Oberseminar, Essen Jan 2000 Chin-Lung Wang

K equivalence in birational geometry

§1. Definition: For normal $\mathbb{Q}$-Gorenstein varieties X, X'
(say, complex projective)

K-partial order $X \geq_K X'$ if $\exists$ Y smooth
st. φ, φ' birational morphism

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

and $\varphi^* K_X \geq \varphi'^* K_{X'}$ → in terms of $K_Y = \varphi^* K_X + E = \varphi'^* K_{X'} + E'$ i.e. $E \leq E'$.

$X =_K X' \iff \varphi^* K_X = \varphi'^* K_{X'}$ "K-equivalent".

well-defined as $\mathbb{Q}$-Weil div. not just $\sim$.

- K-equiv. including

I. ~~D-flop~~: i.e. terminal $X \dashrightarrow X'$ (if exists)

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & \bar{X} & \end{tikzcd}$$

By def. KMM

f log-extr. contr. of ℓ with

$D.\ell < 0$ and $K_X.\ell = 0$ st. $NE(X'/\bar{X}) = \langle \ell' \rangle$

$D'.\ell' > 0$ (and $K_{X'}.\ell' = 0$) is this auto?

(flop ⇒ K-equiv
flip ⇒ $X >_K X'$)

Fujita
Kollár
Wang –

II. Birational minimal models, or more gen.

(rel.) ~~(rel.)~~ bi-rat'l map $f : X \dashrightarrow X'$ st.

K_X and $K_{X'}$ are nef (rel. nef) along exceptional loci.
(not nec. rel. min.)

III. Different coh. small resolutions of a singular space:

Wisniewski
totaro

$\varphi : X \to \bar{X}$ small if

$\operatorname{codim}_{\mathbb{R}} \{ y \in \bar{X} \mid \dim_{\mathbb{R}} \varphi^{-1}(y) = i \} > 2i$ in $\bar{X}$.

Goal: Find invariants invariant under K-equiv.

Theorem (W-) '97: K-equiv smooth proj manifolds have the same Betti & Hodge numbers.

Kollár; Batyrev for C-Y. Wang general. (Huybrechts: Hyperkähler ??)

Kollár 3-folds minimal model (ter. $\mathbb{Q}$-fact)

Q 1. $\exists$ projective canonical isom.? eg. $T := \varphi'_* \varphi^*$. the coh. correspondence? $[Y] \in H^n(X \times X')$ Künneth.

Q 2. Singular Case ??

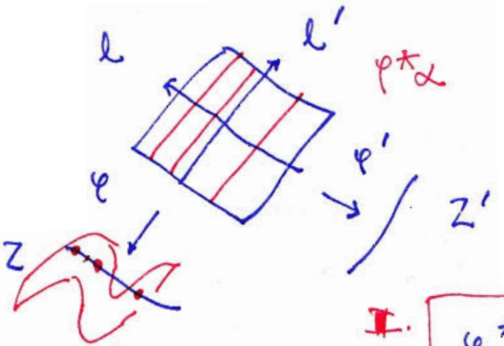
Q 3. Kähler ??? ~~Q4~~: Quantum!

§2. Simplest example : Easy Flop in 3 folds

$$\begin{tikzcd} & Y \supset E = \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X \supset Z = \mathbb{P}^1 & & X' \supset Z' = \mathbb{P}^1 \end{tikzcd}$$

$(-1,-1)$ curve

φ = blow up of Z in X etc.



On H^2 (divisors) :

$\varphi'_* \varphi^* : H^2(X) \xrightarrow{\sim} H^2(X')$

by proper transf. (isom. in codim 1)

but : for $\alpha \longleftrightarrow \alpha'$

I. $$\boxed{\begin{aligned} &\varphi^*\alpha + (Z.\alpha)E = \varphi'^*\alpha' \\ &\text{and} \quad Z.\alpha = -Z'.\alpha' \end{aligned}}$$

must vanishes $Z.\alpha$ times on E to be a pull back of φ'

So $\varphi^* H^2(X) \neq \varphi'^* H^2(X')$

(and $K_X . Z = 0 = K_{X'} . Z'$)

On H^4 (curves) :

II. $$\boxed{\begin{aligned} \varphi^*[Z] &= \ell - \ell' \\ \varphi^*[Z'] &= \ell' - \ell \end{aligned}}$$

1st thing to prove via Fulton.

so $\varphi^* H^4(X) = \varphi^* H^4(X') \Rightarrow \varphi'_* \varphi^* : H^4(X) \xrightarrow{\sim} H^4(X')$

but : $\varphi'_* \varphi^* [Z] = -[Z']$. ef. not preserved.

- ~~Poincaré pairing~~ is preserved under $T = \varphi'_* \circ \varphi^*$.

In fact, T^* (adjoint of T) $= \varphi_* \circ \varphi'^*$

In our case, $T^* = T^{-1}$.

- ~~Cubic form~~ is NOT preserved : (Ring structure)

$$\alpha'^3 = (\varphi'^*\alpha')^3 = (\varphi^*\alpha + (Z.\alpha)E)^3$$

$$= \alpha^3 + \cancel{3(\varphi^*\alpha)^2 (Z.\alpha)E} + 3\,\underline{\varphi^*\alpha}.(Z.\alpha)^2 \underline{E^2} + (Z.\alpha)^3 \underline{E^3}$$

III. $E^2 = E|_E = -(\ell + \ell')$.

$E^3 = (E|_E)^2 = 2$.

$$\Rightarrow \boxed{\alpha'^3 = \alpha^3 - (Z.\alpha)^3}.$$

nice guy!

§3. Proof of thm (Betti case):

the given K-equiv. diag. is defined over $S \subset \mathbb{C}$

S : finite gen. alg. $/\mathbb{Z}$

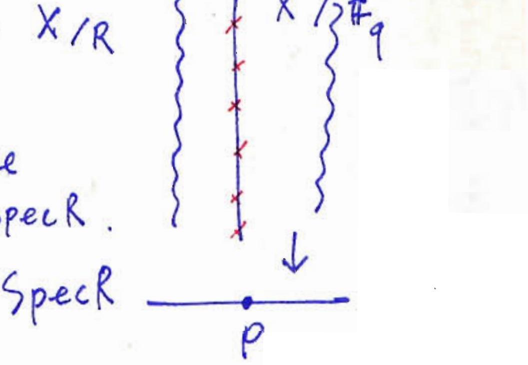
the diag. has "good reduction" $\forall'$ maximal ideal $m \subset S$.

$$(*)\quad \begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

consider formal nbd (completion) at a prime p.

$R = \hat{S}_p$, (so $R/p \cong \mathbb{F}_q$ some q). then $(*)$ becomes a diagram over $\operatorname{Spec} R$. let $\mathbb{K}$ = quot. field)

let $\bar{X}$ be the scheme $/\mathbb{F}_q$: $|a| := q^{-v_p(N_{\mathbb{K}/\mathbb{Q}_p}(a))}$ p-adic valuation



lemma (Weil):

for U smooth R-scheme, Ω nowhere zero holo. n-form (dim = n) over Spec R.
(or r-pluri canonical form)

$$\int_{U(R)} |\Omega|^{1/r} = \frac{|\bar{U}(\mathbb{F}_q)|}{q^n}.$$

Since RHS indep. of choices of Ω (Zariski loc. gen of K_X)

hence a well-defined measure: for X smooth : $m_X(X(R)) = \dfrac{|\bar{X}(\mathbb{F}_q)|}{q^n}$.

For singular $\mathbb{Q}$-Gorenstein var. X.

for a cpt open $S \subset U_i(R)$ (Zariski open set), have $m_X(S) := \int_S |\Omega_i|^{1/r} = \int_{\varphi^{-1}(S)} |\varphi^*\Omega_i|^{1/r}$

notice $\varphi^{-1}(S) \subset Y(R)$.

$m_X(X(R)) < \infty \Leftrightarrow$ log-term.

Now $\forall$ cpt open $T \subset Y(R)$:

$$\int_T |\varphi^*\Omega|^{1/r} = \int_T |\varphi'^*\Omega'|^{1/r}$$ int. on $\mathbb{K}$-analytic manifolds.

bec. they have the same order on Exceptional set :

$K_Y = \varphi^* K_X + E = \varphi'^* K_{X'} + E'$. K-equiv $\Rightarrow$ $E \equiv E'$.

~~change of variable formula~~ $\Rightarrow$ $m_X(X(R)) = m_{X'}(X'(R))$

$\Rightarrow |\bar{X}(\mathbb{F}_q)| = |\bar{X}'(\mathbb{F}_q)|$

$\Rightarrow |\bar{X}(\mathbb{F}_{q^k})| = |\bar{X}'(\mathbb{F}_{q^k})|$ $\forall k \in \mathbb{N}$ (via cyclotomic ext. of $\mathbb{K}$)

$\Rightarrow$ Same zeta function. ($\Rightarrow$ same Euler char.)

Deligne (Weil conj.) $\Rightarrow$ Same ℓ-adic betti numbers.
($\equiv$ top. Betti # .) $\square$

Notice this is true $\forall'$ m.

§4. Proof of thm (Hodge case):

This needs motivic integration (Kontsevich, Denef-Loeser also Batyrev).

let $\mathcal{M} = K_0(Sch_{\mathbb{C}})$ Grothendieck ring of cpx. v. with reduced str. closed se $[X] = [X-Z] + [Z]$, $[X \times Y] = [X]\cdot[Y]$

$\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$, Lefschetz motive

$p \in \mathbb{Z}$; $F^p := \{[S]\cdot\mathbb{L}^{-i} \mid \dim S - i \le -p\} \subset \mathcal{M}[\mathbb{L}^{-1}]$

$\cdots \supset F^0 \supset F^1 \supset \cdots$ decreasing filtration

~~Motivic measure~~: $\boxed{\mu_X : \mathcal{B}(\mathcal{L}(X)) \longrightarrow \widehat{\mathcal{M}[\mathbb{L}^{-1}]}}$

(defined by boolean condition ord_t and coefficients of lowest degree polynomials of) all semi-algebraic set

the (Nash) space of formal curves in X $= \varprojlim \mathcal{L}_m(X)$:

$\mathcal{L}_m(X) = Mor_{k\text{-sch}}(\operatorname{Spec} k[t]/t^{m+1}, X)$

$\mathcal{L}_{m+1}(X) \longrightarrow \mathcal{L}_m(X)$

$\pi_m : \mathcal{L}(X) \longrightarrow \mathcal{L}_m(X)$. both maps are surj if X is smooth.

s.t. if A is ~~stable at~~ level k (i.e.

- A = union of fibers of π_k
- $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piece-wise trivial fibration over $\pi_m(A)$ with fiber $\mathbb{A}^n$, $\forall m \ge k$.)

$$\boxed{\mu_X(A) = [\pi_k(A)]\cdot\mathbb{L}^{-k.n}}$$ indep. of choice of k.

In particular, $\mu_X(\mathcal{L}(X)) = [X] \in \mathcal{M}$.

Fact: every s.alg. set A can be partitioned into stable A_i, $i \in \mathbb{N}$ (mod $\cap F^p$)

for smooth varieties X $\cup\{\infty\}$

then via change of variable: (Hard Part)

For $\alpha : A \to \mathbb{Z}$ simple i.e. all fibers stable $\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{k\in\mathbb{Z}} \mathbb{L}^{-k}\mu(\alpha^{-1}(k))$

$$[X] = \int_{\mathcal{L}(X)} \mathbb{L}^0 \, d\mu_X = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t E} \, d\mu_Y$$

for singular X, this is called the motivic measure of X.

$$= \int_{\mathcal{L}(X')} \mathbb{L}^0 \, d\mu_{X'} = [X']$$

Here $ord_t I : \mathcal{L}(X) \longrightarrow \mathbb{N} \cup \{0\}$ is a "~~simple function~~"

$$ord_t I(\gamma) := \min_{g\in I} \deg_t (g\circ\gamma(t))$$

~~Deligne's Mixed Hodge Theory~~ $\Rightarrow$ $\chi^{p,q}$ $= \sum(-1)^i h^{p,q}(H^i_c(\cdot,\mathbb{C}))$ factors thr. K_0 construction, also $\chi^{p,q} = 0$ on $\cap F^p$.

$\cdots \to H^i_c(X-Z) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(X-Z) \to \cdots$

So: $h^{p,q}(X) = \chi^{p,q}([X]) = \chi^{p,q}([X']) = h^{p,q}(X')$. $\square$

§5. Negative Aspect in the Singular case.

- Can we expect K-equiv. singular mfd have the same int. coh.? $IH^{\cdot}(X)$.

Should we repeat Kawamata-Kollár's procedure for 3-dim'l case here!! for 3-dim'l min. models.

Tonic Varieties:

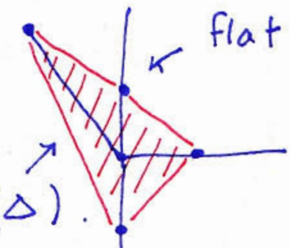
lemma: let $X = X(\Delta)$, $X' = X'(\Delta')$

then $X =_K X' \Leftrightarrow \text{shed}(\Delta) \equiv \text{shed}(\Delta')$.

Here $\text{shed}(\Delta) :=$ union of ~~primitive cones~~. eg.

$\text{shed}(\sigma) := \text{convex}(0, e_1, \cdots, e_r)$

ie. Δ and Δ' are different triangulations of the same polyhedron.



pf: let f_X be the piecewise linear funct. corr to $K_X = -\sum_{\tau \in \Delta_1} D_\tau$

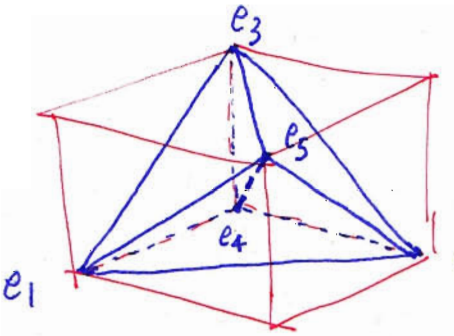
ie. $f_X(\tau) = -1$, then extend linearly, then

for $Y \xrightarrow{\varphi} X$, $Y = X(\tilde{\Delta})$, $X = X(\Delta)$

$K_Y = \varphi^* K_X + \sum_{\sigma \in \tilde{\Delta}_1} (f_X - f_Y)(\sigma) \cdot \tilde{D}_\sigma$

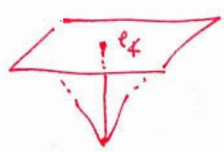
Rmk: $X(\Delta)$ is smooth $\Leftrightarrow$ each cone $\sigma \in \Delta$ is spanned by a $\mathbb{Z}$ basis. $\text{shed}(\Delta)$.

A singular flop in dim = 4:

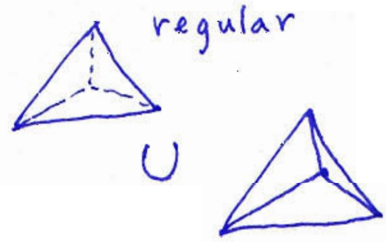


3 regular cones

think as



K-equiv



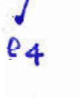
- toric v. are orbifold $\Rightarrow IH^{\cdot} \equiv H^{\cdot}$ (Steenbrink, Danilov)
- for toric v. b_i determines f_i = # of i-dim'l cones

$(h_i = b_{2i} = \sum_{k=i}^{n} (-1)^{k-i} C^k_i f_{n-k}.)$

$\Rightarrow$ for the above, $\dim IH^{\cdot}(X(\Delta)) \neq \dim IH^{\cdot}(X'(\Delta'))$

In fact,

is not small. bec



corr. to a surface. ie. $i = 4$. codim $> 2i = 8$ $\times$.

Rmk: More on toric, see M. Reid

Rmk: $\exists$ K-equiv. 4-dim'l smooth toric. V. $X(\Delta)$, $X'(\Delta')$ st. $X(\Delta)$, $X'(\Delta')$ <u>can not</u> be connected by a sequence of regular flops!!

So to find <u>canonical isom.</u> <u>can't</u> follow Kollár's 3-dim'l pf!!

§6. A proposal via L^2 cohomology : (Wang ~ 95')

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

ω $\qquad\qquad$ ω'

Kähler forms

K equivalence

$\Longleftrightarrow \quad \varphi^* c_1(X) = \varphi'^* c_1(X')$

$$[\varphi^*(-\partial\bar\partial \log \omega^n)] = [\varphi'^*(-\partial\bar\partial \log \omega'^n)]$$

$$\partial\bar\partial \log \varphi^*\omega^n - \partial\bar\partial \log \varphi'^*\omega'^n = \partial\bar\partial f$$

C^∞ on Y

i.e. $\quad (\varphi^*\omega)^n = e^f (\varphi'^*\omega')^n$

$\Longleftrightarrow$ quasi-equivalent (degenerate) volume forms on Y

$$H^i_{L^2}(Y-E, \varphi^*\omega) \simeq H^i_{L^2}(X-Z, \omega) \simeq H^i(X)$$

If $\varphi^*\omega \sim \varphi'^*\omega'$ quasi-isometric (which is NOT true)

then get $H^i(X) \simeq H^i(X')$ done.

Otherwise : Need a "Rotation"
of degenerate metrics in Y to
connect $\varphi^*\omega$ to $\varphi'^*\omega'$.

Q : Is that possible to use complex Monge–Ampere eq'n?

Rmk : Viehweg loves this idea!

※

§7. Quantum Version:

Recall the easy flop

$$\begin{tikzcd}[column sep=small] & Y \supset \mathbb{P}^1\times\mathbb{P}^1 = E \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X \supset \mathbb{P}^1 = Z \arrow[dr, "f"'] & & X' \supset \mathbb{P}^1 = Z' \arrow[dl, "f'"] \\ & \bar{X} \ni p \,:\, \mathrm{ODP} & \end{tikzcd}$$

then for $\alpha \in H^2(X)$

$\alpha' = \varphi'_* \varphi^* \alpha \in H^2(X')$

has: $\underline{\alpha'^3 = \alpha^3 - (Z.\alpha)^3 . \mathbf{1}}$

Cubic form

for general $\begin{array}{ccc} X & \supset & C \\ \downarrow & & \downarrow \\ \bar{X} & \ni & p \end{array}$ cDV (ter. sing.)

locally may deform into several $(-1,-1)$ curves Z_i
$\downarrow$
p_i

then $\underline{\alpha'^3 = \alpha^3 - (\eta.\alpha)^3 \cdot \sum_{d>0} n_d \cdot d^3}$, where

- $\eta \in H^4(X)$ is the primitive class st. $f_* \eta = 0$
  (ie. $[C] = \lambda \eta$)
- n_d = # of $(-1,-1)$ curves Z_i with $[Z_i] = d\eta$.

the correction term is related to GWR-inv.
(via Aspinwall-Morrison formula).

(observed by Wilson)

In fact:

Thm (Ruan-Li): 3-dim'l flop (nonsingular) induces isom. of quantum cohomology.

CONJECTURE: K-equivalence smooth varieties have canonically isomorphic quantum coh.
(raised by Ruan in 1996 ICM talk for minimal models)

Rmk: All 3-dim'l result are proved also in the Kähler case.

Rmk: Batyrev has a version for the toric case without the canonical isom. (Rigorous?)