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K-equivalence in Birational Geometry

2002 2/18 at Tokyo (Iitaka's 60's)

Definition: Two normal $\mathbb{Q}$-Gorenstein varieties X, X' are K-equivalent, denoted by $X =_K X'$, if $\exists$ diagram

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

with Y smooth, φ, φ' proper birational

s.t. $\varphi^* K_X =_{\mathbb{Q}} \varphi'^* K_{X'}$.

Example: let X, X' be birational $\mathbb{Q}$-Gorenstein varieties with at most canonical singularities with $X \setminus Z \cong X' \setminus Z'$, K_X nef on Z, $K_{X'}$ nef on Z', then $X =_K X'$.
(This follows from the Hodge index theorem)

Main Problem: Study relation between X and X' when $X =_K X'$.

Theorem (Kawamata, Kollár)
In dimension three, X, X' birational $\mathbb{Q}$-factorial ter. minimal ~~(with at most can. sing.)~~ $\Rightarrow$ $X \dashrightarrow X'$ can be decomposed into sequences of ~~(canonical)~~ flops.

- ~~Moreover~~, terminal flops preserving germs of singularities, cohomologies (groups) and intersection cohomology groups. Moreover, flops can be performed in families.

Remark: The same proofs works for 3-dim'l $\mathbb{Q}$-factorial, terminal X, X' with $X =_K X'$.

Main Difficulity: Do not have classification for ter. sing. when dim ≥ 4.

Integration Formalism:

Basic idea: $X =_K X' \ni$ $K_Y = \varphi^* K_X + E$
$= \varphi'^* K_X + E'$

so $E = E'$. ie. φ, φ' have the same holomorphic Jacobian factor

if $\exists$ integration theory st. its change of variable formula respect E:

$$\int_X d\mu_X = \int_Y J(E)\, d\mu_Y = \int_{X'} d\mu_{X'}$$

then the invariant "$\int_X d\mu_X$" is preserved under K-equiv.

1st example: old result of Batyrev, Wang for betti.

Theorem: X, X' sm projective and $X =_K X'$ implies that $h^{p,q}(X) = h^{p,q}(X')$.

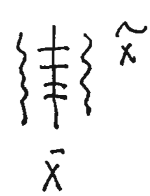
pf: (sketch) take integral model over R: f.g. alg $/\mathbb{Z}$.
$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd} \quad (*)$$

$\forall'$ maximal ideal $p \triangleleft R$, $(*)$ has good reduction mod p. let
$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \tilde{X} & & \tilde{X}' \end{tikzcd}$$
be the corresponding completion at p

$S := \widehat{R_p}$.
$R/p \cong \mathbb{F}_q$. $q = p^r$.
then p-adic integral $\int_{\tilde{X}(S)} d\mu_X = \dfrac{|\bar{X}(\mathbb{F}_q)|}{q^n}$

when $\tilde{X}$ smooth over S.

$n = \dim X$.



by extention of scalar, we get $\forall' p$

$$Z_p(\overset{\bullet}{X}, t) = Z_p(\overset{\bullet}{X}', t) \qquad \text{zeta functions}$$

($\Rightarrow b_i(X) = b_i(X')$ by Weil conj)

To proceed, for simplicity, assume $X \xleftarrow{\varphi} Y \xrightarrow{\varphi'} X'$ defined over $\mathbb{Q}$ (or number field K)

look at ℓ-adic rational Galois repr: $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $H^i_{et}(X, \mathbb{Q}_\ell)$, $H^i_{et}(X', \mathbb{Q}_\ell)$ W_0, W' (fix a prime ℓ)

then $\forall' p \neq \ell$, get from zeta: st $*$ has good reduction

$$H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell)^{ss} \underset{/\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}{\simeq} H^i_{et}(\bar{X}'_p, \mathbb{Q}_\ell)^{ss}$$

by density thm $\Rightarrow$ $H^i_{et}(X, \mathbb{Q}_\ell)^{ss} \simeq H^i_{et}(X', \mathbb{Q}_\ell)^{ss}$ over $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

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now view X, X' as varieties over $\mathbb{Q}_\ell$

Using Faltings' p-adic Hodge theory: (switch ℓ to p)

$$\bigoplus \left(\hat{\bar{\mathbb{Q}}}_p \otimes_{\mathbb{Q}_p} H^{m-i}(X, \Omega^i)(-i)\right) \underset{/\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}{\simeq} \hat{\bar{\mathbb{Q}}}_p \otimes_{\mathbb{Q}_p} H^m_{et}(X, \mathbb{Q}_p)$$

Apply s.s. shows that

$h^{p,q}$ are det. by $H^m_{et}(X, \mathbb{Q}_p)^{s.s.} \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Remark: This can also be proved by using motivic integration (Denef-Loeser, Kontsevich, Batyrev).

2nd example: chern numbers (or complex genera)

Simple Q to think: how to show $\chi(X) = \chi(X')$ directly?

eg.
$$\int_X c_n(X) = \int_Y [\,?\,]\, c_n(Y)$$

Def: $\mathbb{P}^r$-flops: $f : X \dashrightarrow X'$ st the exceptional set Z:

$$\begin{array}{l} X \supset Z \leftarrow F \cong \mathbb{P}^r \\ \quad\ \downarrow \psi \\ \quad\ S \end{array}$$

has a $\mathbb{P}^r$-bundle str. over sm S and $N_{Z/X}|_F \cong \mathcal{O}_F(-1)^{r+1}$

the map f is given by blowing up X along Z to get Y with exl. div E a $\mathbb{P}^r \times \mathbb{P}^r$ bundle over S, then blowing down E along another fiber direction.

. classical flops := $\mathbb{P}^1$-flops.

Theorem (Totaro, Ann of Math 2000 ?):
The most general chern numbers inv under $\mathbb{P}^1$-flops consists of the complex elliptic genera.

Rmk: Hirzebruch's pf using Atiyah-Bott localization thm for $Q(x) = x f(x)$ (ie. $F(x) = 1/f(x)$) then

inv. $\Leftrightarrow$ $F(x+y)\big(F(x)F(-x) - F(y)F(-y)\big)$
$= F'(x)F(y) - F'(y)F(x)$

then solve this into cpx elliptic genera.

pf of Thm 1 & 2: Totaro $\Rightarrow$

$\varphi_{cpx\text{-}ell}$ = " $\Omega^U \twoheadrightarrow \Omega^U / I_F = R$ "

we proved that $\varphi_{cpx\text{-}ell}$ is inv under K-equiv, hence $\Omega^U / I_K \longrightarrow \Omega^U / I_F$

but $I_K \supset I_F$ hence must be $\cong$ and $I_K = I_F$.