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ON THE INCOMPLETENESS OF THE WEIL-PETERSSON METRIC ALONG DEGENERATIONS OF CALABI-YAU MANIFOLDS

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Introduction

The classical Weil-Petersson metric on the Teichmüller space of compact Riemann surfaces is a Kähler metric, which is complete only in the case of elliptic curves [Wo]. It has a natural generalization to the deformation spaces of higher dimansional polarized Kähler-Einstein manifolds. It is still Kähler, and in the case of abelian varieties and K3 surfaces, the Weil-Petersson metric turns out to coincide with the Bergman metric of the Hermitian symmetric period domain, hence is in fact "complete" Kähler-Einstein [Sc].

The completeness is an important property for differential geometric reason. Motivated by the above examples, one may naively think that the completeness of the Weil-Petersson metric still holds true for general Calabi-Yau manifolds (compact Kähler manifolds with trivial canonical bundle). However, explicit calculation done by physicists (eg. Candelas *et al.* [Ca] for some special nodal degenerations of Calabi-Yau 3-folds) indicated that this may not always be the case.

The notion of completeness depends on the precise definition the "moduli space". However, through our analysis, it would become clear that the Weil-Petersson metric is in general incomplete if one sticks

† Some inaccuracies in §3 has been corrected in this reproduced version.

on “moduli” of smooth varieties. In fact, our aim in this paper is to characterize all finite distance degenerations and then to describe the possible picture of the completion. It is found to be closely related to the minimal model program in birational geometry. The results proved here could be regarded as a first attempt toward the completion problem.

There are two parts of this paper. In the first part, §1 – §2, we start with Tian's description of the Weil-Petersson metric as the Chern form of the Hodge bundle F^n. This fits naturally into the framework of variation of Hodge structures and the Weil-Petersson can be formally defined in this setting. By applying Schmid's theory of limiting mixed Hodge structures, we obtain in (1.1) our first Hodge-theoretic criterion:

Theorem A. *The center of a degeneration of polarized Hodge structures of weight n with $F^n \cong \mathbf{C}$ has finite Weil-Petersson distance if and only if $NF^n_\infty = 0$.*

Here N is the nilpotent monodromy and F^n_∞ is the limiting filtration. In §2, we return to the geometric situation, namely the semi-stable degeneration of polarized Calabi-Yau manifolds. As a simple application of the Clemens-Schmid exact sequence [C], we get in (2.5) the

Theorem B. *The central fiber X has finite Weil-Petersson distance if and only if some irreducible component $X_i \subset X$ has $H^{n,0} \neq 0$. This is equivalent to that there is exact one component with $h^{n,0} = 1$.*

This theorem is also claimed in a recent preprint of Hayakawa [H]. However, the proof given there seems to be incomplete and too complicate. As a corollary, we deduce in (2.10) the following theorem which we believe to be very close to the final picture of the completion problem:

Theorem C. *Let X be a Calabi-Yau varieties which admits a smoothing to Calabi-Yau manifolds. If X has only canonical singularities then X has finite Weil-Petersson distance along the base.*

In the second part, §3 – §4, we deal with a refined question: up to a base change, is the finite distance degeneration birationally equivalent to a smooth family? This is exactly what happens in the case of K3 surfaces [Ku] — this is also one reason that one usually regards the Weil-Petersson metric on the “moduli space” of K3 surfaces to be complete. In general, is there any “essential non-trivial finite distance degeneration”? The simplest examples would be those degenerations with monodromy not of finite order. In §3, we show that it is the case for nodal degenerations of 3-folds.

In §4, which is perhaps the most technical part of this work, we study the "expected most general cases" of finite distance degenerations, i.e. degenerations with canonical singularities, via various results in 3-fold birational geometry. The main result is:

Theorem D. *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a terminal Gorenstein 3-fold $\mathcal{X}_0$ with $K_{\mathcal{X}_0}$ nef. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$ with $\mathcal{X}_t = \mathcal{X}'_t$ for $t \neq 0$.*

This applies in particular to terminal degenerations of Calabi-Yau 3-folds (see (4.11)) and hence we conclude that smoothable Calabi-Yau 3-folds with nontrivial terminal singularities all provide essential finite distance points. Since some of these degenerations are known to have C^∞ trivial monodromies, Theorem D then gives a negative answer to the so-called "filling-in problem" in dimension 3.

§0 Background Material

Here we briefly recall some basic definitions and well-known properties about Hodge theory and the Weil-Petersson metrics that will be used in this paper. More details could be found in [G, GS, K, S, T1].

0.1. Polarized VHS and the period map. It is well known that from the theory of harmonic forms that the m-th primitive cohomology of a compact Kähler manifold (X^n, ω) admits a "polarized real Hodge structure" of weight m. That is, for $V := P^m_{\mathbf{R}} \otimes \mathbf{C} \subset H^m_{\mathbf{C}}$, one has

$$V = \bigoplus\nolimits_{p+q=m} P^{p,q}, \qquad P^{p,q} = \overline{P^{q,p}}. \tag{0.2}$$

Equivalently, this can be expressed in terms of the Hodge filtration $F : V = F^0 \supset F^1 \supset \cdots \supset F^m$ with $F^p = \bigoplus_{i \geq p} P^{i,m-i}$. Moreover, for $m \leq n$, the Hodge-Riemann bilinear form

$$Q(u, v) := (-1)^{\frac{m(m-1)}{2}} \int_X u \wedge v \wedge \omega^{n-m} \tag{0.3}$$

polarizes V in the sense that F satisfies the Hodge-Riemann bilinear relations:

$$\begin{aligned} &\text{I.} \quad Q(F^p, F^{m+1-p}) = 0 \quad \text{and} \\ &\text{II.} \quad Q(Cv, \bar{v}) > 0, \end{aligned} \tag{0.4}$$

where C, the Weil operator, acts on $P^{p,q}$ by multiplying $\sqrt{-1}^{p-q}$.

Varying the above data (V, F, Q), a family of polarized Kähler manifolds $\mathcal{X} \to S$, ie. $[\omega(s)]$ is locally constant, gives rise to a "polarized VHS over S": $(\mathbf{V}, \mathbf{F}, \nabla, Q)$ with ∇ the flat connection. This data satisfies the Griffiths transversality relation

$$\nabla \mathbf{F}^p \subset \mathbf{F}^{p-1} \otimes \Omega^1_S. \tag{0.5}$$

It follows that $\bar{\partial}\mathbf{F}^p \subset \mathbf{F}^p$ and hence that $\mathbf{F}^p$'s form holomorphic subbundles of the flat bundle $\mathbf{V}$, which is polarized by the flat bilinear form Q. This can also be described in terms of the "period map":

The period domain D is the classifying space of all Hodge filtrations F of V that are polarized by Q. We have $D = G/K$ where $G = \mathrm{Aut}(P_{\mathbf{R}}, Q)$ and K the stabilizer of a point. It comes naturally with the tautological homogeneous vector bundle $\mathbf{F}^p \subset \mathbf{V} := V \times D$. The compact dual $\check{D}$ is the set of all the F's which satisfy only the first Hodge-Riemann relation. It contains D as an open subvariety. The family $\mathcal{X} \to S$ gives rise to the period map $\phi : S \to \Gamma\backslash D$ with Γ the representation of $\pi_1(S)$ in G. It is clear that $\phi^*\mathbf{F}^p$ gives the holomorphic vector bundle mentioned above and (0.5) translates into the horizontality of ϕ:

$$d\phi : T_S \to \bigoplus_{p+q=m} \mathrm{Hom}(P^{p,q}, P^{p-1,q+1}) =: T^h_D, \tag{0.6}$$

where T^h_D is called the horizontal tangent bundle.

We can formalize the above situation and define the polarized VHS as a locally liftable horizontal holomorphic map $\phi : S \to \Gamma\backslash D$ with Γ a representation of $\pi_1(S)$ in $G \cap \mathrm{Aut}(H_{\mathbf{Z}})$, where $H_{\mathbf{Z}}$ is an integral lattice such that $P_{\mathbf{R}} \subset H_{\mathbf{R}}$. In the case $F^n \cong \mathbf{C}$, we will also consider the "n-th flag period map" $\phi^n : S \to \Gamma\backslash \mathbf{P}(V)$ which in fact contains almost all the information that we will need in this paper.

0.7. Semi-stable degenerations and the monodromies. We are interested in the case of a degeneration $\mathcal{X} \to \Delta$ of polarized Kähler n-folds. By this we mean that $\mathcal{X}$ is a Kähler $(n+1)$-fold and $\mathcal{X} \to \Delta$ is a proper flat holomorphic map with the general fiber $\mathcal{X}_t$, $t \neq 0$, a smooth Kähler n-fold. Notice that the resulting family over the punctured disk has a polarization induced from the Kähler form on $\mathcal{X}$.

$\mathcal{X} \to \Delta$ is called semi-stable if $\mathcal{X}_0$ is a reduced divisor with normal crossings in $\mathcal{X}$. By a theorem of Mumford [K], every degeneration has a semi-stable reduction by a sequence of blow-ups and base-changes. In general, $\mathcal{X} \to \Delta$ is called a degeneration of certain type if $\mathcal{X}_0$ has only singularities of that type. And by "$\mathcal{X} \to \Delta$ is a smoothing of $\mathcal{X}_0$", we will mean that $\mathcal{X} \to \Delta$ is a proper flat family with smooth $\mathcal{X}_t$ for $t \neq 0$ but without assuming the complex space $\mathcal{X}$ to be smooth.

Now a generator of $\pi_1(\Delta^\times) \cong \mathbf{Z}$ induces the so called Picard-Lefschetz transformation – the monodromy T on $H^m_{\mathbf{Z}}$, which is known to be quasi-unipotent. Under the semi-stable asssumption, T will be unipotent and we will consider the associated nilpotent operator $N := \log T$ acting on $H^m_{\mathbf{Q}}$ (and therefore on $V \subset H^m_{\mathbf{C}}$ since the polarization class is invariant under T). The quasi-unipotent statement is also known to be true for any polarized VHS [S]. In this paper, we will usually assume that T is unipotent by allowing a base change implicitly.

0.8. Schmid's theory on limiting MHS. For a polarized VHS $\phi : \Delta^\times \to \langle T\rangle\backslash D$; the map ϕ lifts to the upper half plane $\Phi : \mathbf{H} \to D$ with the coordinates $t \in \Delta^\times$ and $z \in \mathbf{H}$ related by $t = e^{2\pi\sqrt{-1}z}$. Set

$$A(z) = e^{-zN}\Phi(z) : \mathbf{H} \to \check{D}, \tag{0.9}$$

(instead of D). Since $A(z+1) = A(z)$, A descends to a function $\alpha(t)$ on $\Delta^\times$. The very first part of Schmid's "nilpotent orbit theorem" says that $\alpha(t)$ extends holomorphically over $t = 0$. The special value $F_\infty := \alpha(0)$ is called the limiting filtration and is in general outside D. However, the nilpotent operator N uniquely defines a "monodromy weight filtration" on V: $0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m-1} \subset W_{2m} = V$ such that $N(W_k) \subset W_{k-2}$ and induces an isomorphism

$$N^\ell : G^W_{m+\ell} \cong G^W_{m-\ell}, \tag{0.10}$$

where $G^W_k := W_k/W_{k-1}$ is the graded piece. These two filtrations F^p_∞ and W_k together define a "polarized mixed Hodge structure" on V in the following sense: the induced Hodge filtration

$$F^p_\infty G^W_k := F^p_\infty \cap W_k / F^p_\infty \cap W_{k-1}, \quad p = 0, \ldots, m \tag{0.11}$$

defines a (pure) Hodge structure of weight k on G^W_k. The operator N acts on them as a morphism of MHS's of type $(-1,-1)$. That is, $N(F^p_\infty G^W_k) \subset F^{p-1}_\infty G^W_{k-2}$. Moreover, for $\ell \geq 0$, the primitive part $P^W_{m+\ell} := \ker N^{\ell+1} \subset G^W_{m+\ell}$ is polarized by $Q(\cdot, N^\ell \cdot)$.

When ϕ comes from geometric situations, by adding together with the non-primitive part, the total cohomology $H^m(\mathcal{X}_t, \mathbf{C})$ still admits non-polarized MHS.

0.12. The Weil-Petersson metric. For a given family of polarized Kähler manifolds $\mathcal{X} \to S$ with Kähler metrics $g(s)$ on $\mathcal{X}_s$, one can define a possibly degenerate hermitian metric G on S as follows: at $s \in S$ with fiber $X = \mathcal{X}_s$, we consider the Kodaira-Spencer map $\rho : T_{S,s} \to H^1(X, T_X) \cong \mathbf{H}^{0,1}_{\bar\partial}(T_X)$ into harmonic forms with respect to $g(s)$; so for v, $w \in T_s(S)$, we may define

$$G(v,w) := \int_X \langle \rho(v), \rho(w)\rangle_{g(s)}. \tag{0.13}$$

When $\mathcal{X} \to S$ is a polarized Kähler-Einstein family and ρ is injective, $G_{WP} := G$ is called the Weil-Petersson metric on S.

When X is a Calabi-Yau manifold, we have (1) Yau's solution to Calabi's conjecture [Y] that X has an unique Ricci flat metric in each Kähler class and (2) the Bogomolov-Tian-Todorov theorem that the Kuranishi space of X is unobstructed [T1, To].

Let $\mathcal{X} \to S$ be a maximal subfamily of the Kuranishi family with a fixed polarization class $[\omega]$, then ρ is clearly injective. Let $g(s)$ be the unique Ricci flat metric in the given polarization. Using the fact that the global holomorphic n-form $\Omega(s)$ is flat with respect to $g(s)$, it was shown in [T1, To] that

$$G_{WP}(v,w) = \frac{Q(C(i(v)\Omega), \overline{i(w)\Omega})}{Q(C\Omega, \bar\Omega)}, \tag{0.14}$$

where $H^1(X, T_X) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \cong H^{n-1,1}$ via the interior product $v \mapsto i(v)\Omega$ is the well-known isomorphism. The tangent space T_S is mapped to $P^{n-1,1}$ isomorphically and hence leads to the fact that the n-th flag period map is an local embedding. So the Weil-Petersson metric is induced from the Hodge metric on the n-th piece of the horizontal tangent bundle. For convienence, let's write $\widetilde{Q} = \sqrt{-1}^n Q$ ($= Q(C\cdot, \bar\cdot)$ on $H^{n,0} = P^{n,0}$). Tian observed that $\widetilde{Q}$ is a Kähler potential of G_{WP}, that is,

$$\omega_{WP} = \frac{\sqrt{-1}}{2}\mathrm{Ric}_{\widetilde{Q}}(H^{n,0}) = -\frac{\sqrt{-1}}{2}\partial\bar\partial \log \widetilde{Q}, \tag{0.15}$$

where ω_{WP} denotes the fundamental real 2-form of G_{WP} (this formula shows in particular that ω_{WP} is independent of the polarization). The proof is essentially part of Griffiths' curvature calculation [G], hence is purely Hodge theoretic. So we can extend the definition of G_{WP} to polarized VHS over S with $h^{n,0} = 1$ by (0.15), although it is only semi-positive. Since it makes sense to talk about geodesics and distances, we will still call it the Weil-Petersson metric.

§1 Hodge Theoretic Criterion for Finite Distance Points

We now give the basic criterion for finite Weil-Petersson distance in the case of one parameter degenerations of polarized Hodge structures $\phi : \Delta^\times \to \langle T\rangle\backslash D$ with $h^{n,0} = 1$:

Theorem 1.1. *The center of a degeneration of polarized Hodge structures of weight n with $F^n \cong \mathbf{C}$ has finite Weil-Petersson distance if and only if $NF^n_\infty = 0$.*

Proof. We will keep the notation from §0. Let $\Phi : \mathbf{H} \to D$ be the lifting. To start the computation, all we need is a good choice of a holomorphic section Ω of $H^{n,0}$. Let $p^n : D \to \mathbf{P}(V)$ be the projection to the F^n part. we have $\Phi^n(z) = (e^{zN}\alpha(t))^n = e^{zN}\alpha^n(t)$. Here $*^n := p^n(*) \in \mathbf{P}(V)$ means the n-th flag. Near $t = 0$, we can consider a vector (local homogeneous coordinates) representation $\mathbf{a}$ of α^n in V. Then $\mathbf{a}(t) = a_0 + a_1 t + \cdots$ is holomorphic in t. We have correspondingly

$$\mathbf{A}(z) = a_0 + a_1 e^{2\pi\sqrt{-1}z} + a_2 e^{4\pi\sqrt{-1}z} + \cdots. \tag{1.2}$$

The crucial point here is that the function $e^{2\pi\sqrt{-1}z} = e^{2\pi\sqrt{-1}x}e^{-2\pi y}$ has the property that all the partial derivatives in x and y decay to 0 exponentially as $y \to \infty$, with rate of decay independent of x. For ease of notation, let h be the function class satisfying the above property and $\mathbf{h}$ the corresponding function class with values in V.

Now let $\Omega(z) = e^{zN}\mathbf{A}(z)$. This is the desired section because vector representations correspond to sections of the tautological line budle of $\mathbf{P}^n$ which pull back to $H^{n,0}$ by Φ. So the Kähler form ω_{WP} of the induced Weil-Petersson metric G_{WP} on $\mathbf{H}$ is given by

$$\omega_{WP} = -\frac{\sqrt{-1}}{2}\partial\bar\partial \log \widetilde{Q}(e^{zN}\mathbf{A}(z), e^{\bar zN}\overline{\mathbf{A}(z)}). \tag{1.3}$$

Since we are in one complex variable, write $G_{WP} = G|dz|^2$, then $G = -(1/4)\triangle \log \widetilde{Q}$. We have $Q(Tu, Tv) = Q(u, v)$, it follows easily that $Q(Nu, v) = -Q(u, Nv)$ and $Q(e^{zN}u, v) = Q(u, e^{-zN}v)$. Since $\mathbf{A} = a_0 + \mathbf{h}$, we have

$$\begin{aligned} \widetilde{Q}(e^{zN}\mathbf{A}, e^{\bar{z}N}\bar{\mathbf{A}}) &= \widetilde{Q}(e^{zN}a_0, e^{\bar{z}N}\bar{a_0}) + h \\ &= \widetilde{Q}(e^{2\sqrt{-1}yN}a_0, \bar{a_0}) + h = p(y) + h, \end{aligned} \tag{1.4}$$

where $p(y)$ is a polynomial in y with

$$d = \deg p(y) = \max\{\, \ell \mid N^{\ell}\alpha_0 \neq 0 \,\}. \tag{1.5}$$

This a consequence of the polarization condition for the mixed Hodge structure (0.8) and the fact that $a_0 \in G_{n+d}$. So

$$\begin{aligned} 4G &= \frac{(p'+h)^2 - (p+h)(p''+h)}{(p+h)^2} = \frac{(p'^2 - pp'') + h}{p^2 + h} \\ &\sim \frac{p'^2 - pp''}{p^2} + h \sim \frac{d^2 - d(d-1)}{y^2} + h = \frac{d}{y^2} + h. \end{aligned} \tag{1.6}$$

Here we have used the fact that $p^{-2}h \in h$. Obviously, if $NF_\infty^n = 0$ then $d = 0$ and $G = h$, so $\int_t^\infty \sqrt{G}\,|dz| < \infty$ for some curve (e.g. $x = c$). When $NF_\infty^n \neq 0$ we have $d \geq 1$ and for y large enough we can make $h < 1/y^3$ uniformly in x, then clearly $\int_t^\infty \sqrt{G}\,|dz| \sim 2\log y\,|_t^\infty = \infty$ for any path with $y \to \infty$. Q.E.D.

Remark 1.7. From the proof of (1.1), we know that in the case of infinite distance, the Weil-Petersson metric is exponentially asymptotic to a scaling of the Poincaré metric. In particular, the (holomorphic sectional) curvature approaches to a negative constant when $t \to 0$. This is exactly the situation what we are familiar with for the moduli space of elliptic curves.

§2 Geometric Criterion for Finite Distance Points

For a semi-stable degeneration, there is a well-known procedure to relate the limiting MHS and Deligne's canonical MHS on the singular cohomology of the central fiber, namely the Clemens-Schmid exact sequence, which generalizes the classical Picard-Lefschetz theory (cf. §3).

Let's briefly recall the constructions (see [C, GS] for more details). Let $X = \cup_i X_i$ be a simple normal crossing variety, for $I = \{i_0, \cdots, i_p\}$, $X_I := X_{i_0} \cap \cdots \cap X_{i_p}$. Also let $X^{[p]}$ be the disjoint union of all X_I with $|I| = p+1$. There is a spectral sequence which computes $H^*(X) = H^*(X, \mathbf{C})$ with $\mathrm{E}_0^{p,q} = \Omega^q(X^{[p]})$, $d_0 := d$ (the exterior differentiation of forms) and $d_1 := \delta$: the restriction operator of forms defined by

$$(2.1) \qquad (\delta\phi)(X_{i_0\cdots i_{p+1}}) := \sum_{j=0}^{p+1} (-1)^j \phi(X_{i_0\cdots \hat{i_j}\cdots i_{p+1}})\big|_{X_{i_0\cdots i_{p+1}}}.$$

Clearly $\mathrm{E}_1^{p,q} = H^q(X^{[p]})$ and the $\mathrm{E}_2^{p,q}$ term is computed from

$$(2.2) \qquad H^q(X^{[p-1]}) \xrightarrow{\delta} H^q(X^{[p]}) \xrightarrow{\delta} H^q(X^{[p+1]}),$$

where the δ's respect Hodge structures. Moreover, it degenerates at E_2. The weight filtration for the resulting MHS on $H^m(X)$ is $W_\ell := \bigoplus_{s\le \ell} \mathrm{E}_2^{m-s,s}$, and the Hodge filtration is the usual one for each factor induced from E_1. Notice that in contrast to the limiting MHS, the canonical MHS has terms $G_\ell H^m(X)$ only for $0 \le \ell \le m$.

The Clemens-Schmid exact sequence for a semi-stable degeneration

$$(2.3) \qquad \begin{array}{cccccccc} \text{smooth} & \mathcal{X}_t & \subset & \mathcal{X} & \supset & \mathcal{X}_0 & (= X = \cup_i X_i) \\ & \downarrow & & \downarrow & & \downarrow & \\ (0 \ne) & t & \in & \Delta & \ni & 0 & \end{array}$$

is an exact sequence of MHS's:
(2.4)

$$\cdots \to H_{2n+2-m}(\mathcal{X}_0) \xrightarrow{j} H^m(\mathcal{X}_0) \xrightarrow{i} H^m \xrightarrow{N} H^m \xrightarrow{k} H_{2n-m}(\mathcal{X}_0) \to \cdots$$

Notice that the inclusion $\mathcal{X}_0 \subset \mathcal{X}$ is a homotopy equivalence, j is induced by inclusion and duality and i is induced by the inclusion $\mathcal{X}_t \subset \mathcal{X} \sim \mathcal{X}_0$. Also H^m denotes the cohomology for the general fiber $\mathcal{X}_t$ and N is the nilpotent monodromy oprator. Moreover, this exact sequence is compatible with the MHS's with types of morphisms $(n+1, n+1)$, $(0,0)$, $(-1,-1)$ and $(-n,-n)$ respectively, where type (p,q) means $F^*G_* \to F^{*+p}G_{*+2q}$. Here MHS for homology is defined by duality: $G_{-\ell}H_q := G_\ell(H^q)^*$ and $F^{-p}G_{-\ell}H_q := \mathrm{Ann}(F^{p+1}G_\ell H^q)$.

When the degeneration of Hodge structures in (1.1) comes from a semi-stable degeneration of Calabi-Yau manifolds $\mathcal{X} \to \Delta$, we have:

Theorem 2.5. *The central fiber $X = \mathcal{X}_0$ has finite Weil-Petersson distance if and only if some irreducible component $X_i \subset X$ has $H^{n,0} \neq 0$. This is equivalent to that there is exact one component with $h^{n,0} = 1$.*

Proof. By the results of Schmid in (0.8), F_∞ and N defines a MHS on $H^n(\mathcal{X}_t)$ for a reference fiber $\mathcal{X}_t$ with $t \neq 0$. It follows from (0.10) that $(\ker N) \cap F^n_\infty \equiv G^W_n F^n_\infty$. So $NF^n_\infty = 0$ if and only if $F^n_\infty = G^W_n F^n_\infty$.

Recall that the "geometric genus formula" [C] says that

$$h^{n,0}(\mathcal{X}_t) \geq \sum\nolimits_i h^{n,0}(X_i), \tag{2.6}$$

and the RHS corresponds to all the invariant cycles in F^n_∞, that is, $(\ker N) \cap F^n_\infty$. Since the LHS of (2.6) has the same dimension as F^n_∞, the eqality holds if and only if $F^n_\infty = (\ker N) \cap F^n_\infty = G^W_n F^n_\infty$, that is, if and only if $NF^n_\infty = 0$.

In our case, Theorem 1.1 says that finite distance is equivalent to $NF^n_\infty = 0$. Since $h^{n,0}(\mathcal{X}_t) = 1$, this is equivalent to that there exist some (and so at most one) component with $h^{n,0} \neq 0$ (and so in fact it must be 1). The proof is now complete.

For the reader's convienence, we sketch the well-known argument for the geometric genus formula. Apply the Clemens-Schmid exact sequences to $F^n G_n H^n$, we get

$$\to F^{-1}G_{-n-2}H_{n+2}(\mathcal{X}_0) \to F^n G_n H^n(\mathcal{X}_0) \to F^n_\infty G^W_n H^n \xrightarrow{N} 0 \tag{2.7}$$

We know by definition that $G_{n+2}H^{n+2}(\mathcal{X}_0) = \mathrm{E}^{0,n+2}_2 = \ker\delta$ with $\delta : H^{n+2}(X^{[0]}) \to H^{n+2}(X^{[1]})$ and that $F^2 H^{n+2}(\mathcal{X}_0) = H^{n+2}(\mathcal{X}_0)$. From this we conclude that $F^2(\ker\delta) \equiv \ker\delta$. So

$$F^{-1}G_{-n-2}H_{n+2}(\mathcal{X}_0) = \mathrm{Ann}(F^2 G_{n+2}H^{n+2}(\mathcal{X}_0)) = 0, \tag{2.8}$$

and (2.7) implies that $F^n G_n H^n(\mathcal{X}_0) \cong F^n_\infty G^W_n H^n$.

Now $\mathrm{E}^{0,n}_2$ is computed from $0 \to \mathrm{E}^{0,n}_1 \to \mathrm{E}^{1,n}_1 \equiv H^n(X^{[1]})$, and the F^n part of the right term is zero. So $F^n G_n H^n(\mathcal{X}_0) = F^n \mathrm{E}^{0,n}_2 = F^n \mathrm{E}^{0,n}_1 = F^n H^n(X^{[0]}) = H^{n,0}(X^{[0]})$.

The resulting isomorphism $F^n_\infty G^W_n H^n \cong H^{n,0}(X^{[0]})$ clearly gives what we want. Q.E.D.

Remark 2.9. Both (1.1) and (2.5) are stated in the one parameter case, but the Weil-Petersson metric distance should be evaluated in the corresponding smoothing component of the central fiber, which is in general of many dimensions. However, finite distance in a special direction implies finite distance in the whole component, so (2.5) indeed provides a suifficient condition for the existence of finite distance points. The converse is not obvious in case the base dimension is bigger than one. We plan to discuss this issue in a forthcoming work.

Now we apply (2.5) to smoothable singular Calabi-Yau varieties. A Calabi-Yau variety is by definition a normal projective variety with trivial canonical (cartier) divisor. Recall that a normal variety X is has canonical (resp. terminal) singularities if K_X is $\mathbf{Q}$-Cartier and there is a (equivalently for any) resolution $f : \widetilde{X} \to X$ such that $K_{\widetilde{X}} =_{\mathbf{Q}} f^* K_X + \sum e_i E_i$ with $e_i \geq 0$ (resp. $e_i > 0$), where E_i's are the exceptional divisors. Canonical singularities in dimension two are exactly RDP's (also called Du Val, A-D-E, Kleinian singularities). Terminal singularities must be of codimension three. In dimension three, they are completely classified by Reid and Mori [R].

Canonical singularities play an important role in birational geometry. In the case of Calabi-Yau 3-folds, birational primitive contractions [W] will create at most canonical singularities. It has been conjectured that the moduli spaces of Calabi-Yau 3-folds (with $h^1(\mathcal{O}) = 0$) of different topological types can be "connected" by performing primitive contractions and smoothings. This statement is also known to be interestng from the point of view of physics. Our next result implies that this can happen only within finite Weil-Petersson distance.

Theorem 2.10. *Let X be a Calabi-Yau varieties which admits a smoothing to Calabi-Yau manifolds. If X has only canonical singularities then X has finite Weil-Petersson distance along the base.*

Proof. For any resolution $f : \widetilde{X} \to X$, we have as in the above that $H^{n,0}(\widetilde{X}, \mathbf{C}) = \Gamma(\widetilde{X}, K_{\widetilde{X}}) = \Gamma(\widetilde{X}, \sum e_i E_i)$ (notice that e_i's are integers). Since E_i's are exceptional, it follows easily that $H^{n,0}(\widetilde{X}, \mathbf{C}) \neq 0$ precisely when X has at most canonical singularities.

Now let $\mathcal{X} \to \Delta$ be a smoothing of X. Take a semi-stable reduction of it, then there is a component in the central fiber of the semi-stable reduction which corresponds to the proper transform of X. Then it has $h^{n,0} = 1$. Now apply Theorem 2.1 and notice that finite distance in a special smoothing implies finite distance in the whole smoothing component. Q.E.D.

Example 2.11. According to [R], hypersurface singularities of monomial type $\sum_i x^{d_i} = 0$ is canonical if and only if $\sum_i 1/d_i > 1$. In the three dimensional case, the finiteness of the Weil-Petersson distance with singularities of this type were known to Candelas *et al.* [Ca] via direct calculations. Theorem 2.9 seems to indicate that canonical singularities may also play significant role in certain physics problems.

Question 2.12. Is the converse of (2.10) true? More precisely, if a degeneration of Calabi-Yau manifolds has finite Weil-Petersson distance, is that true this degeneration is birational to another degeneration such that the central fiber is an irreducible Calabi-Yau variety with only canonical singularities? This would be an important step toward the completion program.

Remark 2.13. The problem of whether a singular Calabi-Yau variety X with canonical singularities has a flat deformation into nonsingular Calabi-Yau's $\mathcal{X}_t$ has already been studied extensively in dimension 3. The first step was taken by Friedman [F1] in the case of ODP's (see also Tian [T2] and [F2]). Recent preprints of Namikawa-Steenbrink and M. Gross have provided quite satisfactory results in this direction. These developments are closely related to Z. Ran's extension of the Bogomolov-Tian-Todorov theorem to the singular case.

Remark 2.14. In fact, all the statements in §1 and §2 are true in the following more general setting. Given a smooth polarized family of varieties $\mathcal{X} \to S$ parametrized by a smooth base S and with $h^0(\mathcal{X}_s, K_{\mathcal{X}_s}) \geq 2$, we may consider the semi-definite metric ω on S given by the Chern form of $\det F^n$, that is, $\det f_* K_{\mathcal{X}/S}$. Using this metric, the main results (1.1), (2.5) and (2.10) generalize immediately. In fact, the same proofs work except notationally more complicated. However, even in the Kähler-Einstein case with K_X ample, this metric is not the Weil-Petersson metric defined in (0.12). There is a complicated relation between the two in terms of certain "Quillen metrics".

§3 Incompleteness I: Nontrivial Monodromy

In §3 and §4, we work in the projective category. For Calabi-Yau varieties, it will be assumed that $h^1(\mathcal{O}) = 0$. In particular, we have excluded the case of abelian varieties.

There exists smoothable Calabi-Yau 3-folds with canonical singularities such that the smoothing comes from a birational contraction of a smooth family over the disk, which induces isomorphisms outside the puncture. These examples are due to Wilson [W] in his deep study of the jumping phenomenon of Kähler cones. More precisely, his proposition 4.4 says that the "type III primitive contraction" with the exceptional divisor a quasi-ruled surface over an elliptic curve provides such an example.

In the surface case, these correspond to smoothings of K3 surfaces with RDP's. By Kulikov's classification theorem [Ku] and the fact that "$NF_\infty^2 = 0$ implies $N = 0$", they are birational to smooth families (up to a base change).

For our purpose, the above examples should not be considered as incomplete points by the following reason: one can include these points by hand – just replace the degeneration by the smooth family by allowing the polarization line bundle to be only big and nef. In the case of K3 surfaces, an equivalent way is to add these points by allowing Ricci-flat orbifold metrics. In fact, this process leads to the (metric) completion of the K3 moduli! However, The situation changes if $\dim X \geq 3$. We will see that there are "nontrivial" examples. By this we mean a degeneration such that the complement of the central fiber can not be completed into a smooth family.

If the monodromy T is not of finite order ($N \neq 0$) then the degeneration is clearly "nontrivial" in the above sense. In this connection, we mention the following classical result of Picard, Lefschetz and Poincaré:

Theorem 3.1. *For a nodal degeneration of smooth n-folds, the monodromy T is trivial except possibly in the middle dimensional cohomology. In the middle dimensional case, we have that*

I. $N^2 = 0$ *if n is odd, and that*

II. $T^2 = I$ *(so $N = 0$) if n is even.*

The standard proof of (3.1) is to write down the explicit formula of T in terms of the "vanishing cycles". However, in order to see whether $N \neq 0$ in the odd case one needs to know whether the vanishing cycle represent nontrivial homology classes, and this is clearly not just a local problem near the singular points. (The fact that the vanishing cycle can be homologically trivial was kindly pointed out to me by J. de Jong.)

As an exercise, We will show that how Theorem 3.1 follows easily from the topological version of the Clemens-Schmid exact sequence and then make some remarks on it.

First of all, a semi-stable reduction can be obtained by first doing a degree 2 base change and then blowing up the ODP's of the total space. So $X^{[0]}$ is the union of n-quadrics and the proper transform X' of the original central fiber, $X^{[1]}$ is the union of $(n-1)$-quadrics and $X^{[2]} = \emptyset$. For $m < n$, we claim that for $\ell \leq m-1$, $G_\ell H^m = 0$. Suppose it has been proved up to $\ell - 1$, then

$$G_{\ell-2n-2}H_{2n+2-m}(\mathcal{X}_0) \to G_\ell H^m(\mathcal{X}_0) \to G_\ell^W H^m \xrightarrow{N} 0. \tag{3.2}$$

If $m - \ell \geq 2$, since $X^{[2]} = \emptyset$ we have that $G_\ell H^m(\mathcal{X}_0) = \mathrm{E}_2^{m-\ell,\ell} = 0$. If $m-1 = \ell$, since $m < n$ we have $\mathrm{E}_2^{1,m-1} = \operatorname{coker}\delta$ with $\delta : H^{m-1}(X^{[0]}) \to H^{m-1}(X^{[1]})$, which is surjective by explicit cohomologies of quadrics (or use hyperplane section theorem). This proves that $G_\ell H^m(\mathcal{X}_0) = 0$ and so $G_\ell^W H^m = 0$ up to $\ell = m-1$. This means that the MHS is pure and so $N = 0$.

For H^n, the same argument shows that $G_\ell^W H^n = 0$ up to $\ell = n-2$, so $N^2 = 0$. Since $\ell < n$, $G_{\ell-2n-2}H_{2n+2-n}(\mathcal{X}_0)$ is the dual of $G_{2(n+1)-\ell}H^{2(n+1)-n}(\mathcal{X}_0)$, which is zero. So $G_{n-1}H^n(\mathcal{X}_0) \cong G_{n-1}^W H^n$ is given by $\operatorname{coker}\delta$ with

$$\delta : H^{n-1}(X^{[0]}) \to H^{n-1}(X^{[1]}). \tag{3.3}$$

Now the middle cohomology of an $(n-1)$-quadric is zero if n is even, so $N = 0$ in this case. Q.E.D.

For the case n is odd, $N \neq 0$ if and only if δ is not surjective. The middle cohomology has rank 2 for an even dimensional quadric. The image of these n-quadrics under δ consist of suitable powers of the hyperplane class, which is also in the image of $H^{n-1}(X')$ if $n \geq 5$ (because for any of these $(n-1)$-quadrics E, $E|_E$ generates $H^2(E)$, which is only one dimensional). Therefore δ is surjective if and only if the induced map

$$\delta' : H^{n-1}(X') \to H^{n-1}(X^{[1]}) \tag{3.4}$$

is surjective. That is, the surjectivity of (3.3) (or (3.4) for $n \geq 5$) is equivalent to the triviality of monodromy N. It is immedeate from this that if the monodromy is trivial then the number of ODP's of the central fiber X has an upper bound given by $b^{n-1}(X)$. However, it is not clear how to get anything more without specifying the varieties under consideration.

In the three dimensional case, there are some explicit computations done by Candelas *et al.* showing that certain nodal degenerations have indeed monodromies not of finite order. Hence one obtains nontrivial finite distance examples. A theoretic proof of this statement for any nodal degenerations turns out to be delicate (even for Calabi-Yau 3-folds). We will give a sketch of it by showing the existence of nontrivial vanishing cycles, following a suggestion by Mark Gross.

Let us assume that our 3-folds are all simply connected. First of all, a nodal threefold $\mathcal{X}_0$ always admits (not necessarily projective) small resolutions $X \to \mathcal{X}_0$ with smooth rational curves $X \supset C_i \to p_i \in \mathcal{X}_0$ contracted to ODP's. In the case of Calabi-Yau threefolds (Gorenstein threefolds with trivial canonical bundle and with $h^1(\mathcal{O}) = 0$), the existence of global smoothing $\mathcal{X} \to \Delta$ of $\mathcal{X}_0$ forces that there are nontrivial relations of $[C_i] \in H_2(X)$ by Friedman's result [F3, F4]. That is, the canonical map $e : \bigoplus_i \mathbf{Z}[C_i] \to H_2(X, \mathcal{Z})$ has nontrivial kernel dimension $s > 0$. Consider the resulting surgery diagram:

$$\begin{array}{c} X \\ \downarrow \\ \mathcal{X}_0 \subset \mathcal{X} \supset \mathcal{X}_t \end{array} \tag{3.5}$$

It has the following local description: let $V_i \ni p_i$ be a contrctible neighborhood of an ODP, $V_i' \subset \mathcal{X}_t$ be the smoothing of V_i and $U_i \subset X$ be the inverse image of V_i. Then

I. U_i is a deformation retract neighborhood C_i and so has the homotopy type of $S^2 \sim D^4 \times S^2$.

II. V_i' has the homotopy type of $S^3 \times D^3$. Where the sections $\sigma_i \sim S^3$ are the so called vanishing cycles.

III. The surgery from X to $\mathcal{X}_t$ is induced from $\partial(D^4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times D^3)$.

Let us assume that there are k ODP's.

An immedeate consequence of (3.5) is the Euler number formula:

$$\chi(X) - k\chi(\mathbf{P}^1) = \chi(\mathcal{X}_0) - k\chi(\mathrm{pt}) = \chi(\mathcal{X}_t) - k\chi(S^3). \tag{3.6}$$

Let W be the "common open set" of X, $\mathcal{X}_o$ and $\mathcal{X}_t$ away from all points p_i's such that W and V_i's cover $\mathcal{X}_t$ etc. A portion of the Mayer-Vietoris sequence of the covering $\{W, V_i'\}$ of $\mathcal{X}_t$ gives

$$0 \to H_3(W) \to H_3(\mathcal{X}_t) \to \bigoplus_i \mathbf{Z}[C_i] \to H_2(X) \to H_2(\mathcal{X}_t) \to 0. \tag{3.7}$$

Hence that $b_2(X) = b_2(\mathcal{X}_t) + (k - s)$.

Take into account of $b_2(\mathcal{X}_0) = b_2(\mathcal{X}_t)$ and $b_4(\mathcal{X}_0) = b_4(X)$ (which also follows from suitable Mayer-Vietoris sequences), simple manipulations with (6.7) shows that $b_3(\mathcal{X}_t) = b_3(\mathcal{X}_0) + s$. Comparing with the (Mayer-Vietoris) sequence defining the vanishing cycles:

$$\bigoplus_i \mathbf{Z}[\sigma_i] \to H_3(\mathcal{X}_t) \to H_3(\mathcal{X}_0) \to 0, \tag{3.8}$$

we conclude that $s > 0$ is the dimension of the sapce of vanishing cycles. Q.E.D.

Remark (3.9) We do not know whether every Calabi-Yau threefold admits nontrivial finite distance degenerations, not to say nodal degenerations. It is also possible for a nontrivial degeneration to have $N = 0$! In [F2], Friedman remarked that there exists families of quintic hypersurfaces in $\mathbf{P}^4$ aquiring an A_2 singularity and have $N = 0$ (due to Clemens) and in fact are even C^∞ trivial outside the puncture after a finite base change. He then asked whether this family can be filled in smoothly up to a base change. Since an A_2 singularity is terminal, the degeneration has finite Weil-Petersson distance from the smooth fibers. Thus we need better method to detect nontrivial finite distance points. This is the main issue of the following section.

§4 Incompleteness II: Birational Geometry

Now we go to the most technical part of this paper. In this section, by using several results of Reid, Kawamata and Kollár in the theory of 3-fold birational geometry along with Friedman's result on the simultaneous resolution of 3-fold double points, a negative answer to the "filling problem" as stated at the end of §3 is given for any projective smoothing of a terminal Gorenstein 3-fold with numerical effective (nef) canonical bundle. As a consequence, any smoothable terminal Calabi-Yau 3-fold provides nontrivial incomplete points of the Weil-Petersson metric. Even if the monodromy is completely (eg. C^∞) trivial! Similar statement is true for the general setting in remark (2.14). Here is the main theorem:

Theorem 4.1. *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a terminal Gorenstein 3-fold $\mathcal{X}_0$ with $K_{\mathcal{X}_0}$ nef. Then $\mathcal{X} \to \Delta$ is not birational to a projective smooth family $\mathcal{X}' \to \Delta$ with $\mathcal{X}_t = \mathcal{X}'_t$ for $t \neq 0$.*

We start with the following important fact (true in any dimension):

Theorem 4.2. *Let $\mathcal{X} \to \Delta$ and $\mathcal{X}' \to \Delta$ be two projective families with smooth general fiber $\mathcal{X}_t = \mathcal{X}'_t$ for $t \neq 0$. Assume that*

I. *$\mathcal{X}$ and $\mathcal{X}'$ have at most terminal singularities and*

II. *$K_{\mathcal{X}}$ (resp. $K_{\mathcal{X}'}$) is nef on the central fiber,*

then the bimeromorphic map which identifies all fibers outside $t = 0$ extends to a map which is an isomorphism in codimension one. In particular, $\mathcal{X}_0$ and $\mathcal{X}'_0$ are birational to each other.

Proof. This is essentially the same as in [K1, lemma 4.3], except that they deal with the case where $\mathcal{X}$ and $\mathcal{X}'$ are both compact projective (and so Δ is not involved). The same proof applies to our relative situation basically because our families are assumed to be projective. So we will just give a sketch of the proof:

Let ϕ be the given bimeromorphic map and $\mathcal{Z}$ be a desingularization of the closure of the graph of ϕ with projection maps $p : \mathcal{Z} \to \mathcal{X}$ and $p' : \mathcal{Z} \to \mathcal{X}'$ over Δ. Clearly we have that $\mathcal{Z}_t \cong \mathcal{X}_t \cong \mathcal{X}'_t$ for $t \neq 0$.

If p and p' have the same exceptional divisors then the p-exceptional set and p'-exceptional set differ only in codimension two or higher, let E be the union of both set. Then we have the following isomorphisms

$$\mathcal{X} - p(E) \cong \mathcal{Z} - E \cong \mathcal{X}' - p'(E), \tag{4.3}$$

which is the extension of ϕ we want.

To see p and p' have the same exceptional divisors, consider the relation between canonical divisors:

$$K_{\mathcal{Z}} = p^* K_{\mathcal{X}} + E_1 + F = p'^* K_{\mathcal{X}'} + E_2 + G, \tag{4.4}$$

where E_i (resp. F, resp. G) denotes the part which are p and p' (resp. p but not p', resp. p' but not p) exceptional. We can then write

$$p^* K_{\mathcal{X}} = p'^* K_{\mathcal{X}'} + G + (E_2 - E_1 - F). \tag{4.5}$$

Because of the existence of relative hyperplane sections over Δ, the key reduction lemma in [K3, (5.2.5.3)] can be adapted for our purpose – it says that we only need to prove the above statement for the surface case; the nef condition is used here. It implies that $E_2 - E_1 - F \geq 0$, hence $F = 0$ and $E_2 \geq E_1$. Reversing the role of p and p' gives $G = 0$ and $E_1 \geq E_2$, so we have in fact $E_1 = E_2$. Since both $\mathcal{X}$ and $\mathcal{X}'$ have terminal singularities, all exceptional divisors must appear in E_i. So the theorem is proved. Q.E.D.

Proof of (4.1). Assume such a smooth family $\mathcal{X}' \to \Delta$ exists. We will check the conditions needed in (4.2). II is clearly satisfied since $K_{\mathcal{X}}|_{\mathcal{X}_0} = K_{\mathcal{X}_0}$, which is nef. To see I, first notice that by a simple fact in commutative algebra, the total space of a small smoothing of Gorenstein singularities is again Gorenstein. We then need the following nontrivial theorem. (Although the statement is not explicitly appeared in [K3], the proof is actually contained in [K3, (17.4), (17.6)], and so will not be given here.)

Theorem 4.6. *The total space of a small smoothing of Gorenstein canonical singularities has at most Gorenstein terminal singularities.*

Since both conditions in (4.2) are satisfied, we know that $\mathcal{X}_0$ is birational to $\mathcal{X}'_0$. We will show that this is impossible.

If $\mathcal{X}_0$ is $\mathbf{Q}$-factorial then $\mathcal{X}_0$ and $\mathcal{X}'_0$ are birationally equivalent minimal models. Recall that a minimal model is a normal variety which is $\mathbf{Q}$-factorial, terminal and has nef canonical class. By Kollár's theorem on flops [K1], they are related by a sequence of flops. But a flop does not change the singularities in the terminal case, so we get a contradiction.

If $\mathcal{X}_0$ is not $\mathbf{Q}$-factorial, a theorem of Reid-Kawamata (see e.g. [K3, (6.7.4)]) says that we still have a projective small morphism $X \to \mathcal{X}_0$ from a ($\mathbf{Q}$-factorial) minimal model X to $\mathcal{X}_0$. X is birational to $\mathcal{X}_0$ and so is birational to $\mathcal{X}'_0$. As before, this implies that X is smooth and it is related to $\mathcal{X}'_0$ by a sequence of flops. By Kollár's result again [K1], X and $\mathcal{X}'_0$ have the same integral homologies and hence have the same homologies as the general fiber $\mathcal{X}_t$ in $\mathcal{X}$.

Consider the following "small contraction-smoothing" diagram:

$$\begin{array}{c} X \\ \downarrow \\ \mathcal{X}_0 \subset \mathcal{X} \supset \mathcal{X}_t \end{array} \tag{4.7}$$

If $\mathcal{X}_0$ has only ODP singularities, (4.7) is nothing but a "surgery diagram" appeared in the Picard-Lefschetz theory. There is a well known explicit formula which relates the homologies of X and $\mathcal{X}_t$ and shows in particular that they can not be the same. We will state this formula in a form suitable for our purpose. The proof is basically a simple

Mayer-Vietoris argument which is entirely the same as that sketched in [F2].

Lemma 4.8. *Given a diagram as above in the C^∞ category such that near each singular point of $\mathcal{X}_0$ it is a "small contraction-smoothing" diagram of a germ of ODP. Let C_i be the rational curves contracted to those ODP's and let $e : \bigoplus_i \mathbf{Z}[C_i] \to H_2(X, \mathbf{Z})$ be the map which associates to each C_i its class in X, then $H_2(\mathcal{X}_t) = \operatorname{coker} e$.*

So, $H_2(\mathcal{X}_t) \cong H_2(X)$ means the image of e is zero, which is impossible because X is projective. This is the desired contradiction in the case when $\mathcal{X}_0$ has only ODP's as singular points.

In the general case, since the singularities are of index one, by Reid's classification they are exactly isolated cDV singular points, that is, one parameter deformation of surface RDP's. By Friedman's result [F1], if $p \in V$ is a germ of an isolated cDV point and $C \subset U$ is the corresponding germ of the exceptional set (which is a curve) contracted to p, then the versal deformation spaces $\mathrm{Def}(p, V)$ and $\mathrm{Def}(C, U)$ are both smooth and there is an inclusion map of complex spaces $\mathrm{Def}(C, U) \to \mathrm{Def}(p, V)$. Moreover, one can deform the complex structure of a small neighborhood of C so that in this new complex structure, C decomposes into several $\mathbf{P}^1$'s and the contraction map deforms to a nontrivial contraction of these $\mathbf{P}^1$'s down to ODP's, while keeping a neighborhood of these ODP's to remain in the versal deformations of the germ $p \in V$.

We can preform this analytic process for all C's and p's simultaneously in each corresponding small neighborhoods and then patch them together smoothly. As a result, we obtain a deformed diagram which satisfies the conditions stated in lemma 4.8:

$$\begin{array}{c} \widetilde{X} \\ \downarrow \\ \widetilde{\mathcal{X}}_0 \subset \widetilde{\mathcal{X}} \supset \widetilde{\mathcal{X}}_t \end{array} \tag{4.9}$$

By our construction, $\widetilde{X}$ is diffeomorphic to X and $\widetilde{\mathcal{X}}_t$ is diffeomorphic to $\mathcal{X}_t$ for $t \neq 0$. The later is true because $Def(p, V)$ is smooth and the constructiuon is local. Now we have again,

$$H_2(\widetilde{\mathcal{X}}_t) \cong H_2(\mathcal{X}_t) \cong H_2(X) \cong H_2(\widetilde{X}). \tag{4.10}$$

This implies that the image of e is zero. Since the original exceptional curve has nontrivial homology class, at least one deformed rational curve has nontrivial homology class. This leads to the desired contradiction again and we are done. Q.E.D.

In the case of Calabi-Yau 3-folds with at most canonical singularities, $h^1(\mathcal{O}) = 0$ implies $h^2(\mathcal{O}) = 0$. Hence any smoothing $\mathcal{X} \to \Delta$ must be projective by the semi-continuity of $h^2(\mathcal{O}) = 0$, and in fact $\mathcal{X}_t$ must still be Calabi-Yau. So we conclude the following:

Theorem 4.11. *Let $\mathcal{X} \to \Delta$ be a smoothing of a terminal Calabi-Yau 3-fold. Then $\mathcal{X} \to \Delta$ is not birational to a smooth family.*

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