Ext of multiply-twisted pluri-can. forms

1/06, 2011 at IMS, Singapore. by C.L. WANG.

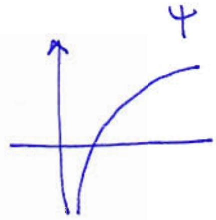
cf. Varolin, Demailly-Hacon-Paun

① Type OT ext. (Siu's version, ext by CWW to log pair (X, D)):

$X \supset D$ sm. $h_D = e^{-\varphi}$ metric on $\mathcal{O}_X(D)$ st. $|s_D| e^{-\varphi} < c_0$

(D, L) OT pair adm. twist.
- $(L, h) \to X$ psef (i.e. $h = e^{-\psi}$, $\Theta_h = \partial\bar\partial\psi \geq 0$ psh.)
- $h|_D$ well-defi. ($\psi|_D \not\equiv -\infty$)
- $\psi + \varphi < c_1$ (φ if φ sm or psh)
- $\mu \Theta_h \geq \Theta_{h_D}$ for some $\mu > 0$

• sheaf of germs of L^2 hol. fcns.



Then every $s \in \Gamma(D, (K_D + L|_D) \otimes \mathcal{J}(h|_D))$

extends to $\tilde{s} \in \Gamma(X, K_X + D + L)$; ie. $\tilde{s}|_D = s \wedge ds_D$ polar set

and $\int_X \langle \tilde{s} \rangle^2_{h_D \otimes h} \leq c \int_D \langle s \rangle^2_{h|_D}$; $c = c(c_0, \mu)$.

② Main Thm (CWW, 2010)

Let $(L_j, h_j) \to X$ be OT pair, $j = 1, \dots, m$

Then every $\sigma \in \Gamma(D, \bigotimes_{j=1}^m (K_D + L_j|_D) \otimes \mathcal{J}_1 \cdots \mathcal{J}_m)$, $\mathcal{J}_j = \mathcal{J}(h_j|_D)$

extends to $\tilde\sigma \in \Gamma(X, \bigotimes_{j=1}^m (K_X + D + L_j))$; ie. $\tilde\sigma|_D = \sigma \wedge (ds_D)^{\otimes m}$.

$\quad\quad \| \; m(K_X + D) + L \quad (L = L_1 + \cdots + L_m)$

Moreover, (L_j, h_j) is allowed to be psef $\mathbb{R}$-divisor.

Goal: Assuming ①, prove integral case of ②: no metric needed

Def'n: Pseudonorms: s measurable section of $mK_X + L \leftarrow h$

$s = f\, dz^1 \wedge \cdots \wedge dz^n$

$\langle s \rangle^{2/m}_h := h(z)^{1/m} |f(z)|^{2/m} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$ measurable (n,n) form

$\langle\langle s \rangle\rangle_h := \int_X \langle s \rangle^{2/m}_h \leq \infty$

Properties i) $s' \in L_{h'}(X, L') \Rightarrow \langle s \otimes s' \rangle^{2/m}_{h \otimes h'} = |s'|^{2/m}_{h'} \langle s \rangle^{2/m}_h$.

ii) $\langle s^\ell \rangle^{2/\ell m}_{h^{\otimes \ell}} = \langle s \rangle^{2/m}_h$.

iii) Hölder: $s_j \in L_{h_j}(X, m_j K + L_j) \Rightarrow$

$\langle\langle s_1 \otimes \cdots \otimes s_r \rangle\rangle^{m_1 + \cdots + m_r}_{h_1 \otimes \cdots \otimes h_r} \leq \langle\langle s_1 \rangle\rangle^{m_1}_{h_1} \cdots \langle\langle s_r \rangle\rangle^{m_r}_{h_r}$.

Pf: Step 1: Reduction to constructing a s.p. metric h_∞ on $m(K_X+D)+\sum_1^m L_j$ (i.e. it is psef), with $|\sigma\wedge ds_D^{\otimes m}|_{h_\infty}\leq$

$$m(K_X+D)+\sum_1^m L_j = K_X+D+\Big(\underbrace{(m-1)(K_X+D)+\sum_1^m L_j}_{\text{call it } L'}\Big)$$

to apply ① to get ②, need s.p. h_0 on L'.

choose A so ample s.t.

(A1) $\forall r=0,1,\cdots,m-1$; $(m-r)A$ is g.b.g.s. $\{t_\ell^{(r)}\}_{\ell=1}^N$.

(A2) $(K_D+L_j|_D+A|_D)\otimes\mathcal{I}_j$ on D is g.b.g.s. $\{s_{j,\ell}\}_{\ell=1}^N$, $\forall j=1,\cdots,m$.

(A3) $\bigotimes_{j=1}^m H^0(D,(K_D+L_j|_D+A|_D)\otimes\mathcal{I}_j)\twoheadrightarrow H^0(D,\sum_{j=1}^m(K_D+L_j|_D+A|_D)\otimes\mathcal{I}_1\cdots\mathcal{I}_m)$.

(A4) All $H^0(D,[m(K_X+D)+\sum L_j+mA]_D)$ extends to X.

Simply take $\quad h_0 = h_\infty^{\frac{m-1}{m}}(h_1\cdots h_m)^{\frac{1}{m}}$

- $h_0|_D$ well-defined.
- $h_0 h_D$ has local wt upper bounded: $h_0 h_D = h_\infty^{\frac{m-1}{m}}\prod_{j=1}^m(h_j h_D)^{\frac{1}{m}}$ (s.p. (psh))
- $M_0\Theta_{h_0}\geq\Theta_{h_D}$ for $M_0=m\mu>0$.

Now $\sigma\in\Gamma(D,\bigotimes_1^m(K_D+L_j|_D)\otimes\mathcal{I}_1\cdots\mathcal{I}_m)$

$$\parallel$$
$$K_D+L'|_D$$

check Finiteness:

$$\langle\sigma\rangle_{h_0}^2=\langle\sigma\rangle_{h_\infty}^{2\frac{m-1}{m}}\langle\sigma\rangle_{h_1\otimes\cdots\otimes h_m}^{\frac{2}{m}}\leq\langle\sigma\rangle_{h_1\otimes\cdots\otimes h_m}^{\frac{2}{m}}$$

means $\sigma\wedge ds_D^{\otimes(m-1)}$ $\quad\sigma\wedge ds_D^{\otimes m}$

$$(A3)\Rightarrow\quad \underbrace{\sigma\otimes t_\ell^{(0)}}_{mA}=\sum_{p=1}^{n_\ell}\tau_\ell^{1p}\otimes\cdots\otimes\tau_\ell^{mp}$$

$\tau_\ell^{jp}\in\Gamma(D,(K_D+L_j|_D+A|_D)\otimes\mathcal{I}_j)$

$$\Big(\sum_{\ell=1}^N|t_\ell^{(0)}|_{h_A^m}^{2/m}\Big)\langle\sigma\rangle_{h_1\cdots h_m}^{2/m}\leq\sum_{\ell=1}^N\sum_{p=1}^{n_\ell}\langle\tau_\ell^{1p}\otimes\cdots\otimes\tau_\ell^{mp}\rangle_{h_1\cdots h_m h_A^m}^{2/m}$$

has lower bound $M_0>0$

$$\Rightarrow\langle\langle\sigma\rangle\rangle_{h_0}\leq\frac{1}{M_0}\sum_{\ell=1}^N\sum_{p=1}^{n_\ell}\langle\langle\tau_\ell^{1p}\rangle\rangle_{h_1h_A}^{1/m}\cdots\langle\langle\tau_\ell^{mp}\rangle\rangle_{h_m h_A}^{1/m}<\infty.$$

General remarks on s.p. metrics:

- Algebraic: $S=\{s_j\in\Gamma(X,L)\}$, $h_S(\sigma)=\dfrac{|\sigma|_h^2}{\sum_j|s_j|_h^2}$, any sm h.
- Analytic: limit process of psh fcn's, like Perron's method.

Step 2. Constructing h_m (modification of Siu, Paun induction)

idea: let $F_k = k(K_X+D) + L^{(k)} + mA$, increase k, fix mA.

$L^{(k)} = q\sum_{j=1}^{m} L_j + (L_1+\cdots+L_r)$, $k = qm + r$

$$\begin{cases} \Lambda_k := \Lambda_r = \{1,\dots,N\}^r \quad r\neq 0 & J\in\Lambda_r,\ S_J^{(r)} := S_{1,j_1}\otimes\cdots\otimes S_{r,j_r} \\ \Lambda_0 = \{0\},\ S_0^{(0)} = 1, & \\ \Lambda_m^* := \{1,\dots,N\}^m & J\in\Lambda_m^*,\ \hat S_J^{(m)} := S_{1,j_1}\otimes\cdots\otimes S_{m,j_m} \end{cases}$$

$(E)_k$: $\sigma_{J,\ell}^{(k)} := \sigma^{\lfloor k/m\rfloor}\otimes S_J^{(r)}\otimes t_\ell^{(r)}$ on D, extends to $\tilde\sigma^{(k)}_{J,\ell}$ on X. $S_k := \{\tilde\sigma^{(k)}_{J,\ell} \mid J\in\Lambda_r,\ \ell=1,\dots,N\}$

claim: (E_k) holds $\forall k\geq m$, with $\langle\langle \tilde\sigma^{(k)}_{J,\ell}\rangle\rangle_{h_D\cdot h_{S_{k-1}}\cdot h_{r^*}} \le C_0$ $\forall k>m$.

where $r^*=r$ if $r\neq 0$, $r^*=m$ if $r=0$.

pf: $k=m$ holds by (A_4). let $k>m$, assume $(E)_{k-1}$.

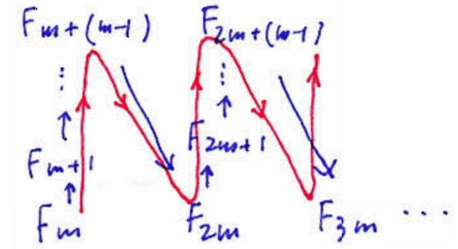
$$F_k = K_X + \underbrace{D}_{h_D} + \underbrace{F_{k-1}}_{h_{S_{k-1}}} + \underbrace{L_{r^*}}_{h_{r^*}}$$

All condi. in OT are easy except $\langle\langle\sigma^{(k)}_{J,\ell}\rangle\rangle_{h_{S_{k-1}}\, h_{r^*}} \le C'$.

case 1: $r\neq 0$.

$$\langle\sigma^{(k)}_{J,\ell}\rangle^2_{h_{S_{k-1}}\cdot h_r} = \frac{\langle \sigma^{\lfloor k/m\rfloor}\otimes S_J^{(r)}\otimes t_\ell^{(r)}\rangle^2_{h\cdot h_r}}{\sum_{J'\in\Lambda_{r-1},\ \ell'=1,\dots,N} \left|\sigma^{\lfloor (k-1)/m\rfloor}\otimes S_{J'}^{(r-1)}\otimes t_{\ell'}^{(r-1)}\right|_h^2}$$

for any smooth metric h.



but $\lfloor \frac{k}{m}\rfloor = \lfloor\frac{k-1}{m}\rfloor$,

write $J=(J_0',j_0)$, $\le \dfrac{|t_\ell^{(r)}|_{h_A^{m-r}}}{\sum_{\ell'=1}^N |t^{(r-1)}_{\ell'}|_{h_A^{m-(r-1)}}}\cdot\langle S_{r,j_0}\rangle^2_{h_r\cdot h_A} \le C_1$.

after $\int_D$, $\forall \ell, r$.

case 2: $r=0$ $\quad\lfloor\frac{k}{m}\rfloor = \lfloor\frac{k-1}{m}\rfloor + 1$. (i.e. $m|k$)

$$\langle\sigma^{(k)}_{0,\ell}\rangle^2_{h_{S_{k-1}}\cdot h_m} = \frac{\left|\sigma^{\lfloor (k-1)/m\rfloor}\otimes\sigma\otimes t_\ell^{(0)}\right|_{h\cdot h_m}}{\sum_{J'\in\Lambda_{m-1},\ \ell'=1,\dots,N}\left|\sigma^{\lfloor\frac{k-1}{m}\rfloor}\otimes S_{J'}^{(m-1)}\otimes t_{\ell'}^{(m-1)}\right|_h}$$

for any sm h.

$$= \frac{|\sigma\otimes t_\ell^{(0)}|^2_{h'}}{\sum_{J\in\Lambda_m^*}|\hat S_{J=(J',j)}^{(m)}|^2_{h'}}\cdot\frac{\sum_{j=1}^N |S_{m,j}|^2_{h_m\cdot h_A}}{\sum_{\ell'=1}^N |t_{\ell'}^{(m-1)}|_{h_A}}$$

h' is the same metric on up/down

by $(A_2)+(A_3)$: $\le C_3$, $\forall \ell$. $\qquad \le C_2$ after $\int_D$.

So let $C' = \max(C_1, C_2C_3)$ dep on σ, $t_\ell^{(r)}$, $S_{j,\ell}$ but not $k\geq m$. □

Step 3 (Siu): Get h_∞ by removing A.

Idea: $h_{S_{qm}}^{1/q}$ on $\frac{1}{q} F_{qm} = m(K_X + D) + L^{(m)} + \frac{m}{q} A$; $q \to \infty$?

In practice, let $\{W_\alpha\}_{\alpha \in I}$ be a trivializing cover of X

$\tilde{f}^{(k)}_{\alpha; T, l} \in \mathcal{O}(W_\alpha)$ representing $\tilde{\sigma}^{(k)}_{T, l}$

claim in step 2 $\Longrightarrow$ $\max\limits_{\substack{x \in W_\alpha \\ \alpha \in I}} \frac{1}{q} \log \sum\limits_{l=1}^{N} \left| \tilde{f}^{(qm)}_{\alpha; 0, l}(x) \right|^2 \le C_0'$.

Jensen, sub-meanvalue

To get h_∞ on $m(K_X + D) + L^{(m)}$ with $|\sigma|_{h_\infty} \le 1$:

wt. on W_α: $\tilde{f}^{(\infty)}_\alpha := \lim\limits_{p \to \infty} \Big(\underbrace{\sup_{q \ge p} \frac{1}{q} \log \sum_{l=1}^{N} \left| \tilde{f}^{(qm)}_{\alpha; 0, l} \right|^2}_{\text{decreasing in } p} \Big)^*$

$()^*$ = upper semiconti. regularization.

$\tilde{f}^{(\infty)}_\alpha$ is also pluri-subharmonic, $\le C_0'$.

it satisfies transition fcn for $m(K_X + D) + L^{(m)}$ since $\frac{mA}{q} \to 0$.

From $\tilde{\sigma}^{(qm)}_{0,l}|_D = \sigma^{(qm)}_{0,l} = \sigma^{\otimes q} \otimes t^{(0)}_l$

$\wedge ds_D^{\otimes qm}$

it is also clear that $\xrightarrow{\hspace{2em}} |\sigma \wedge ds_D^{\otimes m}|_{h_\infty} \le 1$. $\square$