Algebraic methods in periodic singular Liouville equations

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Abstract

Throughout this talk, let

$$E = E_\tau = \mathbb{C}/\Lambda_\tau, \qquad \Lambda = \Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$$

be a flat tori, where $\tau \in \mathbb{H}$.

I will discuss non-linear singular Liouville equation on E_τ of the form

$$\triangle u + e^u = 4\pi \sum_{j=1}^{n} d_j \delta_{p_j},$$

where $d_j > 0$, $p_j \in E_\tau$ are distinct points and δ_{p_j} is the delta measure at p_j. Let $d := \sum_{j=1}^{n} d_j$ be the total singular strength.

This is also the equation of conic metric with constant curvature $K = 1$ outside p_j's.

Under developing maps, it corresponds to the unitary monodromy problem of generalized Lamé equations

$$w'' = \Big(\sum_{j=1}^{n} \eta_j(\eta_j+1)\wp(z-p_j) + \sum_{j=1}^{n} A_j \zeta(z-p_j) + B\Big) w.$$

where $\eta_j := d_j/2$.

Under the integrality assumption $d_j \in \mathbb{N}$, the equation is an algebraically integrable system and methods in algebraic geometry and modular forms can be brought in to study the detailed structures of the moduli spaces of solutions.

When the total strength d is odd, the structure behaves stably in τ.

When d is even, it depends on τ in a delicate manner.

In this talk, I will report only on the case $n = 1$.

Contents

Partly based on joint works with C.-L. Chai and C.-S. Lin.

Green functions on tori

- The Green function $G(z,w)$ on $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the unique function on $E \times E$ which satisfies

$$-\triangle_z G(z,w) = \delta_w(z) - \frac{1}{|E|}$$

  and $\int_E G(z,w)\, dA = 0$.

- Translation invariance of $\triangle_z$ implies $G(z,w) = G(z-w,0)$ and it is enough to consider $G(z) := G(z,0)$. Asymptotically

$$G(z) = -\frac{1}{2\pi} \log|z| + O(|z|^2).$$

- G can be explicitly solved in terms of elliptic functions.
- Let $z = x + iy$, $\tau := \omega_2/\omega_1 = a + ib \in \mathbb{H}$ and $q = e^{\pi i \tau}$ with $|q| = e^{-\pi b} < 1$. We have the odd theta function

$$\vartheta_1(z;\tau) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}.$$

- (Neron): On E_τ (notice the τ dependence),

$$G(z;\tau) = -\frac{1}{2\pi} \log \left| \frac{\vartheta_1(z;\tau)}{\vartheta_1'(0;\tau)} \right| + \frac{1}{2b} y^2 + C(\tau).$$

- The structure of G is fundamental for us. E.g.

$$\nabla G(z) = 0 \iff \frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi} \left((\log \vartheta_1)_z + 2\pi i \frac{y}{b} \right) = 0.$$

- Recall the Weierstrass elliptic functions with periods Λ:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda^\times} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big),$$

$$\zeta(z) := -\int^z \wp = \frac{1}{z} + \cdots, \qquad \sigma(z) := \exp\int^z \zeta(w)\,dw = z + \cdots.$$

- σ is entire, odd with a simple zero on lattice points and

$$\sigma(z+\omega_i) = -e^{\eta_i(z+\frac{1}{2}\omega_i)}\sigma(z),$$

where $\eta_i = \zeta(z+\omega_i) - \zeta(z) = 2\zeta(\frac{1}{2}\omega_i)$ are the quasi-periods.

- Indeed

$$\sigma(z) = e^{\eta_1 z^2/2}\frac{\vartheta_1(z)}{\vartheta_1'(0)}.$$

Hence $(\log\vartheta_1(z))_z = \zeta(z) - \eta_1 z$.

- We set $\omega_1 = 1$, $\omega_2 = \tau = a + bi$, $\omega_3 = \omega_1 + \omega_2$, and $z = x + yi = r\omega_1 + s\omega_2 = (r + sa) + sbi$.
- By Legendre's relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$,

$$\begin{aligned}(\log\vartheta_1)_z + 2\pi i\frac{y}{b} &= (\zeta(z) - \eta_1 z) + 2\pi i s \\ &= \zeta(z) - \eta_1 r - \eta_1 s\omega_2 + (\eta_1\omega_2 - \eta_2)s \\ &= \zeta(z) - r\eta_1 - s\eta_2.\end{aligned}$$

- Hence $\nabla G(z) = 0$ if and only if

$$-4\pi G_z = \zeta(r\omega_1 + s\omega_2) - (r\eta_1 + s\eta_2) = 0.$$

- **Question:** How many critical points can G have in E? What is the dependence of it in $\tau \in \mathbb{H}$?

- The 3 half periods are trivial critical points. Indeed,

$$G(z) = G(-z) \Rightarrow \nabla G(z) = -\nabla G(-z).$$

  Let $p = \frac{1}{2}\omega_i$ then $p = -p$ in E and so $\nabla G(p) = -\nabla G(p) = 0$.
- Other critical points must appear in pair $\pm z \in E$.

Example (Maximal principle)

For rectangular tori E: $(\omega_1, \omega_2) = (1, \tau = bi)$, $\frac{1}{2}\omega_i, i = 1, 2, 3$ are precisely all the critical points.

Example ($\mathbb{Z}_3$ symmetry)

For the 60 degree torus E with $\tau = \rho := e^{\pi i/3}$, there are 2 more points

$$p = \tfrac{1}{3}\omega_3, \qquad -p = -\tfrac{1}{3}\omega_3 \equiv \tfrac{2}{3}\omega_3.$$

Periodic singular Liouville equations

- The geometry of G plays a fundamental role in the non-linear mean field equations. On a flat torus E it takes the form ($\rho \in \mathbb{R}_+$)

$$\triangle u + e^u = \rho\delta_0.$$

- It is the mean field limit of Euler flow in statistic physics. It is also related to the self-dual condensation of abelian Chern-Simons-Higgs model (Nolasco and Tarantello 1999).
- When $\rho \notin 8\pi\mathbb{N}$, it was been proved by C.-C. Chen and C.-S. Lin (CPAM 2014) that the Leray-Schauder degree is

$$d_\rho = n + 1 \quad \text{for} \quad 8n\pi < \rho < 8(n+1)\pi,$$

which is independent of the shape (moduli) of E.

- The first interesting case (critical value) is when $\rho = 8\pi$ where the degree theory fails completely.

Theorem (Lin–W, Existence criterion via ∇G for $n = 1$)

For $\rho = 8\pi$, the mean field equation on a flat torus $E = \mathbb{C}/\Lambda$

$$\triangle u + e^u = 8\pi\delta_0$$

has solutions if and only if the G has more than 3 critical points.

Moreover, each extra pair of critical points $\pm p$ corresponds to an one parameter family of solutions u_λ, where $\lim_{\lambda\to\infty} u_\lambda(z)$ blows up precisely at $z \equiv \pm p$.

- **Structure of solutions.**
- Liouville's theorem says that any solution u of $\triangle u + e^u = 0$ in a simply connected domain $\Omega \subset \mathbb{C}$ must be of the form

$$u = \log \frac{8|f'|^2}{(1+|f|^2)^2},$$

where f, called a developing map of u, is meromorphic in Ω.

- It is straightforward to show that for $\rho = 8\pi\eta \in \mathbb{R}$,

$$S(f) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = u_{zz} - \frac{1}{2}u_z^2 = -2\eta(\eta+1)\frac{1}{z^2} + O(1).$$

I.e., any developing map f of u has the same Schwartz derivative $S(f)$, which is elliptic on E. Hence there is a $B \in \mathbb{C}$ such that

$$S(f) = -2(\eta(\eta+1)\wp(z) + B).$$

- By the theory of ODE, locally $f = w_1/w_2$ for two solutions w_i of the Lamé equation $L_{\eta,B}\, y = 0$:

$$y'' + \frac{1}{2}S(f)y = y'' - (\eta(\eta+1)\wp(z) + B)y = 0.$$

- Furthermore, for any two developing maps f and $\tilde{f}$ of u, there exists $S = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in PSU(2)$ such that

$$\tilde{f} = Sf := \frac{pf - \bar{q}}{qf + \bar{p}}$$

- So, solutions to the mean field equation correspond to Lamé equations with **unitary projective monodromy groups**.

- Geometrically the Liouville equation is simply the prescribing Gauss curvature equation in the new metric $g = e^{w} g_0$ over D, where $w = u/2 - \log \sqrt{2}$ and g_0 is the Euclidean flat metric on $\mathbb{C}$:

$$K_g = -e^{-u} \triangle u = 1. \tag{1}$$

- It is then clear the inverse stereographic projection $\mathbb{C} \to S^2$

$$(X, Y, Z) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)$$

provides solutions to (1) with conformal factor

$$e^{w} = e^{\frac{1}{2}u - \frac{1}{2}\log 2} = \frac{2}{1 + |z|^2}.$$

- Starting from this special solution for $D = \Delta$, the unit disk, general solutions on simply connected domain D can be obtained by using the Riemann mapping theorem via a holomorphic map
$$f : D \to \Delta.$$
- The conformal factor is then the one as expected:
$$e^u = \frac{8|f'|^2}{(1+|f|^2)^2}.$$
- The problem is to glue the local developing maps to a "global one". This is a monodromy problem on the once punctured torus $E^\times = E\backslash\{0\}$. Since it is homotopic to "8", we have
$$\pi_1(E^\times, x_0) = \mathbb{Z} * \mathbb{Z}$$
being a free group of rank two.

Lemma (Developing map for $\eta = \frac{1}{2}\ell \in \frac{1}{2}\mathbb{Z}$)

Given Λ, for $\rho = 4\pi\ell$, $\ell \in \mathbb{N}$, by analytic continuation across Λ, f is glued into a meromorphic function on $\mathbb{C}$. (Instead of on $E = \mathbb{C}/\Lambda$.)

- **First constraint from the double periodicity:**

$$f(z+\omega_1) = S_1 f, \quad f(z+\omega_2) = S_2 f$$

  with $S_1 S_2 = \pm S_2 S_1$ (abelian projective monodromy).

- **Second constraint from the Dirac singularity:**
  (1) If $f(z)$ has a zero/pole at $z_0 \notin \Lambda$ then order $r = 1$.
  (2) $f(z) = a_0 + a_{\ell+1}(z-z_0)^{\ell+1} + \cdots$ is regular at $z_0 \in \Lambda$.

- **Type I (Topological) Solutions $\Longleftrightarrow \ell = 2n+1$:**

$$f(z+\omega_1) = -f(z), \qquad f(z+\omega_2) = \frac{1}{f(z)}.$$

Then $g = (\log f)' = f'/f$ takes the form

$$g(z) = \sum_{i=1}^{l} (\zeta(z-p_i) - \zeta(z-p_i-\omega_2)) + c$$

which is elliptic on $E' = \mathbb{C}/\Lambda'$, $\Lambda' = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ with the only (highest order) zeros at $z_0 \equiv 0 \pmod{\Lambda}$ of order $\ell = 2n+1$.

- The equations $0 = g(0) = g''(0) = g^{(4)}(0) = \cdots$ implies that f is an even function (a non-trivial symmetric function argument). So f has simple zeros at $\pm p_1, \ldots, \pm p_n$ and $\omega_1/2$.

- The remaining equations $0 = g'(0) = g'''(0) = g^{(5)}(0) = \cdots$ leads to the polynomial system for $\wp(p_i)$'s:

Theorem (Type I integrability, $\rho = 4\pi(2n+1)$)

(1) *For* $\rho = 4\pi\ell$, $\ell = 2n+1$. *All solutions are of type I and even.* f *has simple zeros at* $\omega_1/2$ *and* $\pm p_i$ *for* $i = 1, \ldots, n$, *and poles* $q_i = p_i + \omega_2$.

(2) *For* $x_i := \wp(p_i)$, $\tilde{x}_i := \wp(q_i)$, *and* $m = 1, \ldots, n$,

$$\sum_{i=1}^{n} x_i^m - \sum_{i=1}^{n} \tilde{x}_i^m = c_m, \quad (x_m - e_2)(\tilde{x}_m - e_2) = \mu,$$

for some constants c_m *and* $\mu = (e_2 - e_1)(e_2 - e_3)$.

(3) *The corresponding Lamé equation* $L_{\eta=n+1/2,B}\, y = 0$ *has finite monodromy group* M *(in fact* $PM = K_4$*) hence there is a polynomial* p_n *of degree* $n+1$ *such that* $p_n(B) = 0$. *(Brioschi-Halphen 1894.)*

This is far more precise than the degree counting.

- **Type II (Scaling Family) Solutions $\Longleftrightarrow \eta = n$ ($\ell = 2n$):**

$$f(z+\omega_1) = e^{2i\theta_1} f(z), \qquad f(z+\omega_2) = e^{2i\theta_2} f(z).$$

- If f satisfies this, $e^{\lambda} f$ also satisfies this for any $\lambda \in \mathbb{R}$. Thus

$$u_\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}$$

  is a scaling family of solutions with developing maps $\{e^{\lambda} f\}$.

- u_λ is a **blow-up sequence**. The blow-up points for $\lambda \to \infty$ (resp. $-\infty$) are precisely zeros (resp. poles) of $f(z)$.

- $g = (\log f)'$ is elliptic on $E = \mathbb{C}/\Lambda$, with highest order zero at $z = 0$ of order $\ell = 2n$.

- $0 = g'(0) = g'''(0) = \cdots = g^{(2n-1)}(0)$ implies that g is even.
- Suppose that $g(z)$ has zeros $\pm p_1, \cdots, \pm p_n$. We may write

$$g(z) = \frac{\wp'(p_1)}{\wp(z) - \wp(p_1)} + \cdots + \frac{\wp'(p_n)}{\wp(z) - \wp(p_n)}$$

  constrained by $0 = g''(0) = \cdots = g^{(2n-2)}(0)$. These give rise to the first $n-1$ equations on $p_1, \ldots, p_n$. ($g(0) = 0$ is automatic.)
- And then

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

  should satisfy the n-th equation on monodromy

$$\int_{L_i} g \in \sqrt{-1}\mathbb{R}, \qquad i = 1, 2.$$

- **Periods integrals.** Let L_1, L_2 be the fundamental 1-cycles. Set

$$F_i(p) := \int_{L_i} \Omega(\xi, p)\, d\xi = \int_{L_i} \frac{\wp'(p)}{\wp(\xi) - \wp(p)}\, d\xi,$$

where $p \notin \frac{1}{2}\Lambda$ and $\frac{\wp'(p)}{\wp(\xi)-\wp(p)} = 2\zeta(p) - \zeta(p+\xi) - \zeta(p-\xi)$.

Lemma (Periods integrals and critical points)

Let $p = r\omega_1 + s\omega_2$, then (modulo $4\pi i\mathbb{N}$)

$$F_1(p) = 2(\omega_1\zeta(p) - \eta_1 p) = 2(\zeta(p) - r\eta_1 - s\eta_2)\omega_1 - 4\pi i s,$$
$$F_2(p) = 2(\omega_2\zeta(p) - \eta_2 p) = 2(\zeta(p) - r\eta_1 - s\eta_2)\omega_2 + 4\pi i r.$$

- Hence

$$\int_{L_i} g\, d\xi = \sum_{j=1}^{n} F_i(p_j) \in \sqrt{-1}\mathbb{R}, \quad i = 1, 2 \Longleftrightarrow \sum_{j=1}^{n} \nabla G(p_j) = 0.$$

- When $\rho = 8\pi$ ($n = 1, \ell = 2$), $p_1 = p, p_2 = -p, g(z) = \Omega(z, p)$.

$$f(z) = f(0) \exp \int_0^z g(\xi)\, d\xi$$

leads to a solution $\iff F_i(p) \in \sqrt{-1}\,\mathbb{R} \iff \nabla G(p) = 0$.

- **Theorem (Uniqueness, Lin–W 2006, Annals 2010)**

For $\rho = 8\pi$, the mean field equation $\triangle u + e^u = \rho\delta_0$ on a flat torus has at most one solution **up to scaling**.

- **Corollary (Number of critical points)**

The Green function has either 3 or 5 critical points.

  - New proof were found in 2016 by Eremenko et. al. using anti-holomorphic dynamics.

Geometry of critical points over $\mathcal{M}_1$

Theorem (Moduli dependence, Chen–Kuo–Lin–W, JDG)

(1) *Let $\Omega_3 \subset \mathcal{M}_1 \cup \{\infty\} \cong S^2$ (resp. Ω_5) be the set of tori with 3 (resp. 5) critical points, then $\Omega_3 \cup \{\infty\}$ is closed containing $i\mathbb{R}$, Ω_5 is open containing the vertical line $[e^{\pi i/3}, i\infty)$.*

(2) *Both Ω_3 and Ω_5 are simply connected with $C := \partial\Omega_3 = \partial\Omega_5$ homeomorphic to S^1 containing ∞.*

(3) *Moreover, the extra critical points are split out from some half period point when the tori move from Ω_3 to Ω_5 across C.*

(4) *(Strong uniqueness) The map $\Omega_5 \to [0,1]^2$ by $\tau \mapsto (t,s)$ for $p(\tau) = r\omega_1 + s\omega_2$ is a bijection onto*

$$\triangle = [(\tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{2})].$$

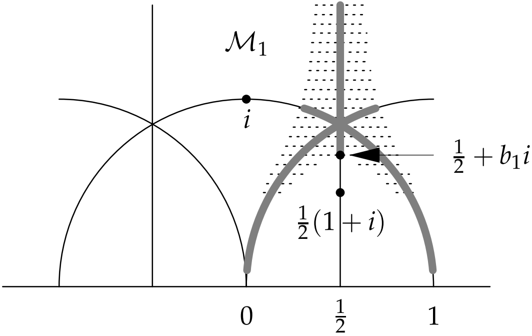

Figure: Ω_5 contains a neighborhood of $e^{\pi i/3}$.

- On the line $\operatorname{Re}\tau = 1/2$ which are equivalent to the rhombuses tori, the proof relies on *functional equations* of ϑ_1.
- The general case uses *modular forms of weight one*.

- **Idea of proof:**

$$\Psi(N) := \#\{\,(k_1,k_2) \mid (N,k_1,k_2) = 1, 0 \le k_i \le N-1\,\}.$$

Consider the weight one modular function for $\Gamma(N)$:

$$\begin{aligned} Z_{N,k_1,k_2}(\tau) &:= \zeta\Big(\frac{k_1\omega_1 + k_2\omega_2}{N};\tau\Big) - \frac{k_1\eta_1 + k_2\eta_2}{N} \\ &= -Z_{N,N-k_1,N-k_2}(\tau) \end{aligned}$$

(first studied by Hecke (1926));

- and the weight $\Psi(N)$ one for full modular group:

$$Z_N(\tau) := \prod_{(N,k_1,k_2)=1} Z_{N,k_1,k_2}(\tau) \in M_{\Psi(N)}(\mathrm{SL}(2,\mathbb{Z})).$$

- Each $\tau \in \mathbb{H}$ with $Z_N(\tau) = 0$ is (at least) a double zero.

- For odd $N \geq 5$, $\nu_i(Z_N) = \nu_\rho(Z_N) = 0$,
- At ∞, Hecke calculated the asymptotic expansion: $\nu_\infty(Z_N) = \phi(N/2) = 0$,
- Then the degree formula for modular forms (Riemann–Roch):

$$(Z_N)_{\mathrm{red}} = \frac{1}{2}\deg Z_N = \frac{1}{2}\sum_p \nu_p(Z_N) = \frac{\Psi(N)}{24}.$$

- Take N prime, this suggests a 1-1 correspondence between Ω_5 and

$$\triangle = [(\tfrac{1}{3}, \tfrac{1}{3}), (\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{2})]$$

under the map $\Omega_5 \to [0,1] \times [0, \frac{1}{2}]$:

$$\tau \mapsto (r, s), \quad \text{where} \quad p(\tau) = r\omega_1 + s\omega_2.$$

- **The actual proof:** Deformations in $r, s \notin \frac{1}{2}\mathbb{Z}$.
- Let $F \subset \mathbb{H}$ be the fundamental domain for $\Gamma_0(2)$ defined by

$$F := \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re}\tau \leq 1, \, |\tau - \tfrac{1}{2}| \geq \tfrac{1}{2}\}.$$

  We analyze solutions $\tau \in F$ for $Z_{r,s}(\tau) = 0$ by varying (r, s).
- For $\tau \in \partial F$, E is a rectangle and the only critical points of G are half periods. So $Z_{r,s}(\tau) \neq 0$ for $\tau \in \partial F$.
- Based on this, we use of the argument principle along the curve ∂F to analyze the number of zeros of $Z_{r,s}$ in F.
- We deduce from the Jacobi triple product formula that

$$\begin{aligned} Z_{r,s}(\tau) = 2\pi i(s - \tfrac{1}{2}) - \pi i \frac{2e^{2\pi i z}}{1 - e^{2\pi i z}} \\ - 2\pi i \sum_{n=1}^{\infty} \left(\frac{e^{2\pi i z} q^n}{1 - e^{2\pi i z} q^n} - \frac{e^{-2\pi i z} q^n}{1 - e^{-2\pi i z} q^n} \right), \end{aligned}$$

  where $z = r + s\tau$.

- Lemma (Asymptotic behavior of $Z_{r,s}$ on cusps)

We have $Z_{t,s}(-1/\tau) = \tau Z_{-s,t}(\tau)$, *and for* $t \in (0,1)$,

$$Z_{r,s}(\tau) = \frac{-1}{\tau} Z_{-s,r}(-1/\tau) = \frac{2\pi i}{\tau}\left(\tfrac{1}{2} - r + o(1)\right)$$

as $\tau \to 0$.

Similarly, $Z_{r,s}(\tau+1) = Z_{r+s,r}(\tau)$, *and for* $r+s \in (0,1)$,

$$Z_{r,s}(\tau) = Z_{r+s,s}(\tau-1) = \frac{2\pi i}{\tau-1}\left(\tfrac{1}{2} - (r+s) + o(1)\right).$$

- Lemma (Non-Vanishing)

  For any $\tau \in \mathbb{H}$, the addition law implies that

  (i) $\zeta(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2)) \neq \frac{3}{4}\eta_1 + \frac{1}{4}\eta_2$.
  (ii) $\zeta(\frac{1}{6}\omega_1 + \frac{1}{6}\omega_2)) \neq \frac{1}{6}\eta_1 + \frac{1}{6}\eta_2$.

  - For (ii), we choose $z = \frac{1}{6}(\omega_1 + \omega_2) = \frac{1}{6}\omega_3$ and $u = \frac{1}{3}\omega_3$. Then

$$0 \neq \frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(\tfrac{1}{2}\omega_3) + \zeta(-\tfrac{1}{6}\omega_3) - 2\zeta(\tfrac{1}{6}\omega_3)$$
$$= -3(\zeta(\tfrac{1}{6}\omega_1 + \tfrac{1}{6}\omega_2) - \tfrac{1}{6}\eta_1 - \tfrac{1}{6}\eta_2).$$

- Claim: Suppose that

$$(r,s) \in [0,1] \times [0,\tfrac{1}{2}] \backslash \{(0,0),(\tfrac{1}{2},0),(0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})\}.$$

  Then $Z_{r,s}(\tau) = 0$ has a solution $\tau \in \mathbb{H}$ if and only if that

$$(r,s) \in \triangle := \{(r,s) \mid 0 < r,s < \tfrac{1}{2}, r+s > \tfrac{1}{2}\}.$$

  Moreover, the solution $\tau \in F$ is unique for any $(r,s) \in \triangle$.
- *Proof*: The cases $(t,s) \notin \triangle$ are excluded by the Lammas. From

$$\nu_\infty(Z_3) + \frac{1}{2}\nu_i(Z_3) + \frac{1}{3}\nu_\rho(Z_3) + \sum_{p \neq \infty, i, \rho} \nu_p(Z_3) = \frac{8}{12},$$

  $Z_{\frac{1}{3},\frac{1}{3}}(\rho) = Z_{\frac{2}{3},\frac{2}{3}}(\rho) = 0 \Longrightarrow \nu_\rho(Z_{(3)}) = 2$ and other terms $= 0$.
  Thus $\tau = \rho$ is a simple root to $Z_{\frac{1}{3},\frac{1}{3}}(\tau) = 0$.

QED

Hyperelliptic geometry and Lamé curves

Theorem (Periods integrals and type II solutions)

Consider the mean field equation $\triangle u + e^u = \rho\delta_0$ *on* $E = \mathbb{C}/\Lambda$.

- *If solutions exist for* $\rho = 8n\pi$, *then there is a unique even solution within each type II scaling family.* ($\ell = 2n$, $a_{n+i} = -a_i$.)
- *The solution* u *is determined by the zeros* $a_1, \ldots, a_n$ *of* f. *In fact*

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}, \qquad f(z) = f(0) \exp \int^z g(\xi)\, d\xi.$$

- $\mathrm{ord}_{z=0}\, g(z) = 2n$ *leads to* $n - 1$ *equations for* $a = \{a_1, \ldots, a_n\}$.
- *The* n*-th equation is given by* $\int_{L_i} g \in \sqrt{-1}\mathbb{R}$, *which is equivalent to*

$$\sum_{i=1}^{n} \nabla G(a_i) = 0.$$

- **The $n-1$ algebraic equations:**
- Under the notations $(w, x_j, y_j) = (\wp(z), \wp(a_j), \wp'(a_j))$,

$$\begin{aligned} g(z) &= \sum_{j=1}^{n} \frac{1}{w} \frac{y_j}{1 - x_j/w} \\ &= \sum_{j=1}^{n} \frac{y_j}{w} + \sum_{j=1}^{n} \frac{y_j x_j}{w^2} + \cdots + \sum_{j=1}^{n} \frac{y_j x_j^r}{w^{r+1}} + \cdots . \end{aligned}$$

- Since $g(z)$ has a zero at $z = 0$ of order $2n$ and $1/w$ has a zero at $z = 0$ of order two, we get

$$\sum_{j=1}^{n} y_j x_j^r = \sum_{j=1}^{n} \wp'(a_j)\wp(a_j)^r = 0, \quad 0 \le r \le n-2.$$

Theorem (Green/polynomial system)

For $\rho = 8n\pi$, $n \in \mathbb{N}$, the n equations for $a = \{a_1, \ldots, a_n\}$ are precisely

$$\wp'(a_1)\wp^r(a_1) + \cdots + \wp'(a_n)\wp^r(a_n) = 0,$$

where $r = 0, \ldots, n-2$, and $\nabla G(a_1) + \cdots + \nabla G(a_n) = 0$.

Theorem (Hyperelliptic geometry/Lamé curve)

For $x_i := \wp(a_i)$, $y_i := \wp'(a_i)$, the first $n-1$ algebraic equations

$$\sum y_i x_i^r = 0, \quad r = 0, \ldots, n-2,$$

defines an affine hyperelliptic curve under the 2 to 1 map $a \mapsto \sum \wp(a_i)$:

$$X_n := \{(x_i, y_i)\} \subset \mathrm{Sym}^n E \longrightarrow (x_1 + \cdots + x_n) \in \mathbb{P}^1.$$

- **The proof relies on its relation to Lamé equations:**

$$f = \exp \int g\,dz = \exp \int \sum_{i=1}^{n} (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\,dz$$

$$= e^{2\sum_{i=1}^{n}\zeta(a_i)z} \prod_{i=1}^{n} \frac{\sigma(z-a_i)}{\sigma(z+a_i)} = (-1)^n \frac{w_a}{w_{-a}},$$

where $w_a(z) := e^{z\sum\zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z-a_i)}{\sigma(z)\sigma(a_i)}$ is **the basic element**.

- Theorem (Explicit map $a \mapsto B_a = (2n-1)\sum \wp(a_i)$)

$a \in X_n$ if and only if w_a and w_{-a} are **two** *solutions of the Lamé equation*

$$\frac{d^2w}{dz^2} - \Big(n(n+1)\wp(z) + (2n-1)\textstyle\sum_{i=1}^{n} \wp(a_i)\Big)w = 0.$$

- This is a long calculation via the polynomial system (omitted).

- **Idea of proof on the hyperelliptic structure on X_n.**
- Consider $y^2 = p(x) = 4x^3 - g_2 x - g_3$, where $(x, y) = (\wp(z), \wp'(z))$, and we set $(x_i, y_i) = (\wp(a_i), \wp'(a_i))$. Consider a basis of solutions to the Lamé equation

$$w'' = (n(n+1)\wp(z) + B)w$$

(for some B) given by $w_a(z)$ and $w_{-a}(z)$.
- Let $X(z) = w_a(z)w_{-a}(z)$. By the addition theorem,

$$X(z) = (-1)^n \prod_{i=1}^{n} \frac{\sigma(z+a_i)\sigma(z-a_i)}{\sigma(z)^2\sigma(a_i)^2} = (-1)^n \prod_{i=1}^{n} (\wp(z) - \wp(a_i)).$$

That is, $X(x) = (-1)^n \prod_{i=1}^{n}(x - x_i)$ is a polynomial in x.

- **Key:** $X(z)$ satisfies the second symmetric power of the Lamé equation:

$$\frac{d^3X}{dz^3} - 4(n(n+1)\wp + B)\frac{dX}{dz} - 2n(n+1)\wp' X = 0.$$

- Hence $X(x)$ is a polynomial solution, in variable x, to

$$p(x)X''' + \tfrac{3}{2}p'(x)X'' - 4((n^2+n-3)x+B)X' - 2n(n+1)X = 0.$$

- X is determined by B and certain initial conditions.

- Write $X(x) = (-1)^n(x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n)$, this translates to a linear recursive relation for $\mu = 0, \cdots, n-1$:

$$\begin{aligned} 0 = 2(n-\mu)(2\mu+1)(n+\mu+1)s_{n-\mu} \\ - 4(\mu+1)Bs_{n-\mu-1} \\ + \tfrac{1}{2}g_2(\mu+1)(\mu+2)(2\mu+3)s_{n-\mu-2} \\ - g_3(\mu+1)(\mu+2)(\mu+3)s_{n-\mu-3}. \end{aligned}$$

- We set $s_0 = 1$.
- For $\mu = n-1$ we get $B = (2n-1)s_1$ as expected.
- Thus all $s_2, \cdots, s_n$, $X(z)$, are determined by s_1, i.e. by B, alone.
- In fact, a slightly more work shows that the set $a = \{a_i\}$ is also determined by B up to sign. Hence $a \mapsto B_a$ is 2 to 1.

QED

Theorem (Chai-Lin-W 2012, CJM 2015)

- *There is a natural projective compactification* $\bar{X}_n \subset \mathrm{Sym}^n E$ *as a, possibly singular, hyperelliptic curve defined by*

$$C^2 = \ell_n(B, g_2, g_3) = 4Bs_n^2 + 4g_3 s_{n-2} s_n - g_2 s_{n-1} s_n - g_3 s_{n-1}^2,$$

in affine coordinates (B, C), *where*

$$s_k = s_k(B, g_2, g_3) = r_k B^k + \cdots \in \mathbb{Q}[B, g_2, g_3]$$

is an universal polynomial of homogeneous degree k with $\deg g_2 = 2$, $\deg g_3 = 3$, *and* $B = (2n-1)s_1$.

- *Thus* $\deg \ell_n = 2n+1$ *and* $\bar{X}_n$ *has arithmetic genus* $g = n$.
- *The curve* $\bar{X}_n$ *is smooth except for a finite number of* τ, *namely the discriminant loci of* $\ell_n(B, g_2, g_3)$, *so that* $\ell_n(B)$ *has multiple roots. In particular* $\bar{X}_n$ *is smooth for rectangular tori.*

- The second technique used in $\rho = 8\pi$ is to use the *method of continuity* to connect to the known case $\rho = 4\pi$ by establishing the non-degeneracy of linearized equations.
- For general ρ, such a non-degeneracy statement is out of reach. However, since solutions u_η always exist for $\rho = 8\pi\eta$, $\eta \notin \mathbb{N}$, it is natural to study the limiting behavior of u_η as $\eta \to n$. If the limit does not blow up, it converges to a solution u for $\rho = 8\pi n$.
- For the blow-up case, we have the connection between the blow-up set and the hyperelliptic geometry of $Y_n \to \mathbb{P}^1$:

Theorem (Chai–Lin–W, CJM 2015)

Suppose that $S = \{a_1, \cdots, a_n\}$ is the blow-up set of a sequence of solutions u_k to with $\rho_k \to 8\pi n$ as $k \to \infty$, then $S \in Y_n := \bar{X}_n \setminus \{\infty\}$. Moreover,

(1) *If $\rho_k \neq 8\pi n$ then S is a branch point ($a = -a$) of $Y_n \to \mathbb{C}$.*
(2) *If $\rho_k = 8\pi n$ for all k, then S is not a branch point of Y_n.*

Pre-modular forms

- Now we study the last equation on $\bar{X}_n$:

$$0 = -4\pi \textstyle\sum_{i=1}^n \nabla G(a_i) = \sum_{i=1}^n Z(a_i). \tag{2}$$

- Consider the rational function on E^n:

$$\mathbf{z}_n(a_1, \ldots, a_n) := \zeta(a_1 + \cdots + a_n) - \textstyle\sum_{i=1}^n \zeta(a_i).$$

- Let $a_i = r_i\omega_1 + s_i\omega_2$, then

$$\begin{aligned} -4\pi \sum \nabla G(a_i) &= \sum (\zeta(a_i) - r_i\eta_1 - s_i\eta_2) \\ &= \zeta(\sum a_i) - (\sum r_i)\eta_1 - (\sum s_i)\eta_2 - \mathbf{z}_n(a) \\ &= Z(\sum a_i) - \mathbf{z}_n(a). \end{aligned}$$

Hence (2) is equivalent to

$$\mathbf{z}_n(a) = Z(\textstyle\sum a_i). \tag{3}$$

- It is thus crucial to study the branched covering map

$$\sigma : \bar{X}_n \to E, \qquad a \mapsto \sigma(a) := \sum_{i=1}^{n} a_i.$$

Theorem (Lin–W 2013, JEP 2017)

(1) $\deg \sigma = \frac{1}{2}n(n+1)$.

(2) *There is a universal (weighted homogeneous) polynomial* $W_n(x) \in \mathbb{C}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][x]$ *of degree* $\frac{1}{2}n(n+1)$ *with*

$$W_n(\mathbf{z}_n) = 0.$$

Moreover, $\mathbf{z}_n \in K(\bar{X}_n)$ *is a primitive generator for the field extension* $K(\bar{X}_n)$ *over* $K(E)$.

(3) *The function* $Z_n(\sigma; \tau) := W_n(Z)$ *is pre-modular of weight* $\frac{1}{2}n(n+1)$*. That is, it is* $\Gamma(N)$*-modular if* $\sigma \in E_\tau[N]$.

- **Idea of proof for (1):** Apply *Theorem of the Cube*: For any three morphisms $f, g, h : V_n \longrightarrow E$ and $L \in \operatorname{Pic} E$,

$$(f+g+h)^*L \cong (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

- Apply to the case $V_n \subset E^n$ which is the ordered n-tuples so that $V_n/S_n = \bar{X}_n$, and $\deg L = 1$. We prove inductively that the map

$$f_k(a) := a_1 + \cdots + a_k$$

has degree $\frac{1}{2}k(k+1)n!$. This is NOT HARD to check for $k = 1, 2$.

- From k to $k+1$, we let $f = f_{k-1}$, $g(a) = a_k$, and $h(a) = a_{k+1}$.
- Then f_{k+1} has degree $n!$ times

$$\tfrac{1}{2}k(k+1) + 3 + \tfrac{1}{2}k(k+1) - \tfrac{1}{2}(k-1)k - 1 - 1 = \tfrac{1}{2}(k+1)(k+2).$$

- **Idea of proof of (2):** Major tool: *tensor product* of two Lamé equations $w'' = I_1 w$ and $w' = I_2 w$, where $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$.
- For $\bar{X}_n(\tau)$ smooth, and a general point $\sigma_0 \in E$, we need to show that the $\frac{1}{2}n(n+1)$ points on the fiber of $\bar{X}_n \to E$ above σ_0 has distinct $\mathbf{z}_n$ values. It is enough to show that for $\sigma(a) = \sigma(b) = \sigma_0$, the condition $\sum \zeta(a_i) = \sum \zeta(b_i)$ implies $B_a = B_b$ (and then $a = b$).
- If $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$, then the product $q = w_1 w_2$ satisfies

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0.$$

- If $a \neq b$, by addition law we find that $Q = w_a w_{-b} + w_{-a} w_b$ is an *even elliptic function* solution, namely a *polynomial* in $x = \wp(z)$. This leads to strong constraints on the corresponding 4-th order ODE in variable x, and eventually leads to a contradiction for generic choices of σ_0.

Indeed,

$$
\begin{aligned}
&p(x)^2 \ddddot{q} + 3p(x)\dot{p}(x)\dddot{q} \\
&\quad + \left(\tfrac{3}{4}\dot{p}(x)^2 - 2(2(n^2+n-12)x + B_a + B_b)p(x)\right)\ddot{q} \\
&\qquad - \left((2(n^2+n-3)x + B_a + B_b)\dot{p}(x) + 6(n^2+n-2)p(x)\right)\dot{q} \\
&\qquad\quad + \left((B_a - B_b)^2 - n(n+1)\dot{p}(x)\right)q = 0.
\end{aligned} \tag{4}
$$

As an even elliptic function, Q takes the form

$$
\begin{aligned}
Q(x) &= C\prod_{i=1}^{n}(\wp(z) - \wp(c_i)) =: C\prod_{i=1}^{n}(x - x_i) \\
&= C(x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^n s_n),
\end{aligned}
$$

The x^{n+2} terms agree automatically. The x^{n+1} degree gives

$$
\sum \wp(c_i) = s_1 = \frac{1}{2}\frac{B_a + B_b}{2n-1} = \frac{1}{2}\left(\sum \wp(a_i) + \sum \wp(b_i)\right).
$$

- Inductively the x^{n+2-i} coefficient in (4) gives recursive relations to solve s_i interns of $B_a + B_b$, $(B_a - B_b)^2$ and g_2, g_3 for $i = 1, \dots, n$.
- Indeed

$$s_i = s_i(B_a + B_b, (B_a - B_b)^2, g_2, g_3) = C_i(B_a + B_b)^i + \cdots$$

  is homogeneous of degree i if we assign $\deg B_a = \deg B_b = 1$ and $\deg g_2 = 2$, $\deg g_3 = 3$.
- There are two remaining consistency equations $F_1 = 0$, $F_0 = 0$ coming from the x^1 and x^0 coefficients in (4).
- In fact $(B_a - B_b)^2$ is a factor of both equations and we may write $F_1(B_a, B_b) = (B_a - B_b)^{2d_1} G_1(B_a, B_b)$ and $F_0(B_a, B_b) = (B_a - B_b)^{2d_0} G_0(B_a, B_b)$.
- If $B_a \neq B_b$ (i.e $\sum \wp(a_i) \neq \sum \wp(b_i)$), then

$$G_1(B_a, B_b) = 0, \qquad G_0(B_a, B_b) = 0,$$

  which has only a finite number of solutions (B_a, B_b)'s, i.e. E_τ's.

Example (of compatibility equations for $n = 2$)

For $n = 2$ we have $s_1 = \frac{1}{6}(B_a + B_b)$ and

$$s_2 = \frac{1}{36}(B_a + B_b)^2 + \frac{1}{72}(B_a - B_b)^2 - \frac{1}{4}g_2.$$

The first compatibility equation from x^1 is (after substituting s_1)

$$\frac{1}{6}(B_a - B_b)^2(B_a + B_b) = 0.$$

The second compatibility equation from x^0 is

$$(B_a - B_b)^2\left(\frac{1}{36}(B_a + B_b)^2 + \frac{1}{72}(B_a - B_b)^2 - \frac{1}{6}g_2\right) = 0.$$

If $B_a \neq B_b$ then $B_b = -B_a$ and then we can solve B_a, B_b:

$$B_a^2 = 3g_2 \implies \wp(a_1) + \wp(a_2) = \pm\sqrt{g_2/3}.$$

Such $a \in \bar{X}_2$ indeed lies at the branch loci of the Lamé curve.

Example ($n = 2$)

For $\mathbf{z}_2(a_1, a_2) = \zeta(a_1 + a_2) - \zeta(a_1) - \zeta(a_2)$, on X_2:

$$\mathbf{z}_2^3(a) - 3\wp(a_1 + a_2)\mathbf{z}_2(a) - \wp'(a_1 + a_2) = 0.$$

On E^2 it has one more term $-\frac{1}{2}(\wp'(a_1) + \wp'(a_2))$. Thus,

$$Z_2(\sigma; \tau) = W_2(Z) = Z^3 - 3\wp(\sigma)Z - \wp'(\sigma).$$

Example ($n = 3$)

For $\mathbf{z} = \mathbf{z}_3(a) = \zeta(a_1 + a_2 + a_3) - \zeta(a_1) - \zeta(a_2) - \zeta(a_3)$, on X_3:

$$\mathbf{z}^6 - 15\wp\mathbf{z}^4 - 20\wp'\mathbf{z}^3 + (\tfrac{27}{4}g_2 - 45\wp^2)\mathbf{z}^2 - 12\wp'\wp\mathbf{z} - \tfrac{5}{4}\wp'^2 = 0.$$

Thus, $Z_3(\sigma; \tau) = W_3(Z)$.

- **Key point:** $Z_1 \equiv Z = -4\pi\nabla G$ is the Hecke modular function. The critical point equation ($\Longleftrightarrow$ type II solutions of MFE) is transformed into zero of pre-modular forms.
- For general $n \geq 1$, we have the equivalences:
  - Solution u to MFE for $\rho = 8\pi n$.
  - Periods integral $\int_{L_j} g \in \sqrt{-1}\mathbb{R}$ ($= \omega_j$ coordinates of $\sum a_i$.)
  - Green equation $\sum_{i=1}^{n} \nabla G(a_i) = 0$ on X_n.
  - $\mathbf{z}_n(a) = Z(\sigma(a))$.
  - Non-trivial zero of $Z_n(\sigma;\tau) := W_n(Z)$.
- Remark on the last one: the branch point $a \in Y_n \backslash X_n$ ($a \neq -a$) satisfies the Green equation trivially.

END