi \searrow \quad \swarrow \psi^+ & \\ & \bar{X} & \end{array}$$

such that f *is an isomorphism in codimension one and* $K_{X^+} + D^+$ *is* ψ^+*-ample.*

The case $D = 0$ is called a **flip**.
The case K_X is ψ-trivial is called a D-**flop**.

Theorem 9 (Mori) *3D flips exist.*

Theorem 10 (Reid, Mori) *A 3D terminal singularity of index* r *has the form* cDV$/\mu_r$*: cDV := isolated singularity* $f(x, y, z) + tg(x, y, z, t) = 0$ *in* $\mathbf{C}^4$ *where* f *is an ADE equation.*

Theorem 11 (Kollár, Mori) *3D flops exist in families. Also 3D birational* $\mathbf{Q}$*-factorial minimal models are related by a sequence of flops.*

Summary of 3D Mori Theory

∞. The MMP works. It ends up with a $\mathbf{Q}$-factorial minimal model.
3. The minimal models are not unique, but any two $\mathbf{Q}$-factorial minimal models X and X' are related by a sequence of flops and flops are completely classified.
2. $Def(X) \cong Def(X')$ canonically.
1. $H^*(X) \cong H^*(X')$, $IH^*(X) \cong IH^*(X')$ compatible with the mix (pure) Hodge structures.
0. X' has the same singularity type as X.

What Can One expect in HD Theory?

"**0**" is wrong in general. "∞" is infinitely hard. But the remaining "**1**", "**2**" and "**3**" do not depend on it. Notice even in 3D, the ring structures in "**1**" is usually different.

K-equivalence Relation

Definition 12 *Two $\mathbb{Q}$-Gorenstein varieties X and X' are K-equivalent, denoted by $X =_K X'$, if $\exists$ smooth Y and birational morphisms ϕ, ϕ':*

$$X \xleftarrow{\ \phi\ } Y \xrightarrow{\ \phi'\ } X'$$

such that $\phi^* K_X = \phi'^* K_{X'}$.

Theorem 13 *If X and X' are birational terminal varieties such that K_X and $K_{X'}$ is nef along the exceptional loci then $X =_K X'$.*

A Geometric Hueristic: For manifolds, this is the same as c_1-equivalent. For ω (resp. ω') Kähler forms on X (resp. X'), we get

$$-\partial\bar{\partial}\log(\phi^*\omega)^n = -\partial\bar{\partial}\log(\phi'^*\omega')^n + \partial\bar{\partial} f.$$

That is, $\phi^*\omega$ and $\phi'^*\omega'$ have quasi-equivalent volume forms. Can one **rotate** $\phi^*\omega$ to $\phi'^*\omega'$ through **Riemannian** metrics while keeping the quasi-equivalence class of degenerate volume forms?

p-adic Integral and Betti/Hodge Numbers

We will assume X and X' smooth from now on. Take an integral model of the K-equivalence diagram, e.g. $\mathcal{X} \to$ SpecS etc. For almost all prime P in S we have good reductions. In such cases, let $R = \widehat{S}_P$. Let U_i's be a Zariski open cover of X such that $K_X|_{U_i}$ is free. Then for a compact open subset $S \subset U_i(R) \subset X(R)$, we define its measure by $(R/P \cong \mathbf{F}_q,\ q = p^r.)$

$$m_X(S) \equiv \int_S |\Omega_i|_p$$

(Independent of Ω.) The p-adic measure of $X(R)$ and $X'(R)$ are the same by the **change of variable formula** and $X =_K X'$. But since

$$m_X(X(R)) = \frac{|\bar{X}(\mathbf{F}_q)|}{q^n},$$

we conclude that X and X' have the same local zeta factors for almost all P. This implies that $h^{p,q}(X) = h^{p \cdot q}(X')$ by Faltings' p-adic Hodge Theory.

Application to The Filling-in Problem

Theorem 14 *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a minimal Gorenstein 3-fold $\mathcal{X}_0$. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$.*

If $\mathcal{X}' \to \Delta$ exists then it must be terminal Gorenstein. So $\mathcal{X} =_K \mathcal{X}'$, in particular they are isomorphic in codimension one, so $\mathcal{X}_0$ is birational to $\mathcal{X}'_0$. If $\mathcal{X}_0$ is $\mathbf{Q}$-factorial then it must be smooth and we already get a contradiction. Otherwise consider a projective small morphism $X \to \mathcal{X}_0$ from a ($\mathbf{Q}$-factorial) minimal model X to $\mathcal{X}_0$. $X \sim \mathcal{X}_0 \sim \mathcal{X}'_0$. Hence X is smooth and $H^*(X) \cong H^*(\mathcal{X}'_0) \cong H^*(\mathcal{X}_t)$.
Consider the **contraction/smoothing** diagram:

$$\begin{array}{c} X \\ \downarrow \\ \mathcal{X}_0 \quad \subset \mathcal{X} \supset \mathcal{X}_t \end{array}$$

If $\mathcal{X}_0$ has only ODP, done by explicit formula for χ or b_i. For general cDV, use symplectic deformations to reduce to the ODP case.

Symplectic Deformation of 3D Flops

– Since index one terminal $\equiv$ isolated cDV $\equiv$ one parameter deformation of surface RDP's. By Friedman's result, if $p \in V$ is isolated cDV and $C \subset U$ is the corresponding germ of the exceptional curve contracted to p, then there is an inclusion $\mathrm{Def}(C, U) \to \mathrm{Def}(p, V)$ and both spaces are smooth.
– One can deform the complex structure of a nbd of C so that C decomposes into several $\mathbf{P}^1$'s and the contraction map deforms to nontrivial contractions of these $\mathbf{P}^1$'s to ODP's, while keeping a nbd of these ODP's to remain in $\mathrm{Def}(p, V)$.
– We can preform this analytic process for all C's and p's simultaneously in each corresponding small nbd and then patch them together smoothly or even **symplectically** (Wilson).

For flops, we may do this process for $X \to \bar{X}$ and $X' \to \bar{X}$ simultaneously to end up with several copies of classical $\mathbf{P}^1$-flops.

Complex Elliptic Genera and Cobordism

For a commutative ring R, an R-genus φ is defined by $Q(x) \in R[\![x]\!]$ through Hirzebruch's multiplicative sequence K_Q (or K_φ).
Let $Q(x) = x/f(x)$. The CEG φ_{ell} is defined by

$$f(x) = e^{(k+\zeta(z))x}\,\frac{\sigma(x)\sigma(z)}{\sigma(x+z)},$$

Theorem 15 *Let φ be the CEG. Then for any algebraic cycle D in X and birational morphism $\phi : Y \to X$ with $K_Y = \phi^* K_X + \sum e_i E_i$, we have*

$$\int_D K_\varphi(c(T_X)) = \int_{\phi^* D} \prod\nolimits_i A(E_i, e_i{+}1)\, K_\varphi(c(T_Y)).$$

where the Jacobian factor is defined by

$$A(t,r) = e^{-(r-1)(k+\zeta(z))t}\,\frac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}.$$

Idea of The Proof

(Residue Theorem) For any cycle D in X and for any blowing-up $\phi : Y \to X$ along smooth center Z with exceptional divisor E, one has for any power series $A(t) \in R[\![t]\!]$:

$$\int_{\phi^* D} A(E)\, K_Q(c(T_Y)) = \int_D A(0)\, K_Q(c(T_X))$$
$$+ \int_{Z.D} \mathrm{Res}_{t=0}\left(\frac{A(t)}{f(t)\prod_{i=1}^r f(n_i - t)}\right) K_Q(c(T_Z)).$$

Here n_i's denote the formal chern roots of the normal bundle $N_{Z/X}$.
The proof makes use of **deformations to the normal cone** to reduce to the case that $X = \mathbf{P}_Z(N \oplus \mathbf{1})$, then apply

$$c(T_Y) = \phi^* c(T_X)\, \phi^* c(\mathcal{Q})^{-1}\, (1{+}E)\, c(\phi^* \mathcal{Q} \otimes \mathcal{O}(-E)).$$

$$\bar{\phi}_* e^k = 0 \quad \text{for} \quad 0 \le k \le r-2$$

$$\bar{\phi}_* e^{(r-1)+k} = (-1)^{(r-1)+k} s_k(N) \quad \text{for} \quad k \ge 0.$$

($s(N) = \sum s_k(N)$ such that $s(N)c(N) = 1$.)

Main Conjectures

Fix a birational map $f : X \dashrightarrow X'$ such that $X =_K X'$. $T := \phi'_* \circ \phi^*$ be the cohomology correspondence determined by $\bar{\mathbf{\Gamma}}_f \subset X \times X'$.
I (canonical isomorphism)

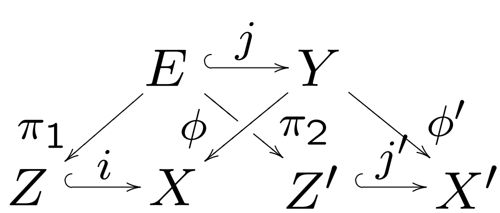
$$T : H^i(X, \mathbf{Q}) \xrightarrow{\sim} H^i(X', \mathbf{Q}),$$

and respects the rational Hodge structures.
II (quantum cohomology/Kähler moduli)
T also induces an isomorphism on the quantum cohomology rings over the extended Kähler moduli spaces.
III (birational complex moduli)
X and X' have canonically isomorphic (at least local) complex moduli spaces.
IV (soft decomposition)
X and X' admit symplectic deformations such that the K-equivalence relation deformed into copies of classical flops.

Topological Evidences

Let Ω^U be the cobordism ring of **stably almost complex manifolds**. An R-valued complex genus is a ring homomorphism $\varphi : \Omega^U \to R$. The cobordism class is determined exactly by all the chern numbers of the stable tangent bundle, i.e. by all its complex genera.

Definition 16 (Classical $\mathbf{P}^k$ Flops) *Let $Z \cong \mathbf{P}^k$ inside an $(n = 2k+1)$-D smooth variety X and $N_{Z/X} = \mathcal{O}_Z(-1)^{\oplus k+1}$. Then $E \cong \mathbf{P}^k \times \mathbf{P}^k$ and one may blow down E in another direction $\phi' : Y \to X'$ to get $j' : Z' = \phi'(E) \hookrightarrow X'$. Z' is also a $\mathbf{P}^k$ with normal bundle $\mathcal{O}_{\mathbf{P}^k}(-1)^{\oplus k+1}$:*



Let I_k be the ideal generated by all $[X] - [X']$.

Conclusion

Theorem 17 (Totaro) $\varphi_{\text{ell}} = (\Omega^U \to \Omega^U/I_1)$.

Theorem 18 (W-) *Let I_K be the ideal generated by $X - X'$ for $X =_K X'$. Then $I_K = I_1$. So Conjecture IV is true up to complex cobordism.*

Recently Huybrechts (supplementaed by a theorem of Demailly and Pann) has shown that birational hyperk"ahler manifolds X and X' admits deformations $\mathcal{X} \to \Delta$ and $\mathcal{X}' \to \Delta$ such that $\mathcal{X}_t \cong \mathcal{X}'$. This is the reason we do not need to consider Mukai flops in **IV**.

It is necessary to include (at least) all I_K to formulate **IV** by dimension reason. **END**