2008 4/8 at Nantes, France

2006 12/20 at IMS, CUHK

Invariance of big quantum ring under flops

(joint with Y. P. Lee and H. W. Lin) 2006, arXiv: math/0608370

Motivations:

X, X' sm proj $/\mathbb{C}$, $X =_K X'$ "K-equivalent" if

$\exists$ $X \xleftarrow{\varphi} Y \xrightarrow{\varphi'} X'$ st. $\varphi^* K_X = \varphi'^* K_{X'}$

φ, φ' birat'l morphisms

eg. birat'l C-Y's or more gen'l birat'l min. models.

~~known~~: $h^{p,q}(X) = h^{p,q}(X')$, but not ring str $[P](X) = [P](X')$, $P = \prod P^{e_i}$

Q.1: (Conj) $\exists$ canonical correspondence $\mathcal{F} \in A^n(X \times X')$ st. $\mathcal{F}: H(X) \xrightarrow{\sim} H(X')$? Motives: $h(X) \xrightarrow{\sim} h(X')$?

"canonical" means $(\mathcal{F}\alpha, \mathcal{F}\beta)_{X'} = (\alpha, \beta)_X$ Poincaré pairing.

Q.2: (Conj) Under $\mathcal{F}$, have isom quantum coh rings $QH(X) \xrightarrow{\sim} QH(X')$ in the sense of analytic continuations over the Kähler moduli.

explanations:

$\{T_i\}$ basis of $H(X)$, $t = \sum t_i T_i$, $\omega \in \mathcal{K}_X$ Kähler cone

$$\langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,\beta} = \int_{[M_{g,n}(X,\beta)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n$$

$$F_0^X = \Phi^X(\omega, t) := \sum_{n=0}^{\infty} \sum_{\beta \in NE(X)} \frac{1}{n!} \langle t, \cdots, t \rangle_{0,n,\beta} \, \underbrace{e^{-2\pi(\omega,\beta)}}_{= q^\beta}$$

convergence issue: Novikov ring $\widehat{\mathbb{C}(NE(X))}$.

~~big quantum product~~

$$T_i *_t T_j := \sum_k \Phi_{ij}{}^k(\omega,t) T^k \quad , \{T^k\} \text{ dual basis under } (\, , \,)_X.$$

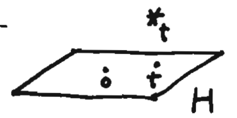
Since $\mathcal{K}_X \cap \mathcal{K}_{X'} = \phi$, $\{\mathcal{K}_{X'} \mid X' =_K X\}$ form a chamber str.

Each X gives a coor system $H(X)$ of a fixed H "motive" "Frobenius manifold"

$\mathcal{F}: H(X) \xrightarrow{\sim} H(X')$ as a (linear) transition fcn

Then $\Phi^X_{ij}{}^k(\omega,t)$ can be analytically conti ~~from~~ $\omega \in \mathcal{K}_X$ to $\mathcal{K}_{X'} \ni \omega$

~~and agrees with $\Phi^{X'}_{ij}{}^k(\omega,t)$~~



A moving prod str on TH.

$$\Phi_{ijk}(\omega,t) = \sum_{n=0}^{\infty} \sum_{\beta \in NE(X)} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle_{0,n+3,\beta}$$

$$\Phi_{ijk}(\omega,t) = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$$

for $t = 0$, $*_0$ = small quantum prod. using only 3 pt fcn's.

Rmk: $\mathcal{F} q^{\beta} = q^{-\beta'}$ if β is a flopped curve "ℓ" but $-\beta' \notin NE(X')$, hence this substitution must be done before we sum up $d\ell$, $d \in \mathbb{N}$ to make Φ analytic in the q^{ℓ} direction.

p. 2

Date

Note

~~equivalently~~:

$\Phi_{ijk}(w,t)$ is well-defined on $\mathcal{K}_X \cup \mathcal{K}_{X'}$ with ~~functional equation:~~ $\mathcal{F}\,\Phi_{ijk}(w,T) = \Phi_{ijk}(w, \mathcal{F}T)$

change coor.

~~History:~~

here $e^{-2\pi(w,\beta)_X} = e^{-2\pi(\mathcal{F}w, \mathcal{F}\beta)_{X'}}$ ($\mathcal{F}w = w$)

(~~for β a flopped curve~~ $\mathcal{F}\beta = -\beta'$)

$= e^{2\pi(w,\beta')_{X'}}$ $\ni \mathcal{F}(q^{\beta}) = q^{-\beta'}$

has convergence prob. if $w \in \mathcal{K}_{X'}$.

~~Def'n~~: simple $\mathbb{P}^r$-flop

$X \supset Z \simeq \mathbb{P}^r$ with $N_{Z/X} = \mathcal{O}(-1)^{\oplus r+1}$

then

$$\begin{tikzcd} & E \subset Y = Bl_Z X \arrow[dl] \arrow[d, "\varphi"'] \arrow[dr] & \\ Z \subset & X \arrow[r, dashed, "f"'] & X' \end{tikzcd}$$

, $E \simeq \mathbb{P}^r \times \mathbb{P}^r$, $N_{E/Y} = \mathcal{O}(-1,-1)$

φ' = contraction along another direction.

Lemma 2. $\alpha_i \in H^{2\ell_i}(X)$, $\ell_1 + \ell_2 + \ell_3 = \dim X = 2r+1$

h = hyp class of $Z = \mathbb{P}^r$, then

reverse

$(\mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3) = (\alpha_1, \alpha_2, \alpha_3) + (-1)^r (\alpha_1, h^{r-\ell_1})(\alpha_2, h^{r-\ell_2})(\alpha_3, h^{r-\ell_3})$

Lemma 1. $\mathcal{F} = [\bar{\Gamma}_f] \in A^{2r+1}(X \times X')$.

For ~~3-folds~~, conj 2 solved by A. Li and Y. Ruan ~2000

~~2 ingredients:~~

(1) symplectic deformation and decomp of K-equiv maps into composite of $\mathbb{P}^1$ flops.

(2) multiple cover formula for $\mathbb{P}^1 = C \subset X$, $N_{C/X} \cong \mathcal{O}(-1)^{\oplus 2}$, $\langle - \rangle^X_{0,d} = \frac{1}{d^3}$

(3) relative G-W inv and degeneration formula.

for (1). Kawamata + Kollár + Friedman

(2). ~~Witten~1992~~ : the classical product defect is corrected by 3-point functions on extr ray C.

~~(3) if $\beta \neq C$, then $\langle \alpha_1, \cdots, \alpha_n \rangle_{g,n,\beta} = \langle \mathcal{F}\alpha_1, \cdots, \mathcal{F}\alpha_n \rangle_{g,n,\mathcal{F}\beta}$~~

intuitively clear by divisor relation

idea of pf: localizations: LLY, Givental: $\quad l+k = 2r-1$

$$e_{1*}\frac{e(U_d)}{z(z-\psi)} = (-1)^{(d-1)(r+1)}\frac{1}{(h+dz)^{r+1}} \Rightarrow \langle T_k h^l\rangle_d = \frac{c_k}{d^{k+3}}.$$

$\psi_j = c_1(L_j)$, univ. cotang. line at x_j. $\quad Z \cong \mathbb{P}^r$, l = line class

We make progresses on (2) and (3):

Thm. (generalized multiple cover formula), for $\mathbb{P}^r$-flops.

$\alpha_1, \dots, \alpha_n \in H^{2l_i}(X)$, $\sum l_i = 2r+1 + (n-3)$, then

$$\langle \alpha_1, \dots, \alpha_n\rangle_{0,n,d} \equiv \int_{\bar M_{0,n}(\mathbb{P}^r, dl)} e_1^*\alpha_1 \cdots e_n^*\alpha_n \cdot e(U_d)$$

obstruction bundle

$$\doteq (-1)^{(d-1)(r+1)} N_{l_1\cdots l_n}\, \underline{\underline{d^{n-3}}}\, (\alpha_1.h^{r-l_1})\cdots(\alpha_n.h^{r-l_n}) \qquad U_d = R^1 ft_* e_{n+1}^* N$$

$$\begin{array}{ccc} \bar M_{0,n+1}(\mathbb{P}^r,d) & \xrightarrow{e_{n+1}} & \mathbb{P}^r \\ \downarrow ft & & \\ \bar M_{0,n}(\mathbb{P}^r,d) & & \end{array}$$

$N_{l_1\cdots l_n}$ const. indep of d, and for $n=2,3$, $N_{l_1\cdots l_n} \equiv 1$. recursively defined.

pf: based on ~~LLY, Givental~~ + Lee-Pandharipande (reconstruction thm)

Cor the quantum product restricted to exceptional classes (extr. ray) are inv. under simple $\mathbb{P}^r$-flops.

Pf: since Poincaré pairing is preserved. we compare n-pt fcn's $n \geq 3$

$$\langle \alpha_1, \alpha_2, \alpha_3\rangle = (\alpha_1,\alpha_2,\alpha_3) + \sum_{d\in\mathbb{N}} \langle \alpha_1,\alpha_2,\alpha_3\rangle_{dl}\, q^{dl} + \sum_{\beta\notin\mathbb{Z}l}\langle\alpha_1,\alpha_2,\alpha_3\rangle_\beta\, q^\beta$$

modulo β with $\beta \notin \mathbb{Z}l$, get

$$(\mathcal{F}\alpha_i, h'^{(r-l_i)})_{X'} = (-1)^{l_i}(\mathcal{F}\alpha_i, \mathcal{F}h^{r-l_i}) = (-1)^{l_i}(\alpha_i, h^{r-l_i})_X$$

$$\langle \mathcal{F}\alpha_1\mathcal{F}\alpha_2\mathcal{F}\alpha_3\rangle^{X'} - \langle\alpha_1,\alpha_2,\alpha_3\rangle^X$$
$$= (\alpha_1,h^{r-l_1})(\alpha_2,h^{r-l_2})(\alpha_3,h^{r-l_3})\left((-1)^r + \frac{(-1)^{2r+1}q^{l'}}{1-(-1)^{r+1}q^{l'}} - \frac{q^l}{1-(-1)^{r+1}q^l}\right)$$

under $\mathcal{F}$, $q^{l'} \longleftrightarrow q^{-l}$ $\Rightarrow$ the RHS $= 0$.

For $n = 3+k \geq 4$ $(k\geq 1)$:

$$\langle \alpha_1,\dots,\alpha_n\rangle = N_{l_1\cdots l_n}(\alpha_1 h_1^{r-l_1})\cdots(\alpha_n h_n^{r-l_n})\sum_{d=0}^\infty (-1)^{(d-1)(r+1)} d^k q^{dl}$$
$$= *** \left(q^l\frac{d}{dq^l}\right)^k\left(\frac{\cancel{(-1)^{r+1}}q^l}{1-(-1)^{r+1}q^l}\right) \quad \text{ie. } \delta^k f\,;\ \delta = q\frac{d}{dq}$$

since $q^{-l}\frac{d}{dq^{-l}} = -q^l\frac{d}{dq^l}$ and the $(\cdots)_X$ and $(\cdots)_{X'}$ differ only by a constant, we get

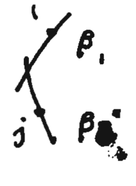
$$\langle \mathcal{F}\alpha_1,\dots,\mathcal{F}\alpha_n\rangle = \langle\alpha_1,\dots,\alpha_n\rangle \quad \forall\, n\geq 4 \qquad \square.$$

under analytic continuation

Key fact: $f(q) := \dfrac{q}{1-(-1)^{r+1}q} = q + (-1)^{r+1}q^2 + (-1)^{2(r+1)}q^3 + \cdots$

Then $f(q) + f(q^{-1}) = (-1)^r$.

L-P: Reconstruction: (for $g=0$)



$$e_i^*L = e_j^*L + (\beta.L)\psi_j - \sum_{\beta_1+\beta_2=\beta}(\beta_1.L)[D_{i\beta_1|j\beta_2}]^{vir}$$

And for $n\geq 3$, $\psi_j = [D_{j|ik}]^{vir}$.

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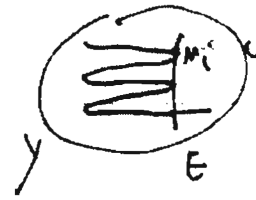
Note

~~Degeneration Analysis~~ / reduction to local models.

~~rel. MK~~ (Y, E), $E \hookrightarrow Y$ sm div.

Top-type: $\Gamma = (g, n, \beta, \rho, \mu)$ $\quad \mu = (\mu_1, \cdots, \mu_\rho) \in \mathbb{N}^\rho$

$$|\mu| = \sum \mu_i = (\beta . E)$$



$A \in H(Y)^{\oplus n}$, $\varepsilon \in H(E)^{\oplus \rho}$

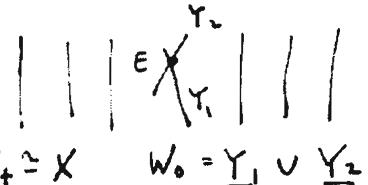
$$\langle A \mid \varepsilon, \mu \rangle^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} e_Y^* A . e_E^* \varepsilon$$

evaluation: $e_Y : \overline{M}_\Gamma(Y,E) \to Y^n$, $e_E : \overline{M}_\Gamma(Y,E) \to E^\rho$

$f :$ $X \overset{\phi}{\leftarrow} Y \overset{\phi'}{\rightarrow} X'$, $X \dashrightarrow X'$ simple $\mathbb{P}^r$-flop

Deformation to the normal cone to X and X':

$$\begin{array}{c} W = Bl_{Z \times \{0\}} X \times \mathbb{A}^1 \\ \Phi \downarrow \\ \mathfrak{X} = X \times \mathbb{A}^1 \end{array}$$



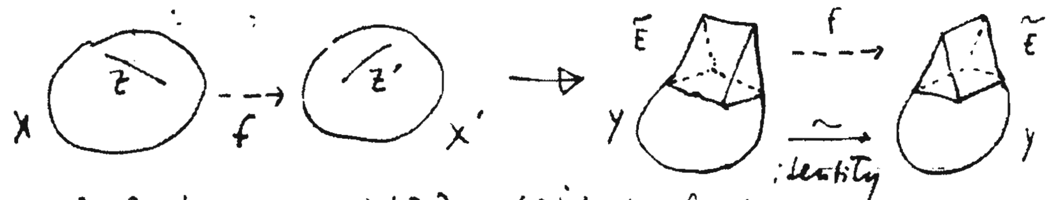
$W_t \cong X$ $\quad W_0 = Y_1 \cup Y_2$

$t \neq 0$

Similarly $\Phi' : W' \to \mathfrak{X}'$. $\quad Y \quad \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O})$

$Y_1 = Y = Y_1'$, $Y_2 \cong \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(-1)^{r+1} \oplus \mathcal{O}) \cong Y_2'$

but the gluings along E are different:



Let $\{e_i\}$ basis of $H(E)$, $\{e^i\}$ dual basis

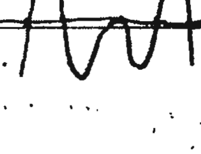
$\{e_I\}$ form a basis of $H(E^\rho)$, $|I| = \rho$, $e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho}$

Thm (Li-Ruan, J. Li, Ionel-Parker)

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta = (\Gamma_1, \Gamma_2, I_\eta)} \frac{m(\mu)}{|\mathrm{Aut}\, \eta|} \langle \alpha(0) \mid e_I, \mu \rangle^{\bullet (Y,E)}_{\Gamma_1} \langle \alpha(0) \mid e^I, \mu \rangle^{\bullet}$$

adm. triple st $\quad$ not nec. conn. in

eg 2 comp.



$\rho = 5$ $\quad$ $\Gamma_1 +_{I_\eta} \Gamma_2$ is conn. of type (g, n, β)

3 comp.

~~Lemma (coh. reduction)~~

Represent $\alpha(0) = (\alpha_1, \alpha_2)$, $(\mathcal{F}\alpha)(0) = (\alpha_1', \alpha_2')$

if $\alpha_1 = \alpha_1'$ then $\mathcal{F}\alpha_2 = \alpha'$.

Prop A. To prove $\mathcal{F}\langle\alpha\rangle^X \cong \langle\mathcal{F}\alpha\rangle^{X'}$, enough to show that

$$\mathcal{F}\langle A \mid \varepsilon, \mu\rangle \cong \langle \mathcal{F}A \mid \varepsilon, \mu\rangle \quad \text{on } \tilde{E} \dashrightarrow \tilde{E}'.$$

where $\langle A \mid \varepsilon, \mu\rangle := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu\rangle_{\beta_2}^{(\tilde{E}, E)} \, I^{\beta_2}$.

Rmk: if $\beta \in NE(X)$, $\beta(0) = (\beta_1, \beta_2)$ then $\beta_1 + \beta_2 = \varphi^*\beta$.

~~Prop B~~. For $\tilde{E} \dashrightarrow \tilde{E}'$, to prove $\mathcal{F}\langle A \mid \varepsilon, \mu\rangle \cong \langle \mathcal{F}A \mid \varepsilon, \mu\rangle$

$\forall A, (\varepsilon, \mu)$, enough to show

$$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \dots, \tau_{k_p}\varepsilon_p\rangle \cong \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \dots, \tau_{k_p}\varepsilon_p\rangle$$

i.e. for descendent inv. with ψ class only in E des. inv. of special type

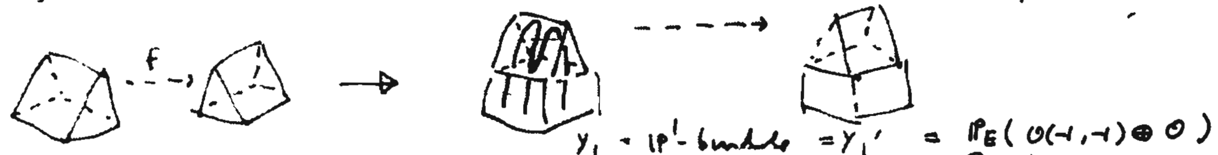
idea of pf: induction on $(|\mu|, n, \rho)$ ~~(ε, μ)~~ reverse ordering

Then $\langle \alpha_1, \dots, \alpha_n, \tau_{\mu_1 - 1} e_{i_1} \cdots \tau_{\mu_p - 1} e_{i_p}\rangle^{\cdot E}$

$$= \sum_{\mu'} m(\mu') \cdot \sum_{I'} \langle \tau_{\mu_1 - 1} e_{i_1}, \dots, \tau_{\mu_p - 1} e_{i_p} \mid e^{I'}, \mu'\rangle^{\cdot(Y_1, E)} \langle \alpha_1, \dots, \alpha_n \mid e_{I}, \mu'\rangle + R$$

"fiber class integral"   conn. inv   lower $|\mu|$ or n

By applying deformation to normal cone to $Z \hookrightarrow \hat{E}$.



Key Point: $\exists!$ highest order term $= c(y)\langle \alpha_1, \dots, \alpha_n \mid e_I, \mu\rangle^{(\tilde{E}, E)}$

Rmk: this is a more precise version of Maulik-Pandharipande

For analytic continuation on local model:

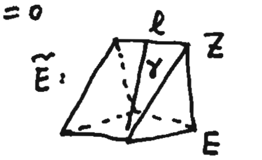
Lemma (Quasi-linearity) $\mathcal{F}\langle \tau_k \xi \cdot \alpha\rangle^{\tilde{E}} = \langle \tau_k \xi' \cdot \mathcal{F}\alpha\rangle^{\tilde{E}'}$ $\tilde{E}$ semi-Fano toric var.

both are finite sum. $\xi = [E] = \xi'$. (via LLY, Givental) $c_1 = (r+2)E + p^* c_1(X)|_Z$

2 generator of Functional Equations: both uses only here requires $g = 0$

$\underline{\underline{\text{GMCF}}} + \underline{\underline{\text{QL}}} \Rightarrow$ local case $\Rightarrow$ Thm. ※

extr. ray   loc. model des. inv. of special type.

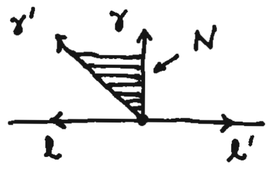


Integral structure: $\langle\alpha\rangle^X \in \mathcal{R} := \widehat{\mathbb{C}[N]}[f]$, $N = \{\beta \in NE(X), \mathcal{F}\beta \in NE(X')\}$

$g = 0$, $n \geq 3$, $\alpha \in H(X)^{\oplus n}$

using $\delta f = f + (-1)^{r+1} f^2$.   Local model:

$$\langle\alpha\rangle^{X_{loc}} \in \mathcal{R}_{loc} = \mathbb{C}[N][f].$$



$\mathcal{F}\gamma = \gamma'$

$\mathcal{F}\ell = [redacted]$