Geometry of the 1st chern class.

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10/26 ; 2002
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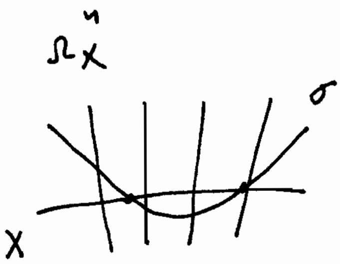
"c_1" : X cpx mfd , TX



$\Omega_X^n = \Lambda^n T^*X$ canonical line bundle

locally $f(z)\, dz_1 \wedge \cdots \wedge dz_n = \sigma \in \Gamma(X, \Omega_X^n)$

$K_X = \mathrm{div}(\sigma) = (f)_0 - (f)_\infty$

canonical divisor

$-[K_X] \in H_{2n-2}(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$

$\quad \shortparallel$

$c_1(X)$ 1st chern class.

~~metric~~ sections σ on $\Omega_X^n \rightsquigarrow$ metric on $\Lambda^n TX$ hermitian

curvature form (1st chern form) $Ric = -\partial\bar\partial \log \sigma \wedge \bar\sigma \in c_1(X)$.

volume form $\longmapsto$ 1st chern form

$\omega = g_{i\bar j}\, dz^i \wedge d\bar z^j \longmapsto -\partial\bar\partial \log \det(g_{i\bar j})$

$\det(g_{i\bar j}) = \omega^n$

conversely, any 1st chern form

$\rightarrow$ an arbitrary volume form $\sigma \wedge \bar\sigma$

can find $(\omega + \partial\bar\partial\varphi)^n = e^f \omega^n$. $\quad = e^f \omega^n$

"Absolute geometry".

"relative geometry of c_1" = "bi-rational geometry"

$Y \xrightarrow{\phi} X$ generic 1-1 (cpx morphism) holomorphic

$K_Y = \phi^* K_X + E$

locally, $dx_1 \wedge \cdots \wedge dx_n = (J\phi)\, dy_1 \wedge \cdots \wedge dy_n$

$J\phi = \det D\phi$

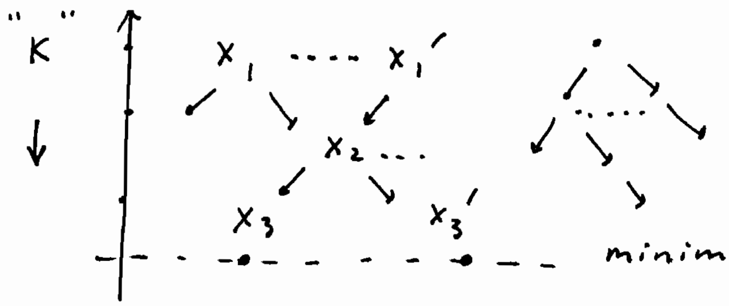
so $\mathrm{div}(J\phi) = E$.

X smooth, $E > 0$

$\quad$ " $\sum e_i E_i$ exc. divisor. $\phi|_{Y \setminus E}$ = isom.

so, "$Y \geq_K X$".

MMP:
minimal model program:



formal def:
X is a minimal model if K_X is nef.
(allow certain sing. st)
- K_X defined
- $E \geq 0$

Fact: All bi-rat'l min model have equal K-level.

Def: K-equiv. or G-equiv.
K-partial ordering.

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X \arrow[rr, dashed, leftrightarrow, "f"'] & & X' \end{tikzcd}$$

st. $\phi^* K_X = \phi'^* K_{X'}$
(i.e. $E = E'$)
$(\Leftrightarrow \varphi^* G(x) = \varphi'^* G(x'))$

Goal: Study G-equiv.

1. motives
2. complex cobordism
3. jet space

1: motive: Assume X proj sm.

consider $K_0(Var_{\mathbb{C}})$:

K-theoretic property:

$$\begin{array}{ccc} Y & \supset & E = \mathbb{P}_Z(N) \\ \downarrow & & \downarrow \\ X & \supset & Z \end{array}$$

$N = N_{Z/X}$
$rk = r$

$$[X] = [X \setminus Z] + [Z]$$
$$= [Y \setminus E] + [E] \cdot [\mathbb{P}^{r-1}]^{-1}$$

"good" change of variable formula

$$K_Y = \varphi^* K_X + (r-1) E$$

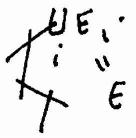
inductively $Y \to X$ composite of blowing ups

$$K_Y = \varphi^* K_X + \sum_{i=1}^m r_i E_i$$

$$\left\{ \begin{array}{l} x_1 = y_1 \\ x_2 = y_1 y_2 \\ \vdots \\ x_r = y_1 y_r \end{array} \right.$$

$$\left\{ \begin{array}{l} x_{r+1} = y_{r+1} \\ \vdots \\ x_n = y_n \end{array} \right.$$ define Z

$$[x] = \sum_{I \subset \{1, \cdots, m\}} [\mathring{E}_I] \prod_{i \in I} [\mathbb{P}^{r_i - 1}]^{-1}$$



$$\mathring{E}_I = \left(\bigcap_{i \in I} E_i \right) \setminus \left(\bigcup_{j \notin I} E_j \right).$$

$$E_\phi = Y \setminus E$$

$$dx_1 \wedge \cdots \wedge dx_r \wedge \cdots dx_n$$
$$= dy_1 \wedge y_1 dy_2 \wedge \cdots y_r dy_r \wedge \cdots dy_n$$
$$= y_1^{r-1} dy_1 \wedge \cdots \wedge dy_n.$$

$X =_K X' \Rightarrow \exists P = \prod_i P_i$ st.
$$[P] \cdot [X] = [P] \cdot [X']$$

use weak factorization thm of Wlodarczyk.

Deligne theory of mixed Hodge str on H_c^i

$\Rightarrow \chi_c^{p,q}$ factors thr K_0-construction
$\Rightarrow H^{p,q}(X) \cong H^{p,q}(X')$ non-canonical.

original approach : p-adic integral
+ Weil conj
+ Čebotarev density
+ p-adic Hodge theory

$$\star\quad \begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X \arrow[rr, dashed, leftrightarrow, "f"'] & & X' \end{tikzcd}$$

/ over $R = \mathbb{Z}[a_i]$ f.g $/\mathbb{Z}$, may assume finite $/\mathbb{Z}$.
take $P \in \operatorname{Spec} R$, good red. ($\forall'$)
take completion over $\hat{R}_p$: (eg. $\mathbb{Z}_p$)

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{Spec} \mathbb{Z}_p \end{array}$$

$$\int_{\mathcal{X}(\mathbb{Z}_p)} d\mu = \frac{|X(\mathbb{F}_p)|}{p^n} \quad \text{(Weil).}$$

p-adic integral
$\parallel$
$$\int_{\mathcal{Y}(\mathbb{Z}_p)} |J\phi|\, d\mu$$

$\Rightarrow |X(\mathbb{F}_p)| = |X'(\mathbb{F}_p)|$
Similarly for $\mathbb{F}_{p^i}$

let $K = Q(R)$, number field, fix prime ℓ
$\operatorname{Gal}(\bar{K}/K)$ modules $V = H^k_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)^{ss}$, $H^k_{et}(X'_{\bar{K}}, \mathbb{Q}_\ell)^{ss} = V'$
$\forall' p \in \operatorname{Spec} R$, good reduction, unramified

(i.e. $\operatorname{Gal}(\bar{K}_p/K_p)$ dep. only on ?) $\operatorname{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$, i.e. Frobenius.

Deligne's Weil conj :
$\Rightarrow \forall' p$, char poly of Fr_p on V, V' equal
density thm $\Rightarrow V \simeq V'$ as $\operatorname{Gal}(\bar{K}/K)$-repr.
→ also true for local repr : replace K by K_p.

Hodge-Tate decomposition :
(Fontaine-Messing)
$$H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_i \left(\mathbb{C}_p \otimes_K H^{m-i}(X_K, \Omega^i)(-i) \right)$$
$$\Rightarrow h^{i,m-i} = \dim_K \left(\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)^{ss}(i) \right)^G \quad \star$$

(Date : 2002/Aug.)

- still another pf use "motivic integration" (later)
- still another "original approach", All above lacking canonical mapping between $H^i(X,\mathbb{C}) \to H^i(X',\mathbb{C})$?

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X & & X' \\ \omega & & \omega' \end{tikzcd}$$

$-\partial\bar{\partial} \log \phi^*\omega^n = -\partial\bar{\partial} \log \phi'^*\omega'^n + \partial\bar{\partial} f$

$\Rightarrow (\phi'^*\omega')^n = e^f (\phi^*\omega)^n$
vol. equiv. deg. Kähler metrics. ?

2: Complex cobordism:

$\prod_i c_i^{n_i}$

Find the "most general" chern numbers preserved under the K-equiv.

$$\Omega_{\oplus} = \{\text{stably almost cpx mfd } Tx \text{ or } Tx \oplus \text{ has cpx str.}\} / \partial\{\text{s.a.c.m}\}.$$

$M_1 - M_2 = \partial W$

Milnor / Novikov: $[M_1] = [M_2] \Leftrightarrow$ Chern #'s are the same

R-Genus: $\Omega \xrightarrow{\varphi} R$ ring hom.

Hirzebruch: multiplicative sequence $Q(x) \in R[[x]]$

" $x/f(x) = 1 + \cdots$

$$\varphi(x) := \int_X K_\varphi(c(T_X)) = \int_X \prod_i Q(x_i)$$

$$c(T_X) = \prod_i (1 + x_i) \quad \text{chern roots.}$$

idea: View $K_\varphi(c(T_X))$ "=" $d\mu_X$

* Residue Thm (Wang 2000): $\phi: Y \to X$ blow-up, $E \to Z$, $N = N_{Z/X}$. $rk = r$.

$$\int_{\phi^* D} A(E)\, d\mu_Y = \int_D A(0)\, d\mu_X + \int_{Z.D} \operatorname{Res}_{t=0}\left(\frac{A(t)}{f(t)\prod_{i=1}^{r} f(n_i - t)}\right) d\mu_Z$$

Z

"good CVF" $\Leftrightarrow$ Functional Equations.

$$c(N_{Z/X}) = \prod_{i=1}^{r}(1 + n_i).$$

for each $r = 2, 3, \cdots$

$A_r(t)$: Jacobian factors.

$\operatorname{Res}_{t=0} = 0$, $r = 2$:

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}.$$

Computations $\Rightarrow$

$$f(x) = e^{kx} e^{S(z)x} \frac{\sigma(x)\sigma(z)}{\sigma(x+z)}$$

parameters k, $\underbrace{w_1, w_2}_{\text{elliptic curve}}$, z / mark pt.

$\cong \mathbb{C}^4$

$$\Rightarrow A_r(t) = e^{-(r-1)(k+S(z))t} \frac{\sigma(t + rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}$$

Called: cpx elliptic genera "CEG"

induction: for CEG φ, CVF: $\underline{rz \neq 0}$

$$\int_D d\mu_X = \int_{\phi^* D} \prod_i A(E_i; e_i + 1) \cdot d\mu_Y.$$

plus weak factorization: $X =_K X' \Rightarrow \varphi(x) = \varphi(x')$.

Main Conclusion:

Ω two ideals

$I_K = \{[X]-[X'] \mid X =_K X'\}$

$I_1 = \{[X]-[X'] \mid X \sim X'$ a classical flop $\}$

$\mathbb{P}^1$

$\mathbb{P}^r$-flop:

$$X \supset Z \qquad \mathbb{P}^r\text{-bundle } \psi \searrow S$$

$$N_{Z/X}\big|_{\psi^{-1}(s)} \cong \mathcal{O}(-1)^{\oplus r+1}$$

codim $= r+1$

$$\begin{tikzcd}
 & & Y \arrow[dll] \arrow[drr] & & \\
 & & E \arrow[dl] \arrow[dr] \arrow[dd] & & \\
X \supset & Z \arrow[dr] & & Z' \arrow[dl] & \subset X' \\
 & & S & &
\end{tikzcd}$$

($E \subset Y$)

Then, $I_K = I_1$ in Ω!

i.e. up to cpx cobordism

$X =_K X'$ via f

f is decomposed into composite of $\mathbb{P}^1$-flops

Rmk: $\dim X = 3$, true for symplectic deformation via MMP.

Q: In general, true for symp. def. into $\mathbb{P}^r$ flops $\forall r \in \mathbb{N}_{\geq 2}$?

3. Jet space

- Direct investigation :

Nash's arc space (1968) :

X cpx mfd , $JX = \{ \gamma : (\mathbb{C}, 0) \to X \}$
germs of curves

$J_m X = JX \mod t^{m+1}$
i.e. $\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1^{(i)}(0) = \gamma_2^{(i)}(0)$ $\forall i = 1, \dots, m$.

Basic observation :

$$Z \subset X \overset{f}{\dashleftarrow\dashrightarrow} X' \supset Z'$$

any birational map. exc. loci

$(JX)^\times := JX \setminus JZ$: $(JX)^\times \simeq (JX')^\times$
1-1, onto
Rieman's ext. thm

Problem : How to compute invariants or geom. obj from this "analytic loop space" ?

Basic structure Thm : $\underbrace{\text{"CVF"}}_{\text{universal}}$

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X & & X' \end{tikzcd}$$

the pf is the same as in the p-adic integral

$\operatorname{ord}_t J\phi : (J\cancel{X}Y)^\times \to \mathbb{N}_{\geq 0}$
$\gamma \mapsto \operatorname{ord}_t \{ (J\phi) \circ \gamma(t) \}$

$(JY)^\times = \coprod_{k \geq 0} S_k$ — γ's with $\operatorname{ord}_t J\phi = k$

$$\begin{array}{c} S_k \\ \downarrow \\ (JX)^\times \end{array}$$

1-1. by Riem ext. not interesting.

but, $m \gg k$: $\pi_m S_k \hookrightarrow (J_m Y)$, $\pi_m S_k \to J_m X$ is a piece-wise $\mathbb{C}^k$-fibration over its image.

$$\left\{ \begin{tikzcd} & \overline{S_k} \arrow[dl] \arrow[dr] & \\ J_m X & & J_m X' \end{tikzcd} \right\}_{k \in \mathbb{N}}$$

Q : Can we use this to prove canonical isom of coh. (Hodge str) ?

Applications : Deformations of birat'l C-Y mfds.

Appendix :

idea of pf of the universal CVF :

locally : $\phi : \mathbb{C}^n \to \mathbb{C}^n$, $\phi(0)=0$

let $\phi(y(t)) = x(t)$, $y(t) \in S_k$

①. $\forall\, v \in \mathbb{C}[\![t]\!]^n$, $\exists!$ solution $u \in \mathbb{C}[\![t]\!]^n$ $\quad l \geq 2k+1$

$$\phi(y(t) + t^{l-k} u) = x(t) + t^l v$$

$$\shortparallel$$

$$\phi(y(t)) + \underbrace{D\phi\,(y(t))}_{A} \cdot \underline{t}^{l-k} u + R \cdot t^{2(l-k)}$$

$J\phi = \det A$ order $= k$.

let $m \geq 2k$, by Newton's iteration or Hensel's lemma.

②. $l = m+1$: $\quad (2l-2k) - l = l-2k \geq 1$

Solve $\phi(\tilde{y}(t) \ \underline{\bmod t^{m+1}}) = x(t) \ \underline{\bmod t^{m+1}}$ =.

by ① solution $\bar{u}$ is a subspace in $(\mathbb{C}^k)^n = \mathbb{C}^{nk}$

Hensel's lemma : equiv.

may assume

$\hat{y}(t) = y(t) + t^{m+1-k} u$

$\bar{u} \equiv u \mod t^k$

$$D\phi(y(t))\, \underline{t}^{(m+1)-k}\, u = 0 \mod \underline{t}^{m+1}$$

i.e. $Au \equiv 0 \mod t^k$

since $\operatorname{ord}_t(A^*) = (n-1)k$ $\ast$

$\Rightarrow$ sln of $\bar{u}$ has dim

$(nk) - (n-1)k = k \quad \square$

To go further, we study the <u>Key structure</u>

$$\begin{tikzcd} & \overline{S_k} \hookrightarrow \mathcal{Y}_m^{\times} \arrow[dl] \arrow[dr] & \\ \mathcal{X}_m^{\times} \arrow[rr, dashed, leftrightarrow, "\text{bi-rat'l}"'] & & \mathcal{X}'^{\times}_m \end{tikzcd}$$

$\underline{m \gg k}$

$$\begin{tikzcd} & S_k \arrow[dl] \arrow[dr] & \\ \mathcal{X} & & \mathcal{X}' \end{tikzcd} \quad \& \quad \begin{tikzcd} & \pi_m(S_k) \arrow[dl] \arrow[dr] & \\ \mathcal{X}_m & & \mathcal{X}'_m \end{tikzcd}$$

image

$$\begin{tikzcd} & \overline{S_k} \arrow[dl] \arrow[dr] & \\ \phi_m(\overline{S_k}) & & \phi'_m(\overline{S_k}) \end{tikzcd}$$

$$\mathcal{X}_W^{\times} = \pi_0^{-1}(W) \setminus Z$$

$$\mathcal{X}_{m\,W}^{\times} = \psi_m^{-1}(W) \qquad \psi_m : \mathcal{X}_m^{\times} \longrightarrow \mathcal{X}_0 = X$$

straightforward pull back

$$\overline{B_k} = \phi_m(\overline{S_k}) \times \phi'_m(\overline{S_k})$$

modulo " $\overline{S_k}$ "

$$\begin{tikzcd} S_k \hookrightarrow \mathcal{Y} \hookleftarrow S'_k \arrow[d, phantom] & \\ \gamma \in \mathcal{X} \quad \xleftarrow{\phi} \quad \xrightarrow{\phi'} \quad \mathcal{X}' \end{tikzcd}$$

$S_k \equiv S'_k$

since $(J\phi) = (J\phi')$

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ A \arrow[dr] & & A' \arrow[dl] \\ & B & \end{tikzcd}$$

γ has a unique pre image in $\mathcal{Y}$, say $\tilde{\gamma}$

so $k(\gamma)$ is well-defined

γ has unique corr. in $\mathcal{X}'$, called γ'

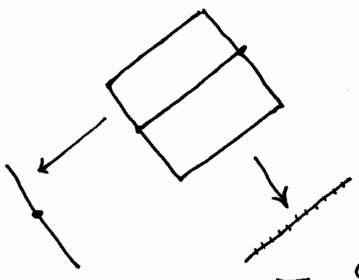
clearly $\gamma \longleftrightarrow \tilde{\gamma} \longleftrightarrow \gamma'$.

$$B = A \times A' / S$$

in ring level

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

$$\begin{tikzcd} & S & \\ A \arrow[ur, "i"] & & A' \arrow[ul, "i'"'] \\ & i(A) \cap i(A') \arrow[ul] \arrow[ur] & \end{tikzcd}$$



$$X_0 = X$$

$$\mathcal{X}_1 \subset X_1 = \mathbb{P}(T_X \oplus \xi)$$

$$T\mathcal{X}_1 \subset X_2 = \mathbb{P}(T_{X_1} \oplus \xi)$$

$$T_{\mathcal{X}_1} \subset \qquad \cdots \text{ etc.}$$

$\mathcal{X}_2 \quad \int$ what's the relation?

<u>End</u>.