Calabi–Yau Without Maximal Degenerations

Chin-Lung Wang
National Taiwan University

40 Years of Calabi–Yau Theory
Jioling, Meizhou
December 23, 2018

Contents

Sheng–Xu–Zuo's Example

- Let $a = \{H_1, \ldots, H_6\} \in \mathfrak{M}$ be a hyperplane arrangement in P^3 and $D_a := \bigcup_{i=1}^{6} H_i$.
- Let $Z_a \to P^3$ the $3:1$ cyclic cover branched along D_a, it has singularities of binomial type

$$y_1^{a\alpha_1} \cdots y_p^{\alpha_p} = x_1 \cdots x_q.$$

- In [SXZ, 2013] they constructed a crepant resolution

$$\begin{array}{ccc} Y_a & \xrightarrow{\phi_a} & Z_a \\ & & \Big\downarrow {\scriptstyle \pi}\ {\scriptstyle 3:1} \\ & & P^3 \end{array}$$

- Y_a is a Calabi–Yau 3-fold with $h^{21}(Y_a) = 3$, $h^{11}(Y_a) = 51$.

- Let $f : \mathscr{Y} := \bigcup_{a\in\mathfrak{M}} Y_a \to \mathfrak{M}$ be the CY family with

$$F^p = f_*\Omega^p_{\mathscr{Y}/\mathfrak{M}}, \qquad \mathcal{H}^{pq} := F^p \cap \overline{F^q}.$$

- Denote the infinitesimal period map by

$$\sigma : T\mathfrak{M} \longrightarrow \mathrm{Hom}(\mathcal{H}^{30}, \mathcal{H}^{21}) \oplus \mathrm{Hom}(\mathcal{H}^{21}, \mathcal{H}^{12}) \oplus \mathrm{Hom}(\mathcal{H}^{12}, \mathcal{H}^{03}).$$

- **Theorem (SXZ, 2013)**
$f : \mathscr{Y} \to \mathfrak{M}$ is a maximal family (moduli) of CY 3-folds such that

$$\sigma_i \circ \sigma_j = 0 \qquad \forall i, j$$

where $\sigma_i = \sigma(\partial_{t^i})$, i.e. f has Yukawa coupling length $\ell = 1$.

Curvature of the Weil–Petersson metric

- Let $\mathcal{H} \to S$ be a polarized VHS of weight n with $h^{n,0} = 1$. Let $g_{WP} = \sum g_{i\bar{j}}\, dt_i \otimes d\bar{t}_j$ with $\omega_{WP} = c_1(F^n, Q)$ on S.

- Theorem (W-1997, Schumacher 1993)

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}) + \frac{\langle \sigma_i \sigma_k\, \Omega, \sigma_j \sigma_\ell\, \Omega \rangle}{\langle \Omega, \Omega \rangle}.$$

- For $n = 3$, it is equivalent to Strominger's formula

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}}) + \sum_{p,q} g^{p\bar{q}}\, F_{pik} \overline{F_{qj\ell}},$$

where $F_{ijk} = \frac{\int_X \partial_i \partial_j \partial_k\, \Omega \wedge \Omega}{\int_X \Omega \wedge \bar{\Omega}}$ is the Bryant–Griffiths–Yukawa cubic form .

Definition

The length of Yukawa coupling $\ell(\rho)$ for a VHS $\rho : \mathcal{H} \to S$ is the largest integer ℓ with $\sigma_{i_1} \cdots \sigma_{i_\ell} \not\equiv 0$ for some $i_1, \ldots, i_\ell$.

- *Mirror Symmetry* requires the existence of *maximal degenerate point*, and it implies that $\ell(\rho) = n$.
- For maximal CY families $f : \mathscr{Y} \to \mathfrak{M}$ with $\ell(\rho_f) = 1$. Then

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}).$$

  That is, locally complex hyperbolic: $\mathfrak{M} \cong B^n_{\mathbb{C}} / \Gamma$.
- Does MS extends over families with $\ell(\rho_f) < n$?
- Say for the SXZ example?
- They actually constructed such examples for all odd $n \geq 3$.

Extending toric MS via confifold transitions

- *Topological MS*: (Y, Y°) is a mirror pair of CY 3-folds if

$$h^{21}(Y) = h^{11}(Y^\circ), \qquad h^{11}(Y) = h^{21}(Y^\circ).$$

- *Classical MS*, or $A \leftrightarrow B$ MS:

$$B(Y) \cong A(Y^\circ), \qquad A(Y) \cong B(Y^\circ).$$

- $A(Y) = QH(Y)$ is the genus zero Gromov–Witten theory on the complexified Kähler moduli

$$\mathscr{K}_Y^{\mathbb{C}} = H^2(Y, \mathbb{R}) \oplus \sqrt{-1}\mathrm{Amp}(Y).$$

- $B(Y) = (\mathscr{H}, \nabla^{GM})$ is the VHS on the complex moduli $\mathscr{M}$.
- The modern SYZ/HMS are not discussed here.

Toric MS

- A lattice polytope $\triangle \subset M_{\mathbb{R}}$, $M \cong \mathbb{Z}^{n+1}$ is reflexive if $0 \in \operatorname{int} \triangle$ and its polar (dual) polytope

$$\triangle^\circ := \{w \in N := M^\vee \mid \langle w, v \rangle \geq -1, \forall v \in \triangle\}$$

is also a lattice polytope, in $N_{\mathbb{R}}$.
- Number of them $N_2 = 16$, $N_3 = 4319$, $N_4 = 473800776, \ldots$ [Kruezer–Skarke, 2000].
- For reflexive pair $(\triangle, \triangle^\circ)$, the toric variety

$$P_\triangle := \operatorname{Proj}\left(\bigoplus_{k\geq 0} \mathbb{C}^{k\triangle \cap M}\right)$$

is Fano with $H^0(K_{P_\triangle}^{-1}) = \bigoplus_{v \in \triangle \cap M} \mathbb{C}\, t^v$; similarly for $P_{\triangle^\circ}$.
- For a general section f, $X_f := \{f = 0\}$ is a CY n-fold.

- Consider $n = 3$ and families $X_f \subset P_\triangle$, $X_g^\circ \subset P_{\triangle^\circ}$.
- Topological MS holds [Batyrev '94].
- $A \leftrightarrow B$ MS holds for "many cases".
- $A(X_f)$ is *determined* by localization data [LLY, G 1999]

$$I_\beta = \frac{\prod_{m=1}^{K^{-1}.\beta}(K_{P_\triangle}^{-1} + mz)}{\prod_{\rho\in\Sigma_1}\prod_{m=1}^{D_\rho.\beta}(D_\rho + mz)}, \qquad \beta \in H_2(X_f, \mathbb{Z}),$$

  where Σ is the (normal) fan of $P_\triangle$.
- $B(X_f)$ is *determined* by the GKZ* system: (1) symmetry operators, (2) for ℓ a relation of $m_i \in \triangle \cap M$ with $\sum \ell_i = 0$,

$$\Box_\ell := \prod_{\ell_i>0} \partial_i^{\ell_i} - \prod_{\ell_i<0} \partial_i^{-\ell_i}.$$

- Observation: Σ_1 = rays from 0 to Vert$(\triangle^\circ)$.
- Existence of max-deg-point [HLY] ($\Rightarrow$ mirror transform).

Conifold transition

- Geometric transition $X \nearrow Y$ (or $Y \searrow X$) of CY 3-folds

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \phi} \\ X & \rightsquigarrow & Z \end{array}$$

  is a *finite WP distance degeneration* from X to Z (with canonical singularities) followed by a *crepant resolution* Y.
- It is a conifold transition when Z_{sing} has only k nodes $p_i : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$. Then $S_i^3 \rightsquigarrow p_i \leftarrow S_i^2$ and

$$\mu + \rho = k.$$

- $\mu = h^{21}(X) - h^{21}(Y) =$ dimension of vanishing cycles in X.
- $\rho = h^{11}(Y) - h^{11}(X) =$ rank of ϕ-exc $(-1, -1)$ curves in Y.

- Assume MS on (X, X°) with $X \nearrow Y$; form $X^\circ \searrow Y^\circ$:

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ \color{red}{X} & \rightsquigarrow & Z \end{array} \qquad\qquad \begin{array}{ccc} & & Y^\circ \\ & & \wr\!\downarrow \\ \color{red}{X^\circ} & \xrightarrow{\psi} & Z^\circ \end{array}$$

- If Z° also has k nodes $\Rightarrow$ topological MS on (Y, Y°).
- **Propositions:*** True for toric mirror candidates (X, X°).
- For classical $A \leftrightarrow B$ MS, needs to lift/categorify $\mu + \rho = k$ to a version on flat connections.
- *Local exchange* on quantum data based on

$$0 \to A(Y)/A(X) \to \underline{\mathbb{C}}^k \to B(X)/B(Y) \to 0$$

(basic exact sequence of weight 2 Hodge structures) holds near $[Z] \in \mathscr{M}_Z$ [LLW 2015].

Connecting SXZ to toric web

- Work in progress with Tsung-Ju Lee.
- Let $s \in H^0(P^3, \mathscr{O}(6))$, $D_s = \{s = 0\}$ smooth for general s:

$$\begin{array}{ccc}
 & & Y_a = Y \\
 & & \downarrow \phi \\
X := Z_s & \overset{s \to \prod \ell_i}{\rightsquigarrow} & Z_a = Z \\
\downarrow 3{:}1 & & \downarrow 3{:}1 \\
P^3 \supset D_s & \rightsquigarrow & P^3 \supset D_a = \bigcup_{i=1}^{6} H_i
\end{array}$$

- This leads to a geometric transition $Y \searrow X$ with

$$h^{21}(X) = 103, \qquad h^{11}(X) = 1.$$

- Z has singularities far more complicate than nodes.

- Q: Is X an anti-canonical hypersurface in toric Fano?
- In general an r-cyclic cover is realized in a line bundle L:

$$\begin{array}{c} \xi = E_\infty \subset \widetilde{P} := P(L \oplus \mathscr{O}) \\ \uparrow \\ L \supset X \\ \Big\downarrow {\scriptstyle r:1} \\ h \subset P^n \supset D_s \end{array}$$

- For D_s connected and reduced (1) X is Cohen–Macaulay, (2) if codim $X_{\text{sing}} \geq 2$ then X is CY $\Longleftrightarrow$

$$L^{\otimes(r-1)} \cong K_{P^n}^{-1}.$$

- Now $r = 3, n = 3, K_{P^3}^{-1} \cong \mathscr{O}(4) \Longrightarrow L \cong \mathscr{O}(2)$.

- $\widetilde{P} = P_{P^3}(\mathscr{O}(2) \oplus \mathscr{O})$ is toric, $\operatorname{Pic} \widetilde{P} = \mathbb{Z}h \oplus \mathbb{Z}\xi$,

$$K_{\widetilde{P}}^{-1} = \pi^* L^{\otimes r} \otimes \mathscr{O}_{\widetilde{P}}(2) = 6h + 2\xi$$

  is ample. Hence $\tilde{P}$ is toric Fano.
- But $X \in |\, 6h + 3\xi \,|$ (locally $y^3 = f(x)$) rather than $K_{\widetilde{P}}^{-1}$. $6h + 3\xi$ is base-point-free but not ample: $X \cap E_\infty = \emptyset$.
- May contract $E_\infty \subset \widetilde{P}$ to a point to get $p \in P$:

$$\begin{array}{ccc} X & \hookrightarrow & \widetilde{P} \supset E_\infty \\ \downarrow = & & \downarrow \varphi \\ X & \hookrightarrow & P \ni p \end{array} \qquad K_{\widetilde{P}} = \varphi^* K_P + E_\infty.$$

- $P = P(1,1,1,1,2)$ singular toric Fano with $X \in |K_P^{-1}|$. (P is the one point compactification of L.)
- $\Rightarrow X^\circ$ exists. Q: Can we construct Y° using $X \nearrow Y$?

A test of Reid's fantasy with quantum data

- Q′: Can we *decompose* $Y \searrow X$ into conifold transitions?
- [Namikawa 2002] Let $S \to P^1$ be a rational elliptic surface with 6 singular fibers of type II (i.e., cuspidal). Then
- $Z = S \times_{P^1} S$ is a CY 3-fold with 6 cA_2 singular points:

$$x^2 - y^3 = u^2 - v^3.$$

- Z admits smoothings to $X = S_1 \times_{P^1} S_2$ with $S_i \to P^1$ having disjoint discr. loci, and a small resolution $\phi : Y \to Z$ exists.
- The ϕ-exc loci *can not be deformed* to a disjoint union of $(-1, -1)$-curves: type II fiber splits to ≤ 2 type I, but a general fiber of ϕ-deformation of $p \in Z_{sing}$ has 3 nodes.
- [S.-S. Wang] This $Y \searrow X$ can be factorized into 2 conifold transitions *up to flat deformations*.

- [Reid 1987] Can all CY 3-folds be connected through (possibly non-projective) conifold transitions?
- If this holds for the SXZ example

  $$Y \searrow X \sim (Y = Y_0) \searrow_{Z_1} Y_1 \searrow_{Z_2} Y_2 \ldots \searrow_{Z_n} (Y_n = X),$$

  then we *may construct* the chain of mirror transitions

  $$X^\circ \searrow Y^\circ \sim (X^\circ = Y_n^\circ) \searrow_{Z_n^\circ} Y_{n-1}^\circ \searrow_{Z_{n-1}^\circ} Y_{n-2}^\circ \ldots \searrow_{Z_1^\circ} Y^\circ.$$

- By *choosing the birational contraction $Y_i^\circ \to Z_i^\circ$* so that the number of nodes of Z_n° coincides with those of Z_n.
- And then the mirror candidate Y° exists.
- And hence the topological MS holds for (Y, Y°).

- For classical MS, the easy direction $A(Y^\circ) \hookrightarrow A(X^\circ)$ holds:

$$(*) \qquad \langle - \rangle_\beta^{Y^\circ} = \sum_{\gamma \mapsto \beta} \langle - \rangle_\gamma^{X^\circ}.$$

- X° is an anti-canonical toric CY 3-fold.
- Explicit calculations $\Longrightarrow^*$ $\langle - \rangle_\beta^{Y^\circ} \neq 0$ for some β. (without actually using the decomposition.)
- $\Longrightarrow A \leftrightarrow B$ MS fails on (Y, Y°)!

- **Conclusion:** Either

(i) Reid's fantasy fails for $Y_{SXZ} \searrow X_{LW}$ or

(ii) the classical MS fails for (Y, Y°).

Remarks

- It might be possible to prove $(*)$ for any geometric transition directly, without the existence of factorization into conifold transitions.
- If so, then the classical MS needs to be corrected when there is no maximally degenerate boundary points.
- The correction might come from the fact that the local transition of excess A and excess B theory can not be continued far away from the transition point $[Z] \in \mathscr{M}_{[Z]}$.
- There might be wall crossing when the vanishing cycles (Special Lagrangian) intersect.

In Celebration of 40 Years of CY Theory

And to Express my Gratitude to Porfessor Yau

Thank you