Mean Field Equations and Alg Geom

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$\boxed{\text{MFE}}$: $\Delta u + e^u = \sum_{i=1}^{r} \rho_i \delta_{p_i}$ on $T = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}$

developing map: Liouville thm $\Rightarrow$

$$u = c + \log \frac{|f'|^2}{(1+|f|^2)^2}$$

f mero on U, simply conn in $T \setminus \{p_i\}$

Schwartz derivative: $v := \log f'$

(*) $S(f) := v'' - \frac{1}{2}(v')^2 = u_{zz} - \frac{1}{2} u_z^2$

indep of f $\Rightarrow$ f is determined up to $PSU(2)$ action

"integrable case" $\rho_i = 4\pi l_i$, $l_i \in \mathbb{N}$, $l := \sum l_i$

$\Rightarrow$ f mero ext over p_i $\forall i$, hence mero on $\mathbb{C}$

(eg Lin-W 07')

$$f(z+\omega_1) = \bar{S}_1 f(z) \qquad f(z+\omega_2) = \bar{S}_2 f(z)$$

$\bar{S}_1 \bar{S}_2 = \bar{S}_2 \bar{S}_1$ in $PSU(2)$

Lemma: for any u, may choose f st one of the following holds

type I : $f(z+\omega_1) = -f(z)$, $f(z+\omega_2) = \dfrac{1}{f(z)}$

type II : $f(z+\omega_1) = e^{i\theta_1} f(z)$, $f(z+\omega_2) = e^{i\theta_2} f(z)$

Basic Monodromy Theory $\eta_i := \frac{1}{2} l_i \in \frac{1}{2}\mathbb{N}$

(*) $\Rightarrow$ $S(f) = -2 \left(\sum_{i=1}^{r} \eta_i(\eta_i+1)\wp(z-p_i) + A_i \zeta(z-p_i) + B \right)$

with $\sum A_i = 0$

ie $f = \dfrac{w_1}{w_2}$, w_i sol of Generalized Lamé equ

$\boxed{\text{GLE}}$: $w'' + \frac{1}{2} S(f)\, w = 0$.

f mero ⟺ GLE has all sol log-free

wrt each p_i, has fund sol's

$$h_1(z) = z^{-\eta_i} g_1(z), \quad h_2(z) = z^{\eta_i+1} g_2(z)$$

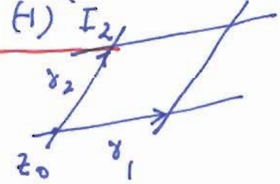
g_i holo at p_i $(z=0)$, $g_i(0)=1$

$\Rightarrow$ local monodromy matrix $\sigma_{p_i} = (-1)^{\overset{\ell_i}{\overset{\parallel}{2\eta_i}}} \begin{pmatrix}1&0\\0&1\end{pmatrix}$ in any basis

Monodromy repr

$$\rho : \pi_1(T \setminus \{p_1, \dots, p_r\}, z_0) \longrightarrow GL(2,\mathbb{C})$$

$$S_i := \rho(\gamma_i) \Rightarrow S_2^{-1} S_1^{-1} S_2 S_1 = \prod_{i=1}^{r} \sigma_{p_i} = (-1)^{\ell} I_2$$



Prop. Consider GLE with log-free sol,

(1) ℓ odd $\Rightarrow$ $PM \cong \underset{\text{Klein 4 gp}}{K_4}$ $\Rightarrow$ type I

(2) ℓ even $\Rightarrow$ Any sol u of MFE (if $\exists$) must be type II

Pf (1) $S_1 v = \lambda v$, $v \neq 0$ $\Rightarrow$ $S_1(S_2 v) = -S_2 S_1 v = -\lambda (S_2 v)$

w.r.t. basis $v, S_2 v$: $S_1 = \begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}$ $S_2 = \begin{pmatrix}0 & b^{\neq 0}\\ 1 & 0\end{pmatrix}$

$\bar\rho : \pi_1(T) \to PGL(2,\mathbb{C})$ has image $\cong K_4$

(2) S_i coming from f must be diagonalizable, hence simult diag since $S_1 S_2 = S_2 S_1$

But then they can not gen type I relation ※

Rmk. In repr th, a basic question is if $M := \operatorname{im} \rho \subset$ Unitary gp for MFE, we ask similar question for PM in $PGL(2,\mathbb{C})$:

Type I $\leftrightarrow$ K_4, Type II $\leftrightarrow$ $PM \subset U(1) \cong S^1$, abelian

Rmk Log-free GLE with ℓ even may give no sol to MFE at all, if both S_i are not diagonalizable

no local criterion!

~~log-free constraint via alg geom.~~

Poly equations for $F_i(A_1, \ , A_r, B)$ $\quad (F_0 = \sum_{i=1}^r A_i = 0)$

$$(*)_1 \quad S(f) = v'' - \frac{1}{2}(v')^2 = -2\Big(\sum_{i=1}^r \eta_i(\eta_i+1)\wp(z-p_i) + A_i\,\zeta(z-p_i) + B\Big)$$

~~local exp at p_i~~ $\quad 1 \le i \le r$

$$f(z) = c_{i,0} + c_{i,l_i+1}(z-p_i)^{l_i+1} + $$

$$v(z) = \log f' = \log(l_i+1)c_{i,l_i+1} + l_i\log(z-p_i) + \sum_{k\ge1} d_{i,k}(z-p_i)^k$$

$$v'(z) = \frac{l_i}{z-p_i} + \sum_{k\ge 0} e_{i,k}(z-p_i)^k$$

comparing coefficients in $(*)_1$ $\qquad l_i = 2\eta_i$

$(z-p_i)^{-2}$ $\quad -l_i - \frac{1}{2}l_i^2 = -2\eta_i(\eta_i+1)$ $\quad$ automatic

$(z-p_i)^{-1}$: $\quad l_i e_{i,0} = 2A_i \Rightarrow e_{i,0} = \dfrac{2A_i}{l_i}$ $\qquad \wp(p_i-p_j)$ etc

$(z-p_i)^0$: $\quad e_{i,1} - \frac{1}{2}\,2\cdot l_i e_{i,1} - \frac{1}{2}e_{i,0}^2 = -2\sum_{j\ne i}\big(\eta_j(\eta_j+1)\overset{\shortparallel}{\wp_{ij}} + A_j\zeta_{ij}\big)$

$\qquad\qquad \overset{\shortparallel}{(1-l_i)e_{i,1}}$ $\qquad\qquad -2B$

If $l_i = 1$, get quadratic equation

$\boxed{QE}$ $\quad A_i^2 = B + \sum_{j\ne i}\zeta_{ij}A_j + \frac{3}{4}\wp_{ij}$

otherwise, $e_{i,1}$ is determined, and continue.

$(z-p_i)^k$, $k\ge1$: $\quad \tilde e_{i,s} := \frac{1}{2}e_{i,s}$, $\hat A_i := \frac{A_i}{l_i}$, $(\tilde e_{i,0} = \hat A_i)$

$$(l_i - (k+1))\,\tilde e_{i,k+1} = -\sum_{t=0}^k \tilde e_{i,t}\,\tilde e_{i,k-t} + L_k$$

$L_k = \sum_{j\ne i}\big(\eta_j(\eta_j+1)\wp_{ij}^{(k)} + A_j\zeta_{ij}^{(k)}\big)$ $\quad$ modular function of

wt. $= k+2$ $\quad$ (~~wt. $A_j, \zeta = 1$~~, ~~$B, \wp = 2$~~)

The critical case $k = l_i - 1$

leads to poly $F_i(A_1,\cdots,A_r,B)$ of $\deg F_i = \overset{k+2}{\overset{\shortparallel}{l_i+1}}$

~~Cor.~~ The case $r=1$, $F_0 = A_1 = 0$, $\quad l = l_1 = 2n+1$ odd

$F_1(B)$ recovers the Brioschi-Halphen poly $P_n(B)$

of deg $(l+1)/2 = n+1$ in B

ie Number of sol of

$$\Delta u + e^u = 4\pi(2n+1)\delta_p \quad \text{is } n+1 \text{ (count w. mult.)}$$

Key observation for general case

The top degree terms of F_i involves only A_i & B

$g_{\ell_i}(A_i, B)$ has wt $\ell_i + 1$ For ℓ_i small :

$$g_0 = A$$

$$g_1 = -(A^2 - B)$$

$$g_2 = \frac{1}{(2!)^2} A (A^2 - 2^2 B)$$

$$g_3 = \frac{-1}{(3!)^2}(A^2 - B)(A^2 - 3^2 B) \quad \cdots$$

Lemma $$g_{\ell_i} = \frac{(-1)^{\ell_i}}{(\ell_i!)^2} \prod_{j=0}^{\ell_i} \left(A_i - (\ell_i - 2j) B^{1/2} \right)$$

Note : This seems to be related to sl_2-repr but my pf uses Riccati & Confluent hyp geom eq'ns

Main Theorem I (Algebraic counting formula)

For $\ell = \sum_{i=1}^{r} \ell_i$ being odd, the log-free parameters $(\{A_i\}, B)$ is a discrete set with cardinality, counted with multi, equals $\frac{1}{2}$ Bezout number :

$$a_{\vec{\ell}} = \frac{1}{2} \prod_{i=1}^{r} (\ell_i + 1)$$

This gives rise to all (type I) sol to MFE

($a_{\vec{\ell}}$ is also the Leray-Schauder top. degree)

Proof For the projective completion in

$$\mathbb{P}^{r+1} \ni [A_0 : A_1 : \cdots : A_r : B],$$

the additional solutions at ∞ has $A_0 = 0$,

Lemma $\Rightarrow$ $A_i = 0 \ \forall i$ with B free

~~Claim:~~ get isolated pt $Q = [0 : 0 : \cdots : 0 : 1]$

Computing multiplicity at Q and pf of isolation

$\ell = \sum \ell_i$ odd, may assume $\ell_r = 2n_r + 1$

On the chart $B \neq 0$, let $x_i = \frac{A_i}{B}$, $i = 0, \cdots, r$

The ~~dominant polynomials at Q~~ are

$$\tilde{f}_0(x) = \sum_{i=1}^{r} x_i$$

$$\tilde{f}_i(x) = \prod_{j=0}^{\ell_i} (x_i - (\ell_i - 2j)\, x_0^{1/2}), \quad 1 \le i \le r$$

$\Rightarrow x_i \sim \mu\, x_0^{1/2}$ with $\mu_i \equiv \ell_i \mod 2$

$$\sum_{i=1}^{r} x_i \sim \sum_i \mu_i\, x_0^{1/2} \equiv \sum \ell_i\, x_0^{1/2} = \ell\, x_0^{1/2} \neq 0$$

since ℓ is odd (unless $x_0 = 0$)

Moreover, $\tilde{f}_r = 0 \Rightarrow x_r^2 = \mu^2 x_0$ for $n_r + 1 = \frac{1}{2}(\ell_r + 1)$ choices of odd μ

For each μ, get $\prod_{j_i=0}^{\ell_i} \left(x_i - \frac{\ell_i - 2j_i}{\mu} x_r\right) =: \prod L_{i,j_i}(x_i, x_r)$ for $i = 1, \cdots, r-1$

$\Rightarrow$ mult. $Q = \frac{1}{2}(\ell_r + 1) \prod_{i=1}^{r-1} (\ell_i + 1)$ *

Main Theorem II (PDE/Poly non-degeneracy correspondence)

For primitive MFE (i.e. $\Delta u + e^u = 4\pi \sum \delta_{p_i}$)

$\ell_i \equiv 1$ (and $\ell = r = 2n+1$, or $2n$ both OK)

A sol u is non-deg "$\Delta\varphi + e^u \varphi = 0$ in $T \Rightarrow \varphi = 0$"

iff on the corr $(\{A_i\}, B)$, the linearized eq'n

$$\boxed{QE'}\qquad \begin{aligned} &\sum_{i=1}^{r} A_i' = 0 \\ &2A_i A_i' = \sum_{j \neq i} \zeta_{ij} A_j' + B', \quad i = 1, \cdots, r \end{aligned}$$

has only trivial sol $(\{A_i'\}, B') \equiv 0$

~~Rmk~~ For ℓ even, the existence (and counting) of type II sol's is hard even for $\Delta u + e^u = 8\pi\delta_0$

It depends on ~~moduli pt~~ $\tau = \omega_2/\omega_1$ (Lin-W, Annals

~~End~~