

QUANTUM FLIPS I: LOCAL MODEL

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ABSTRACT. We study analytic continuations of quantum cohomology under simple flips $f : X \dashrightarrow X'$ along the extremal ray quantum variable q^ℓ . The inverse correspondence $\Psi = [\Gamma_f]^*$ by the graph closure gives an embedding of Chow motives $[\hat{X}'] \hookrightarrow [\hat{X}]$ which preserves the Poincaré pairing. We construct a deformation $\hat{\Psi}$ of $\Psi = [\Gamma_f]^*$ which induces a non-linear embedding

$$QH(X') \hookrightarrow QH(X)$$

in the category of F -manifolds into the *regular integrable loci* of $QH(X)$ near $q^\ell = \infty$. This provides examples of *functoriality of quantum cohomology* beyond K -equivalent transformations. In this paper, we focus on the case when X and X' are (projective) local models.

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0. INTRODUCTION

The theory of quantum cohomology emerged about three decades from the study of physics on Calabi–Yau 3-folds. The mathematical foundations have been established and many essential tools including localization techniques and degeneration formulas were developed. These lead to fruitful results on explicit computations of quantum cohomology and enumerative geometry. However, one of the very basic property of usual cohomology, the functoriality under natural morphisms, is still generally lacking for quantum cohomology.

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0.1. Structure of quantum cohomology. Let X be a complex projective manifold and $\overline{M}_n(X, \beta)$ the moduli space of stable maps from n -pointed rational nodal curves to X with image class $\beta \in NE(X)$, the Mori cone of effective one cycles. For $i \in [1, n]$, let $e_i : \overline{M}_n(X, \beta) \rightarrow X$ be the evaluation map. The $g = 0$ Gromov–Witten potential

$$F(\mathbf{t}) = \langle\langle - \rangle\rangle(\mathbf{t}) := \sum_{n, \beta} \frac{q^\beta}{n!} \langle \mathbf{t}^{\otimes n} \rangle_{n, \beta}^X = \sum_{n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_n(X, \beta)]^{vir}} \prod_{i=1}^n e_i^* \mathbf{t}$$

is a formal function in $\mathbf{t} \in H = H(X)$ and the Novikov variables q^β 's. We call $\mathcal{R} := \mathbb{C}[[q^\bullet]]$ the (formal) Kähler moduli and denote $H_{\mathcal{R}} = H \otimes \mathcal{R}$.

Let $\{T_\mu\}$ be a basis of H and $\{T^\mu := \sum g^{\mu\nu} T_\nu\}$ the dual basis with respect to the Poincaré pairing $g_{\mu\nu} = (T_\mu, T_\nu)$. Denote $\mathbf{t} = \sum t^\mu T_\mu$. The big quantum ring $(QH(X), *)$ is a \mathbf{t} -family of rings $Q_{\mathbf{t}}H(X) = (T_{\mathbf{t}}H_{\mathcal{R}}, *_\mathbf{t})$ defined by

$$\begin{aligned} T_\mu *_\mathbf{t} T_\nu &:= \sum_{\epsilon, \kappa} \partial_\mu \partial_\nu \partial_\epsilon F(\mathbf{t}) g^{\epsilon\kappa} T_\kappa = \sum_{\kappa} \langle\langle T_\mu, T_\nu, T^\kappa \rangle\rangle(\mathbf{t}) T_\kappa \\ (0.1) \quad &= \sum_{\kappa, n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \langle T_\mu, T_\nu, T^\kappa, \mathbf{t}^{\otimes n} \rangle_{n+3, \beta}^X T_\kappa. \end{aligned}$$

The WDVV associativity equations equip $H_{\mathcal{R}}$ a structure of formal Frobenius manifold over \mathcal{R} . It is equivalent to the flatness of the Dubrovin connection

$$\nabla^z = d - \frac{1}{z} A := d - \frac{1}{z} \sum_{\mu} dt^\mu \otimes T_\mu *_\mathbf{t}$$

on the formal relative tangent bundle $TH_{\mathcal{R}}$ for all $z \in \mathbb{C}^\times$. The connection matrix A_μ for $z\nabla_\mu$ is z -free ($= T_\mu *$). This uniquely characterizes the constant frame $\{T_\mu\}$ among all other frames $\{\tilde{T}_\mu\}$ with $\tilde{T}_\mu \equiv T_\mu \pmod{\mathcal{R}}$.

Indeed, let \mathcal{D}^z be the ring of differential operators generated by $z\partial_i$ with coefficients in $\mathbb{C}[z][[q^\bullet, \mathbf{t}]]$. The \mathcal{D}^z module associated to $z\partial_i \mapsto z\nabla_i$ is isomorphic to $\mathcal{D}^z J$ generated by the J function: let ψ be the class of cotangent line at the first marked section, then

$$J(\mathbf{t}, z^{-1}) := 1 + \frac{\mathbf{t}}{z} + \sum_{\beta, n, \mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z - \psi)}, \mathbf{t}^{\otimes n} \right\rangle_{n+1, \beta}^X$$

which encodes invariants with one descendent insertion. The topological recursion TRR implies (the quantum differential equation QDE)

$$z\partial_\mu z\partial_\nu J = \sum_{\kappa} A_{\mu\nu}^\kappa z\partial_\kappa J.$$

In practice, one might be able to find element $I(\mathbf{t}_1) \in \mathcal{D}^z J$ but only along some restricted variables $\mathbf{t}_1 \in H_1 \subset H$. If it happens that H_1 generates H (either in classical product or quantum product), then often one may compute $J(\mathbf{t})$ and/or ∇^z effectively. For a toric manifold X , such an I function can be found through the \mathbb{C}^\times -localization data with $\mathbf{t}_1 \in H^{\leq 2}(X)$.

0.2. Statement of results. Which parts of the structure are functorial?

Y. Ruan raised the problem around 1997. Since then, partial inspiring progresses were made on K -equivalent manifolds (crepant birational transformations) where the functoriality is the simplest possible one, namely the equivalence of quantum cohomology under analytic continuations along the Kähler moduli.

The purpose of this paper is to go beyond the setting of K -equivalence and to understand the categorical framework to formulate the functoriality. The simplest non K -equivalent birational maps “preserving the Kähler moduli” are smooth ordinary flips.

A birational map $f : X \dashrightarrow X'$ is called a *simple* (r, r') *flips*, with $r > r' > 0$, if the exceptional loci of f

$$Z \cong P^r \subset X, \quad Z' \cong P^{r'} \subset X'$$

have the local properties:

$$N_{Z/X} \cong \mathcal{O}_{P^r}(-1)^{\oplus r+1}, \quad N_{Z'/X'} \cong \mathcal{O}_{P^{r'}}(-1)^{\oplus r'+1}.$$

The flip f is achieved by blowing up X along Z to get $Y = \text{Bl}_Z X$, with exceptional divisor $E \cong P^r \times P^{r'}$, and then by contracting E along P^r to get X' . In this first instalment of this project, we are mainly concerned with the (projective) *local models* of the simple (r, r') flips. That is,

$$X_{loc} = P_{P^r} \left(\mathcal{O}(-1)^{r+1} \oplus \mathcal{O} \right), \quad X'_{loc} = P_{P^{r'}} \left(\mathcal{O}(-1)^{r'+1} \oplus \mathcal{O} \right).$$

In particular, $\dim X_{loc} = \dim X'_{loc} = r + r' + 1$ and X_{loc} is Fano.

It was shown in [8] that the graph closure of f^{-1} defines a correspondence $\Psi = [\bar{\Gamma}_f]^*$ which identifies the Chow motive of X' as a sub-Chow motive of X . While $\Psi : H(X') \rightarrow H(X)$ *preserves the Poincaré pairing*, it does *not preserve the classical ring (cup product) structure*; see [8, § 2.3].

The simple flips allow two limits. When $r = r'$, this is a simple flop. It was shown in [8] that $\Phi = [\Gamma_f]_*$ induces an isomorphism $H(X) \cong H(X')$ as vector spaces with bilinear pairing. and the “anomaly” of Φ with respect to the ring structure is cancelled by Gromov–Witten invariants associated to the extremal rays $\ell \subset X$ and $\ell' \subset X'$. This is understood as analytic continuations along the Kähler moduli. In fact the *big quantum cohomology rings* are isomorphic $QH(X) \cong QH(X')$ under analytic continuations induced by Φ . The other limit is $r' = 0$ and this is the case of blowing down. Most of our discussions and results apply to these two limiting cases.

In the case of flips we will show that $QH(X')$ can still be regarded as a sub-theory of $QH(X)$ in a canonical, though *non-linear*, manner.

First of all, there is a basic split exact sequence (cf. Lemma 1.1)

$$0 \rightarrow K \rightarrow H(X) \xrightarrow{\Phi} H(X') \rightarrow 0$$

with splitting map $\Psi : H(X') \rightarrow H(X)$. The kernel space (vanishing cycles) K has dimension $d := r - r'$ and is orthogonal to $\Psi H(X')$.

Secondly, the Dubrovin connection can be analytic continued “along the Kähler moduli” to a connection $\Phi\nabla$ under the rule $\Phi q^\beta = q^{\Phi\beta}$ with $\beta \in NE(X)$. As $\Phi\beta$ might not be effective, indeed $\Phi\ell = -\ell'$, analytic continuations are generally required. By the very construction,

$$\nabla_\mu = \partial_\mu - \frac{1}{z} T_\mu^*$$

has only (formal) *regular singularities* at $q_i = 0$ in Kähler moduli via the standard identification of *divisorial coordinates* t^i and Novikov variables q_i (which follows from the divisor axiom in GW invariants):

$$q_i = e^{t^i}, \quad \partial_i = q_i \frac{\partial}{\partial q_i}.$$

The resulting connection $\Phi\nabla$ turns out to be analytic in the extremal ray variable q^ℓ and contains *irregular directions* along the divisor $q^\ell = \infty$, that is $q^{\ell'} = 0$, corresponding precisely to the kernel subspace K .

This suggests strongly the possibility of extracting the Dubrovin connection ∇' on $TH'_{\mathcal{R}'}$, where $H' = H(X')$ and $\mathcal{R}' = \mathbb{C}[[NE(X')]]$, from $\Phi\nabla$ by “removing the K directions”—since after all ∇' is expected to be regular.

Indeed, in the next step, it is shown that there is an *eigen-decomposition*

$$(0.2) \quad TH \otimes \mathcal{R}'[1/q^{\ell'}] = \mathcal{T} \oplus \mathcal{K}$$

into irregular eigenbundle \mathcal{K} which extends K over $\mathcal{R}'[1/q^{\ell'}]$ and the regular eigenbundle $\mathcal{T} = \mathcal{K}^\perp$ which is precisely the orthogonal complement of \mathcal{K} . From WDVV equations, both \mathcal{T} and \mathcal{K} are shown to be integrable distributions (cf. Proposition 4.15). The integrable submanifold $\mathcal{M}_{q'}$ passing through the section $(q' \neq 0, \mathbf{t} = 0)$ is then the proposed manifold corresponding to $QH(X')$.

The decomposition (0.2) has the flavor of Magrange’s theorem on formal decomposition of meromorphic connections. Unfortunately $\Phi\nabla$ turns out has essential singularities along $q^{\ell'} = 0$ in the naive way. So in practice we start with the *small quantum cohomology* $Q_0H(X)$ and establish (0.2) in that case first, since $\Phi\nabla$ is then meromorphic of Poincaré rank one along $q^{\ell'} = 0$ (cf. Lemma 2.9, 2.10).¹

If one now restrict to the local models, the *Picard–Fuchs equations* arising from \mathbb{C}^\times -localizations become available. It turns out that X and X' share the same Picard–Fuchs equations after analytic continuations, and this forms the initial step to compare \mathcal{T} and $QH(X')$. Technically there are non-trivial *Birkhoff factorizations* and *generalized mirror transforms* involved to go from Picard–Fuchs equations to Dubrovin connections. Still, at the end the functoriality turns out to be quite satisfactory: the product structure can be preserved by deforming the embedding Ψ along the underlying

¹Another way is to utilize the “adic” topology given by the Mori cone near $q^{\ell'} = 0$: modulo any q^β , the irregularity is of finite order. We do not take this approach here.

Frobenius manifold, if one is willing to give up the conservation of bilinear pairing. This is known as the F -structure.

Theorem 0.1. *For the local model $f : X \dashrightarrow X'$ of simple (r, r') flips, there is an \mathcal{R}' -point $\sigma_0(q') \in H_{\mathcal{R}'}$ and an embedding $\widehat{\Psi}(q', \mathbf{s})$ over \mathcal{R}' :*

$$\sigma_0(q') + \mathbf{s} \mapsto \widehat{\Psi}(q', \mathbf{s}) : H(X')_{\mathcal{R}'} \longrightarrow \mathcal{M} \hookrightarrow H(X)_{\mathcal{R}'},$$

where $\mathbf{s} \in H(X')$, such that

- (1) $(\widehat{\Psi}, \sigma_0)$ restricts to $(\Psi : H' \hookrightarrow H, 0)$ when modulo q^{ℓ} ,
- (2) outside the divisor $q^{\ell} = 0$, the big quantum products on the corresponding tangent spaces are preserved (i.e. $\widehat{\Psi}$ is an F -embedding):

$$\langle\langle \widehat{\Psi}_{\mu}, \widehat{\Psi}^i, \widehat{\Psi}_j \rangle\rangle^X(\widehat{\Psi}(q', \mathbf{s})) = \langle\langle T'_{\mu}, T'^i, T'_j \rangle\rangle^{X'}(\sigma_0(q') + \mathbf{s}).$$

In particular, there is a ring decomposition

$$\mathcal{Q}_{\widehat{\Psi}(q', \mathbf{s})} H(X) \cong \mathcal{Q}_{\sigma_0(q') + \mathbf{s}} H(X') \times \mathbb{C}^{r-r'}.$$

This can be described in more geometric, but perhaps less precise, terms. One first identify $q^{\ell} = 1/q^{\ell'}$, and this extend the $A_{q^{\ell}}^1$ to $P_{q^{\ell}}^1$. Note that $QH(X)$ over the Novikov ring \mathcal{R} is analytic with respect to the variable q^{ℓ} and in general formal with respect to other Novikov variables. One can regard $QH(X)_{\mathcal{R}}$ as an analytic family of Frobenius manifold over $A_{q^{\ell}}^1$. Near $q^{\ell} = \infty$, Theorem 0.1 states that there is a family of codimension $d = r - r'$ integrable submanifolds which extends to $q^{\ell} = \infty$ as a family of F -manifolds. Furthermore, in a neighborhood of $q^{\ell} = \infty$, this family of F -manifold is isomorphic to the F -manifold given by $QH(X')$.

To give a brief sketch of the proof to Theorem 0.1, we note that a more precise, and slightly stronger, statement is that $\widehat{\Psi}$ induces an affine (but not Frobenius) embedding over $\mathcal{R}'[1/q^{\ell}]$:

$$(0.3) \quad (TH'_{\mathcal{R}'[1/q^{\ell'}]}, \nabla') \xrightarrow{d\widehat{\Psi}} (TH_{\mathcal{R}'[1/q^{\ell'}]}, \nabla)|_{\mathcal{M}} \longrightarrow \mathcal{K} \cong N_{\widehat{\Psi}}.$$

A key step is to prove (0.3) in the special case $\mathbf{s} = 0$ (cf. Proposition 4.5), which implies Theorem 0.1 for $\mathbf{s} = 0$ (cf. Theorem 4.12). For general $\mathbf{s} \in H(X')$, we make use of the local model assumption to get semi-simplicity of \mathcal{M} and $QH(X')$, and then construct the map $\widehat{\Psi}$ by matching the corresponding canonical coordinates (cf. (4.35)).²

While the main results are formulated for the local models, we keep our presentation in theoretic terms whenever possible, with an eye towards future results for global flips. Indeed, a large part of the proofs works for

²By Proposition 4.5, the \mathcal{R}' -point $\sigma_0(q')$, and hence the embedding $\widehat{\Psi}(q', \mathbf{s})$, is unique if we impose also equation (0.3) at $\mathbf{s} = 0$ in the statement of Theorem 0.1. Nevertheless we expect that the uniqueness should hold without this additional constraint.

more general flips with explicit local structures (e.g. toric flips). Nevertheless all of our results are essentially constructive and effective. The explicit frame leading to Birkhoff factorization and the exact form of connection matrices for the Dubrovin connection for $Q_0H(X)$ is given. The explicit algorithm for block-diagonalization leading to the eigenbundle decomposition is also given.

As an example to illustrate the ideas involved in the proof of the main theorem, we give the computational details for the $(2, 1)$ flip in the last section (§6, notably Theorem 6.4, Lemma 6.8, Corollary 6.9, Theorem 6.12).

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1. FROM PICARD–FUCHS TO SMALL \mathcal{D}^z -MODULES

In this section we study projective local models of (r, r') flips. The classical aspect on cohomology is discussed in §1.1. The basic properties of small quantum cohomology are discussed in §1.2 (Picard–Fuchs ideals) and §1.3 (first order PDE system).

1.1. Classical cohomology and correspondence. We have

$$X = P_{P^r}(\mathcal{O}(-1)^{r'+1} \oplus \mathcal{O}), \quad X' = P_{P^{r'}}(\mathcal{O}(-1)^{r+1} \oplus \mathcal{O}).$$

By Leray–Hirsch, the cohomology ring of X has the following presentation

$$H(X) = \mathbb{Z}[h, \xi] / (h^{r+1}, \xi(\xi - h)^{r'+1}),$$

where h (resp. ξ) is the hyperplane class of P^r (resp. $X \rightarrow P^r$). $H(X)$ has rank $R = (r+1)(r'+2)$ with \mathbb{Z} -basis

$$(1.1) \quad h^i(\xi - h)^j, \quad i \in [0, r], j \in [0, r'+1].$$

Such a presentation of basis is called a *canonical presentation*.

Notice that $[Z] = (\xi - h)^{r'+1}$, and for $i \in [0, r]$

$$(1.2) \quad k_i := (h|_Z)^i = h^i(\xi - h)^{r'+1} = (-1)^i(\xi - h)^{r'+i+1}$$

is the class of codimension i linear subspace in Z .

Similar description holds for X' by switching the roles of r and r' :

$$H(X') = \mathbb{Z}[h', \xi'] / (h'^{r'+1}, \xi'(\xi' - h')^{r+1}),$$

which has rank $R' = (r'+1)(r+2)$. We denote a canonical basis by

$$(1.3) \quad T'_{(i,j)} := (\xi' - h')^i h'^j, \quad i \in [0, r+1], j \in [0, r'].$$

It was shown in [8, §2.3] that the Chow motive of X' is a sub-motive of that of X by the correspondence $\Psi = \Phi^* = [\bar{\Gamma}_{f-1}]$ from X' to X , where $\Phi = [\bar{\Gamma}_f]$. Moreover Ψ preserves the Poincaré pairing. Also $\Phi h = \xi' -$

h' , $\Phi\zeta = \zeta'$, and Φ restricts to an isomorphism on the ideal $\zeta.H(X)$ with inverse Ψ . The following lemma summarizes it using the basis elements in (1.1) and (1.2). Denote

$$(1.4) \quad d := R - R' = r - r'.$$

Lemma 1.1. *The kernel K of $\Phi : H(X) \rightarrow H(X')$ is a free abelian group of rank d , generated by*

$$k_i \quad \text{with } i \in [0, d-1].$$

The image of Ψ in degree j is the full $H^{2j}(X)$ if $j \notin [r'+1, r]$. For $j \geq r'+1$, the image in $H^{2j}(X)$ has a basis given by

$$(1.5) \quad T_{(j-i,i)} := \Psi(T'_{(j-i,i)}) = h^{j-i}(\zeta - h)^i + (-1)^{r'-i} k_{j-(r'+1)},$$

where $i \in [0, r']$, and the first term vanishes if $j - i \geq r + 1$.

The pair (Φ, Ψ) leads to an orthogonal splitting of $H(X)$:

$$(1.6) \quad 0 \longrightarrow K \longrightarrow H(X) \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{array} H(X') \longrightarrow 0.$$

Proof. It is clear that K has a basis given by κ_i with dimension $r - i \geq r' + 1$.

For $j \geq r' + 1$ and $i \in [0, r']$, we compute

$$\begin{aligned} \Psi(T'_{(j-i,i)}) &= \Psi((\zeta' - h')^{j-i} h'^i) \\ &= \Psi((\zeta'^{j-i} - \dots + (-1)^{j-i-1} C_{j-i-1}^{d-i} \zeta' h'^{j-i-1} + (-1)^{j-i} h'^{j-i}) h'^i) \\ &= \Psi(\zeta'^{j-i} h'^i - \dots + (-1)^{r'-i} C_{r'-i}^{j-i} \zeta'^{j-r'} h'^{r'}) \\ &= \zeta^{j-i} (\zeta - h)^i - \dots + (-1)^{r'-i} C_{r'-i}^{j-i} \zeta^{j-r'} (\zeta - h)^{r'} \\ &= \zeta^{j-i} (\zeta - h)^i - \dots + (-1)^{r'-i} C_{r'-i}^{j-i} \zeta^{j-r'} (\zeta - h)^{r'} + \\ &\quad \dots + (-1)^{j-i-1} C_{j-i-1}^{j-i} \zeta (\zeta - h)^{j-1} \\ &= (\zeta - h)^i (\zeta - (\zeta - h))^{j-i} - (-1)^{j-i} (\zeta - h)^j \\ &= (\zeta - h)^i h^{j-i} - (-1)^{j-i} (-1)^{j-r'-1} h^{j-r'-1} (\zeta - h)^{r'+1} \\ &= h^{j-i} (\zeta - h)^i + (-1)^{r'-i} k_{j-(r'+1)}. \end{aligned}$$

If $j \in [r'+1, r]$, this is orthogonal to $k_{r-j} = h^{r-j} (\zeta - h)^{r'+1}$ by (1.2). \square

Instead of (1.1), we will use the elements in Lemma 1.1 as our basis.

1.2. Small quantum \mathcal{D} -modules via the Picard–Fuchs systems. The *small quantum cohomology* $Q_0H(X) = (T_0H_{\mathcal{R}}, *_0)$ encodes 3-point invariants by (0.1). The fundamental class axiom and divisor axiom shows that for $\mathbf{t} = \hat{\mathbf{t}} = t^0 T_0 + D \in H^{\leq 2}(X)$ where $D \in H^2(X)$:

$$\langle\langle T_\mu, T_\mu, T^\kappa \rangle\rangle(\hat{\mathbf{t}}) = \sum_{\beta \in NE(X)} q^\beta e^{D \cdot \beta} \langle T_\mu, T_\mu, T^\kappa \rangle_{3, \beta}.$$

Thus we may couple together the Novikov variables and the divisor variables and interpret directional derivatives ∂_D as derivatives in q^{β} 's. The subspace $H^{\leq 2}(X)$ is referred as the *small parameter space* and the product $*_{\dagger}$ is equivalent to $*_0$. Often we write $*_{small}$ to denote either one of them. The coupled variables are especially suitable for applying (generalized) mirror theorems arising from localization techniques.

For a simple (r, r') flip $f : X \dashrightarrow X'$, the local models X and X' are both toric manifolds. The small quantum \mathcal{D} -modules for toric manifolds are generated by the I function which encodes localization data on stable map moduli spaces. The genus zero Gromov–Witten theory can then be constructed from this \mathcal{D} -module $\mathcal{D}^z I$ via the so called BF/GMT procedure. This will be discussed in the next section.

For the moment we focus on $\mathcal{D}^z I$ and study the corresponding GKZ differential system. In the case of iterated projective bundles, the GKZ system reduces to the Picard–Fuchs system which can be written down easily.

We start with the X side. Let $D = t^1 h + t^2 \xi$ be the divisor variable, ℓ and γ be the fiber curves for $Z \rightarrow \text{pt}$ and $X \rightarrow Z$ respectively. Denote by

$$q_1 = q^\ell e^{t^1}, \quad q_2 = q^\gamma e^{t^2}$$

the Novikov variables coupled with the “small parameters”, and $\partial_i = \partial/\partial t^i$.

The I -function is given by

$$(1.7) \quad I = I^X = e^{D/z} \times \sum_{d_1, d_2} q_1^{d_1} q_2^{d_2} \frac{1}{\prod_1^{d_1} (h + mz)^{r+1} \prod_1^{d_2-d_1} (\xi - h + mz)^{r'+1} \prod_1^{d_2} (\xi + mz)}$$

where z is a formal parameter, and the middle factor goes up as

$$(1.8) \quad (\xi - h)^{r'+1} \prod_{m=d_2-d_1+1}^{-1} (\xi - h + mz)^{r'+1} = k_0 \prod_{m=1}^{d_1-d_2-1} (h + mz)^{r'+1}$$

when $d_2 - d_1 < 0$. It is annihilated by the following Picard–Fuchs (box) operators

$$(1.9) \quad \begin{aligned} \square_\ell &:= (z\partial_1)^{r+1} - q_1(z\partial_2 - z\partial_1)^{r'+1}, \\ \square_\gamma &:= (z\partial_2)(z\partial_2 - z\partial_1)^{r'+1} - q_2, \end{aligned}$$

The Novikov variables q^{β} 's can now be ignored since there is no convergence issue in dealing with equations (1.9). Hence we may treat $(q_1, q_2) \in \mathbb{C}^2$ as variables and identify $\partial_i = q_i \partial / \partial q_i$.

Since $r > r'$, the PF system for X

$$\square_\ell I = 0, \quad \square_\gamma I = 0$$

is regular holonomic on \mathbb{C}^2 of rank R .

On the X' side we have similar notions of $D' = s^1 h' + s^2 \xi'$, ℓ' , γ' ,

$$q'_1 = q^{\ell'} e^{s^1}, \quad q'_2 = q^{\gamma'} e^{s^2}$$

and $\partial_{i'} = \partial/\partial s^i$. The I -function is given by

$$(1.10) \quad I' = I^{X'} = e^{D'/z} \times \sum_{d'_1, d'_2} q_1^{d'_1} q_2^{d'_2} \frac{1}{\prod_1^{d'_1} (h' + mz)^{r'+1} \prod_1^{d'_2 - d'_1} (\zeta' - h' + mz)^{r+1} \prod_1^{d'_2} (\zeta' + mz)},$$

where a similar rule as in (1.8) applies to the case $d'_2 - d'_1 < 0$.

The Picard–Fuchs operators for X' which annihilates I' are

$$(1.11) \quad \begin{aligned} \square_{\ell'} &:= (z\partial_{1'})^{r'+1} - q_1' (z\partial_{2'} - z\partial_{1'})^{r+1}, \\ \square_{\gamma'} &:= (z\partial_{2'}) (z\partial_{2'} - a\partial_{1'})^{r+1} - q_2', \end{aligned}$$

and the PF system for X' is

$$\square_{\ell'} I' = 0, \quad \square_{\gamma'} I' = 0.$$

Since $\Phi(t^1 h + t^2 \zeta) = -t^1 h' + (t^1 + t^2) \zeta'$, we have

$$s^1 = -t^1 \quad \text{and} \quad s^2 = t^1 + t^2.$$

Then $z\partial_{1'} = z\partial_2 - z\partial_1$ and $z\partial_{2'} = z\partial_2$. Also

$$(1.12) \quad \Phi(q_1) = 1/q_1', \quad \Phi(q_2) = q_1' q_2'.$$

Lemma 1.2. *The \mathcal{D}^z -module defined by the Picard–Fuchs ideal of X is isomorphic to that of X' over $\mathbb{C}[q_1, q_1^{-1}, q_2] \cong \mathbb{C}[q_1', q_1'^{-1}, q_2']$.*

Proof. The operators in (1.11) for X' , written in the variables on X , are

$$(1.13) \quad \begin{aligned} \square_{\ell'} &:= (z\partial_2 - z\partial_1)^{r'+1} - q_1' (z\partial_1)^{r+1}, \\ \square_{\gamma'} &:= (z\partial_2) (z\partial_1)^{r+1} - q_2'. \end{aligned}$$

Hence $\square_{\ell'} = q_1^{-1} \square_{\ell}$ and $\square_{\gamma'} = z\partial_2 \square_{\ell} - q_1 \square_{\gamma}$. \square

However, the behavior of the PF system on X' is bad. Since $r > r'$, the expression of $\square_{\ell'}$ in (1.11) shows that it has an irregular singularity at $q_1' = 0$. This is also reflected by the analytic behavior of the I functions:

Lemma 1.3. *On the X side, the function I^X is an entire function in q_1 , while on the X' side the function $I^{X'}$ is divergent, hence only formal, in q_1' .*

Proof. The convergence radii can be easily deduced from the explicit formulae above. For X , (1.8) shows that when d_1 is large (with d_2 fixed) there is a $(d_1)!$ factor appearing in the denominator of the coefficient of $q_1^{d_1}$. On the other hand, for X' with large d_1' the $(d_1)!$ factor appears in the numerator. The lemma then follows from the ratio test of convergence. \square

Remark 1.4. In principle we may still go from the PF system for X' to get a *formally regular system* with coefficients in formal series by working on the completion of $\mathbb{C}[[NE(X')]]$ and by applying (1.11) inductively.

1.3. The first order linear system. In general, a *quantized version* of the basis given in Lemma 1.1 allows us to rewrite the higher order PDEs (PF system on X) in terms of systems of first order PDE's (cf. [3, 9])

$$(1.14) \quad (z\partial_i - C_i(q_1, q_2, z))S = 0, \quad i = 1, 2.$$

such that the matrices C_i 's are power-series in q_1, q_2 and z . Here S is the $R \times R$ fundamental solution matrix. In the one variable case this is the standard process to transform an n -th order scalar ODE to a first order system.

In the current local case, we have $c_1(X) = (r - r')h + (r' + 2)\xi$ and X is a Fano manifold. Then C_i 's are indeed polynomials in q_1, q_2 and z .

Remark 1.5. From (1.7), and (1.8), the z degree for $\beta = d_1\ell + d_2\gamma \in NE(X)$ is given by

$$\begin{cases} -((r - r')d_1 + (r' + 2)d_2) & \text{if } d_2 \geq d_1, \\ -((r - r')d_1 + (r' + 2)d_2) - (r' + 1) & \text{if } d_2 < d_1, \end{cases}$$

which is ≤ -3 in all cases. Hence $I = J_{small}$, the small J function on X , and no mirror transform is needed. However, for the full matrix system (1.14) there could still be non-trivial z -dependence if the frame (quantized basis) is chosen incorrectly.

For X' , $c_1(X') = -(r - r')h' + (r + 2)\xi'$ contains both positive and negative directions and the situation is necessarily complicated as explained in Remark 1.4.

The precise determination of C_i 's will be achieved in the next sections. Here we list only the basic properties of them on X .

Lemma 1.6. *As a scalar function in q_1 and q_2 , each entry of C_i is sub-linear. Indeed, the entries of C_i can be written as linear combinations of 1, q_i and q_1q_2 .*

Proof. The only time q_i occurs is when one uses the i -th equation in (1.9). The first relation, which involves $(z\partial_1)^{r+1}$, can be used at most once. For q_2 , the worst case is when one uses the first equation in computing

$$z\partial_1 \left((z\partial_1)^r (z\partial_2 - z\partial_1)^j I \right), \quad j \in [0, r' + 1],$$

which gives a factor of $q_1(z\partial_2 - z\partial_1)^{r+1}$ to the right of $(z\partial_2 - z\partial_1)^j$. One moment's thought concludes that each final resulting monomial can be at most linear in q_1q_2 . \square

Corollary 1.7. *The system (1.14) is regular singular at $q_1 = 0$ and irregular singular of Poincaré rank 1 at $q_1 = \infty$. It is ordinary at any other value of q_1 .*

Proof. Note that $\partial_i = q_i\partial_{q_i}$. Therefore (1.14) is regular singular at $q_1 = 0$. From (1.12), $\Phi(q_1q_2) = q_2'$ and the (sub)-linearity guarantee that it is no worse than rank 1 irregular singularity at $q_1' = 0$ (that is $q_1 = \infty$). \square

Note that

$$(1.15) \quad \deg q_1 = r - r' = d, \quad \deg q_2 = r' + 2, \quad \deg z = 1.$$

Each entry of C_i is then homogeneous in the following sense. Consider the (i, j) -th entry of C_k . Let d_i be the degree of the i -th basis element. Then

Lemma 1.8 (Homogeneity).

$$\deg(C_k)_{ij} = d_j - d_i + 1.$$

Consequently, the highest degree of any entry is $r + r' + 2$. In fact only the $(1, R)$ -th one has this degree.

Proof. $z\partial_k$ increases degree by 1 and $(C_k)_{ij}$ sends the j -th element to the i -th element. \square

Definition 1.9 (Hopf–Möbius stripe). By (1.12), the parameter space M where (q_1, q_2) lies is identified with the total space of $\mathcal{O}_{P^1}(-1)$, which will be called the *Hopf–Möbius stripe*, or the q^ℓ -compactified Kähler moduli.

We rephrase Lemma 1.6 as follows.

Corollary 1.10. *The Picard–Fuchs system defines a meromorphic connection, with parameter z , on a trivial rank R vector bundle over the Hopf–Möbius stripe $M = \mathcal{O}_{P^1}(-1) \rightarrow P^1$, with $q_1 = 1/q'_1$ being the coordinate of the base P^1 .*

The connection is regular singular along the divisor $q_1 = 0$ and irregular singular of Poincaré rank 1 along $q'_1 = 0$. Furthermore, the irregular singularity does not occur in the differentiation in the fiber direction q_2 .

2. THE GW SYSTEM FOR Q_0H

The (small) Dubrovin connection of X , which is a toric Fano manifold, can be written down directly by choosing the quantum frame carefully (cf. Definition 2.8). This gives the Gromov–Witten invariants for two-point primary invariants without starting at the one-point descendent J function.

Since the explicit form of the Dubrovin connection is not strictly necessary, we choose to work in a slightly more theoretic manner which is precise enough to study the eigenvalue functions of h^*_{small} and ξ^*_{small} and to identify the bundle directions leading to irregular singularities near $q^\ell = \infty$, namely the kernel space K (cf. Lemma 2.9).

2.1. Abstract structures of QH . In order to deal with Dubrovin connection in a non-constant frame, which is essential in our proof, we recall some standard structures attached to the quantum cohomology rings.

The following is well-known

Lemma 2.1. *The Dubrovin connection $\nabla^z := d - z^{-1} \sum_i dt^i \otimes T_i^*$ is compatible with the polarized (or Hermitian) pairing: for H -valued Laurent series $a(z), b(z)$,*

$$(2.1) \quad ((a(z), b(z))) := (a(z), \overline{b(z)}) := (a(z), b(-z)).$$

Proof. On one hand

$$\theta := \partial_i(\sum a_j T_j, \sum \bar{b}_k T_k) = \sum((\partial_i a_j) \bar{b}_k + a_j (\partial_i \bar{b}_k))(T_j, T_k)$$

On the other hand,

$$(\nabla_i^z a, \bar{b}) + (a, \overline{\nabla_i^z b}) = \theta + \sum a_j \bar{b}_k \left(\frac{-1}{z} (T_i * T_j, T_k) + \frac{1}{z} (T_j, T_i * T_k) \right) = \theta,$$

where the Frobenius property is used. \square

The Dubrovin connection ∇^z on any constant frame T_i 's (cohomology basis) has its 0-th order operator the matrix A_k of T_k^* , and has the quantum differential equation $z\partial_k z\partial_j J = \sum_i (A_k)_j^i z\partial_i J$. Hence the fundamental solution matrix $S = (S_j) = (S_j^i)$ with $z\partial_k S = A_k S$ is determined by the adjoint relation

$$\left(\sum_i S_j^i T_i, T_k \right) = (T_j, z\partial_k J).$$

That is,

$$S_j^i = (T_j, \sum_k g^{ik} z\partial_k J) = \sum_{k,l} g^{ik} g_{jl} z\partial_k J^l.$$

In terms of a non-constant frame $\tilde{T}_j = \sum_i T_i p_j^i(q, \mathbf{t}, z)$ as power series in q, \mathbf{t} and z , the corresponding fundamental solution matrix Z satisfies $S = PZ$ with $P = (p_j^i)$, and the equation becomes $z\partial_k Z = \tilde{A}_k(q, \mathbf{t}, z)Z$ with

$$(2.2) \quad \tilde{A}_k = -zP^{-1}\partial_k P + P^{-1}A_k P.$$

Remark 2.2. Even if P , or equivalently \tilde{T}_i 's, is independent of z , the connection matrices \tilde{A}_k might still be z -dependent if $P(q, \mathbf{t})$ is not constant in (q, \mathbf{t}) . On the other hand, for a (formal) change of variables $(q, \mathbf{t}) \mapsto (\tilde{q}, \tilde{\mathbf{t}})$ we get a linear change

$$(2.3) \quad A_k \mapsto \tilde{A}_l = \sum_k A_k (\partial t^k / \partial \tilde{t}^l)$$

which is z -independent if A_k 's are. When both operations are performed the connection matrices $\tilde{A}_k(\tilde{q}, \tilde{\mathbf{t}}, z)$'s are usually complicated.

- (1) A typical case for this to occur is the connection matrix obtained from the I function. In that case one uses Birkhoff factorization (BF) to recover the frame \tilde{T}_j to get z -independent connection matrices and use generalized mirror transform (GMT) to recover the change of coordinates if any. This is discussed in §4.
- (2) The block diagonalization/decomposition of connections gives another instance of this construction. This is discussed in §3.

We will study non-constant frames arising from combinations of these two “gauge transformations”.

As a linear map, the matrix of T_k^* in the basis (non-constant frame) \tilde{T}_j 's is given by $P^{-1}A_k P$ instead of \tilde{A}_k . Hence

$$(2.4) \quad T_k * \tilde{T}_j = \sum_i \tilde{T}_i (\tilde{A}_k)_j^i + \sum_i \tilde{T}_i (P^{-1} z\partial_k P)_j^i.$$

In particular, on the $\deg z = 0$ component we get

$$(2.5) \quad T_k * \tilde{T}_j(0) = \sum_i \tilde{T}_i(0) \tilde{A}_{kj}^i(0).$$

In terms of GW invariants we have

$$\tilde{A}_{kj}^i(0) = \sum_l \tilde{g}^{il}(0) \langle\langle T_k, \tilde{T}_j(0), \tilde{T}_l(0) \rangle\rangle = \langle\langle T_k, \tilde{T}_j(0), \tilde{T}^i(0) \rangle\rangle,$$

where

$$(2.6) \quad \tilde{T}^i := \tilde{g}^{il} \tilde{T}_l$$

is the dual frame with respect to the polarized pairing

$$\tilde{g}_{ij} := (\langle\langle \tilde{T}_i, \tilde{T}_j \rangle\rangle).$$

The pairing becomes symmetric when we restrict to $z = 0$.

Remark 2.3. While (2.5) holds for any frame $\{\tilde{T}_j(q, \mathbf{t}, z)\}$, a special frame such that $\tilde{A}_k(q, \mathbf{t}, z)$ is z -independent is of fundamental importance. In fact it is unique up to a constant transformation matching the constant basis $\tilde{T}_j \pmod{NE(X)}$ with the original one. Nevertheless, as we shall see later (cf. (4.9)), non-trivial non-constant frames with z -independent \tilde{A}_k do exist when we consider analytic continuations in certain q variables toward infinities. Of course in that case \tilde{T}_j is not defined near $q = 0$.

2.2. Dubrovin connection on $Q_0H(X)$. The system defined by the (small) Dubrovin connection of X is

$$(2.7) \quad (z\partial_i - A_i(q_1, q_2))S = 0, \quad i = 1, 2,$$

where $A_1 = h*_{small}$ and $A_2 = \zeta*_{small}$ are the matrices defined by the (small) quantum product. Notice the characteristic feature that A_i 's are independent of z by definition.

Lemma 2.4. *For Fano X , A_i 's are polynomial functions in q_1 and q_2 .*

Proof. By (1.15) we know that degrees of q_1 and q_2 are both strictly positive. Therefore, we have the polynomiality in q_j . \square

The (small) Dubrovin connection on X extends meromorphically over the parameter space M , with regular singularity on the fiber divisor $q_1 = 0$ and possibly irregular singularity on the fiber $q_1' = 0$. A detailed determination is given in this subsection.

Before doing so, we first describe the eigenvalue functions $\lambda(q_1, q_2)$ of $h*_{small}$ and $\mu(q_1, q_2)$ of $\zeta*_{small}$ in terms of the Picard–Fuchs system (1.9). It is important to notice that, since h and ζ are of degree one, by definition the eigenvalue functions are also of degree one.

Since X is toric Fano, no mirror transformation is needed for small quantum cohomology and we get

$$(2.8) \quad \lambda^{r+1} = q_1(\mu - \lambda)^{r'+1}, \quad \mu(\mu - \lambda)^{r'+1} = q_2.$$

Then we clearly have a simple relation

$$(2.9) \quad \mu\lambda^{r+1} = q_1q_2,$$

and we may use (2.9) to eliminate μ in (2.8) to get the equation for λ :

$$(2.10) \quad \lambda^{(r+1)(r'+2)} = q_1(q_1q_2 - \lambda^{r+2})^{r'+1}.$$

Since $(r+1)(r'+2) - (r+2)(r'+1) = R - R' = r - r' = d > 0$, all the solutions $\lambda(q_1, q_2)$ are analytic in q_1, q_2 as expected.

It is also clear from (2.9) and (2.10) that the small quantum product on X is generically semi-simple. Since the semi-simplicity is an open condition, we conclude also the generic semi-simplicity for big quantum product.

Remark 2.5. In [6], Iritani proved that the big quantum cohomology of any smooth projective toric variety is convergent and generically semi-simple.

However, under the analytic continuation $x = q'_1 = 1/q_1$, $y = q'_2 = q_1q_2$ to the locus $x = 0$, equation (2.10) becomes

$$(2.11) \quad \begin{aligned} 0 &= x\lambda^{(r+1)(r'+2)} - (y - \lambda^{r+2})^{r'+1} \\ &= x\lambda^R - (-1)^{r'+1}\lambda^{R'} - \sum_{j=1}^{r'+1} (-1)^{r'+1-j} C_j^{r'+1} y^j \lambda^{R'-(r+2)j}. \end{aligned}$$

The leading terms $\lambda^{R'}(x\lambda^d - (-1)^{r'+1})$ lead to the following.

Lemma 2.6. *Near $x = 0$, there are $d = r - r'$ singular eigenvalue functions*

$$\lambda_i(x^{1/d}, y) = \omega^i x^{-1/d} + \dots$$

*of h^*_{small} , where $\omega^d = (-1)^{r'+1}$. The corresponding eigenvalue for ξ^*_{small} is*

$$\mu_i(x^{1/d}, y) = \omega^{-(r+1)j} x^{(r+1)/d} y + \dots$$

We will see that they correspond to the space of vanishing cycles K .

Definition 2.7 (Naive quantization frame). (cf. [9]) We use the notations of naive quantizations when a cohomology class is represented by a product of divisors in a canonical manner. Namely for any divisor D we set $\hat{D} = z\partial_D$ as a directional derivative, and for a class $a = \prod_i D_i^{e_i}$ under the fixed canonical presentation we set $\hat{a} = \prod_i \hat{D}_i^{e_i}$ as a higher order derivative.

It is easy to see that $\hat{a}e^{D/z} = ae^{D/z}$ where $D = \sum t^i D_i \in H^2(X)$ is a general divisor. In particular $\hat{a}I \equiv ae^{D/z} \pmod{NE(X)}$.

Definition 2.8 (The Ψ -corrected quantization frame). The quantized basis corresponding to the kernel of Φ is chosen to be the naive ones

$$(2.12) \quad \kappa_i := \hat{k}_i I = \hat{h}^i (\hat{\xi} - \hat{h})^{r'+1} I,$$

where $i \in [0, d-1]$.

For classes in the image of Ψ , a correction term will be inserted as follows: for $\mathbf{e} = (e_1, e_2)$ with $e_1 \in [0, r+1]$ and $e_2 \in [0, r']$, we define

$$(2.13) \quad v_{\mathbf{e}} := \hat{h}^{e_1} (\hat{\xi} - \hat{h})^{e_2} I + \delta_{(e_1, e_2)} (-1)^{r'-e_2} \hat{k}_{e_1+e_2-(r'+1)} I,$$

The proof of the remaining part of Lemma 2.9 as well as a complete proof of Lemma 2.10 are straightforward computations based on the Picard–Fuchs equations (1.9), similar to the one in (2.14). They are written in §5.1.

Here we emphasize that the difference between C_1^{21} and C_2^{21} on the constant terms is due to the fact that elements in K can not contain the ξ factor.

3. DECOMPOSITION OF Q_0H VIA BLOCK-DIAGONALIZATION

We learned from Lemma 2.9 and Lemma 2.10 that for simple (r, r') flips (with $r > r'$) the Dubrovin connection is irregular of Poincaré rank 1 at $q^\ell = \infty$. Over the Hopf–Möbius strip M , the Dubrovin connection is a system of first order PDE's of two variables $x = q'_1 = 1/q_1$ and $y = q'_2 = q_1q_2$. Recall that $R = \text{rank } H(X)$, $R' = \text{rank } H(X')$, and $d = \text{rank } K = r - r'$ (so that $R = R' + d$). In the q'_1 direction, it takes the form

$$zq'_1 \frac{\partial}{\partial q'_1} S = AS,$$

where S is the fundamental solution matrix and A is the connection matrix of size $R \times R$. A is entire in q_1 but has a simple pole at $q'_1 = 0$.

The solution of ODE of this type was developed by Sibuya, Malgrange, Wasow etc. (cf. [12, 11]), and completed in early 1970's. One key step is to block-diagonalize the matrix A , starting from the worst singularity. It turns out that this “classical” procedure produces an ideal of quantum multiplication generated by K , which is however NOT an ideal in $H(X)$! By the flatness of the Dubrovin connection we may simultaneously block-diagonalize all quantum multiplication matrices.

3.1. Block diagonalization. We have $A_j = C_j$, $j = 1, 2$. From Lemma 2.9,

$$C_1^{22} = \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1}q_1 \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} = \frac{1}{x} \begin{bmatrix} 0 & 0 & \cdots & (-1)^{r'+1} \\ x & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & x & 0 \end{bmatrix}.$$

We will now work on the irregular system of partial differential equations in variables (x, y) with a parameter z . The irregularity comes only from x , and it is thus necessary to keep track of the lowest order entries in x in the connection matrix. By §5.1.1, the only non-zero row in C_1^{21} where the lowest (constant) order entry occurs comes from the first row given in (5.2). For convenience, we *drop the explicit dependence on y* from the notations below when no confusion will likely arise.

A transformation is needed to bring C_1^{22} into its “semisimple” form: let $u = x^{1/d}$, we modify the constant frame in Definition 2.8 to $\{T_i\}$ with

$$(3.1) \quad \{T_i\}_{i=0}^{R'-1} = \{T_e\}, \quad \{T_{R'+i}\}_{i=0}^{d-1} = \{u^i k_i\}_{i=0}^{d-1}.$$

Lemma 3.1 (Shearing). *Let*

$$Y(x) = \text{diag}(1^{R'}, u^0, u^1, \dots, u^{d-1}).$$

The equation

$$(3.2) \quad zx \frac{\partial}{\partial x} S = C_1 S,$$

after the substitutions $S = YW$ and $x = u^d$, becomes

$$(3.3) \quad zu \frac{\partial}{\partial u} W = D_1(u)W,$$

where D_1 can be written in the block form as

$$(3.4) \quad \begin{aligned} D_1^{11} &= d \cdot C_1^{11}, \\ D_1^{12} &= d \cdot C_1^{12} \cdot \text{diag}(u^0, u^1, \dots, u^{d-1}), \\ D_1^{21} &= d \cdot \text{diag}(u^0, u^{-1}, \dots, u^{-d+1}) \cdot C_1^{21}, \\ D_1^{22} &= \frac{d}{u} \cdot \begin{bmatrix} 0 & 0 & \dots & (-1)^{r'+1} \\ 1 & -z \frac{1}{d} u & \dots & 0 \\ & \ddots & \ddots & \\ 0 & \dots & 1 & -z \frac{d-1}{d} u \end{bmatrix}. \end{aligned}$$

Furthermore, D_1^{21} is a power series in u . Thus, (3.3) is irregular of Poincaré rank 1 in u , and the irregular part only appears in the (2, 2) block.

Proof. The computation of the sheared connection matrix D is straightforward. The last assertion about the regularity of D_1^{21} follows from Lemma 2.9 that the constant term of C_1^{21} only appears in the first row, with other entries being zero. This concludes the proof. \square

Remark 3.2. For the equation related to C_2 , $zy \frac{\partial}{\partial y} S = C_2 S$, we note that C_2 is holomorphic in x and y . After the shearing the equation becomes

$$(3.5) \quad zy \frac{\partial}{\partial y} W = D_2 W,$$

such that $D_2^{21} = d \text{diag}(u^0, u^{-1}, \dots, u^{-(d-1)}) C_2^{21}$. By Lemma 2.10 the non-trivial entries in C_2^{21} must divide $q_2 = xy$. Therefore, D_2 is still holomorphic in u and y .

We note that the lowest degree term of D_1 in u is of the (block) form

$$\begin{bmatrix} 0 & 0 \\ 0 & D_1^{22}(0) \end{bmatrix} \frac{du}{u}$$

such that

$$(3.6) \quad D_1^{22}(0) = \begin{bmatrix} 0 & 0 & \dots & (-1)^{r'+1} \\ 1 & 0 & \dots & 0 \\ & \ddots & & \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

Therefore, $D_1(0)$ has R eigenvalues, including 0 with multiplicity R' and d distinct nonzero eigenvalues coming from $D_1^{22}(0)$. The latter group consists of d distinct solutions of $\omega^d = (-1)^{r'+1}$. (As we have seen in Lemma 2.6.)

Following the classical procedure as in [12], together with the flatness of the Dubrovin connection, we conclude that

Proposition 3.3. *The connection matrices C_1 and C_2 can be simultaneously block diagonalized, such that the $(2, 2)$ blocks is completely diagonalized.*

Furthermore, the block-diagonalization frame $\{\tilde{T}_i\}_{i=0}^{R'-1}$ can be chosen so that \tilde{T}_i has the initial term T_i in u . Consequently the bundle \mathcal{T} spanned by \tilde{T}_i with $i \in [0, R' - 1]$ and \mathcal{K} spanned by \tilde{T}_j with $j \in [R', R - 1]$ are orthogonal to each other.

Proof. Since the $(1, 1)$ block and $(2, 2)$ block do not share any eigenvalues, the block diagonalization is possible. The complete diagonalization of the $(2, 2)$ block follows from the fact that all eigenvalues of $D_1(0)$ are different in the $(2, 2)$ block.

As explained also in Remark 3.2, we can use the same shearing transformation matrix for C_1 and C_2 . We need to simultaneously diagonalize the sheared counterparts (i.e. the $(2, 2)$ blocks) of C_1 and C_2 . This is doable as they form part of the flat connection. The flatness together with suitable boundary condition makes the process possible.

To be precise, let $\tilde{T}_i(u, y, z)$ be the frame leading to block diagonalization for ∇_1 such that \tilde{T}_i has T_i as the initial term. Then for $i \in [0, R' - 1]$,

$$\nabla_1 \tilde{T}_i = \sum_{j=0}^{R'-1} e_{1i}^j \tilde{T}_j$$

for some power series $e_{1i}^j(u, y, z)$ and $\tilde{T}_i(0, y, z) = T_i$.

We claim that the sub-bundle \mathcal{T} spanned by \tilde{T}_i , $i \in [0, R' - 1]$ is also closed under ∇_2 , i.e. $\nabla_2 \tilde{T}_i \in \mathcal{T}$. For the initial value along $u = 0$ we have $\nabla_2 \tilde{T}_j(0, y, z) = \nabla_2 T_j$. By Lemma 2.10, the block C_2^{21} vanishes since $q_2 = xy = 0$ along $u = 0$. Hence $\nabla_2 T_j \in \mathcal{T}$. Now

$$\nabla_1(\nabla_2 \tilde{T}_i) = \nabla_2 \nabla_1 \tilde{T}_i = \sum_{j=0}^{R'-1} (\partial_2 e_{1i}^j) \tilde{T}_j + \sum_{j=0}^{R'-1} e_{1i}^j (\nabla_2 \tilde{T}_j).$$

The uniqueness theorem of ODE in u then implies that $\nabla_2 \tilde{T}_i \in \mathcal{T}$.

The bundle \mathcal{T}^\perp under the pairing (2.1) is closed under ∇_i , a fact which follows from Lemma 2.1 easily. Indeed for all $v \in \mathcal{T}$ and $w \in \mathcal{T}^\perp$, we have

$0 = \partial_i((v, w)) = ((\nabla_i v, w)) + ((v, \nabla_i w)) = ((v, \nabla_i w))$. Hence $\nabla_i w \in \mathcal{T}^\perp$. This in particular implies that $\mathcal{K} = \mathcal{T}^\perp$.

The proof that $\nabla_2 \tilde{T}_j$ is proportional to \tilde{T}_j for $j \in [R', R - 1]$ is similar and thus omitted. \square

Remark 3.4. The initial term of the frame which leads to the block diagonalization is the starting frame in (3.1). The initial terms of the frame further diagonalizes the $(2, 2)$ block corresponds to the eigenvectors of $D_1^{22}(0)$ in (3.6) under the starting frame. Let K_j be the eigenvector with eigenvalue ω_j where $\lambda^d - (-1)^{r'+1} = \prod_{j=0}^{d-1} (\lambda - \omega_j)$. Then it is easy to see that

$$(3.7) \quad K_j = \sum_{i=0}^{d-1} \omega_j^{-i} u^i k_i, \quad j \in [0, d-1].$$

Proposition 3.5. *After the block-diagonalization, the $(1, 1)$ block of equation (3.3) can be written in terms of x , instead of $u = x^{1/d}$, and we get*

$$zx \frac{\partial}{\partial x} Z = \tilde{C}_1^{11} Z, \quad zy \frac{\partial}{\partial y} Z = \tilde{C}_2^{11} Z,$$

where \tilde{C}_j^{11} 's are power series in x, y and z .

Proof. We will concentrate on \tilde{C}_1 where most of the action happens. Then the question is essentially reduced to an ODE in variable u , with y, z acting as parameters. In the following the dependence on y is mostly suppressed since it does not participate in the formal process involving u and D_1 . The steps involved are to apply the algorithm described in [12, §11].

For notational convenience we rewrite (3.3) as follows

$$(3.8) \quad zu^2 \frac{\partial}{\partial u} W = \bar{D}(u, z)W,$$

where

$$\bar{D} = \sum_{l=0}^{\infty} \bar{D}_l u^l$$

as a matrix-valued power series in u . In particular, the subscripts now stand for the exponent of power series for the duration of this proof (and the 1 and 2 of D is temporarily suppressed). Similarly, let

$$P(u) = \sum_{l=0}^{\infty} P_l u^l$$

with $P_0 = I$ and P_l being off-block-diagonal for $l > 0$. Now we perform a gauge transformation

$$W = PZ$$

with new frame

$$(3.9) \quad (\tilde{T}_0, \dots, \tilde{T}_{R-1}) = (T_0, \dots, T_{R-1})P$$

to equation (3.8), aiming to get the connection matrix in the block diagonalized form. That is,

$$(3.10) \quad zu \frac{\partial}{\partial u} Z = \frac{E}{u} Z, \quad \text{with} \quad E^{12} = 0 = E^{21}.$$

By writing

$$E = \sum_{l=0}^{\infty} E_l u^l,$$

this is equivalent to solving the following system of algebraic equations inductively [12]:

$$\begin{aligned} E_l^{11} &= -H_l^{11}, \\ E_l^{22} &= -H_l^{22}, \\ P_l^{12} &= -H_l^{12} (\bar{D}_0^{22})^{-1}, \\ P_l^{21} &= (\bar{D}_0^{22})^{-1} H_l^{21}, \end{aligned}$$

where

$$H_l := \sum_{s=1}^{l-1} P_s E_{l-s} - \sum_{s=0}^{l-1} \bar{D}_{l-s} P_s - z(l-1)P_{l-1}$$

is determined by P_s and E_s for $s \leq l-1$. Note that $\bar{D}_0^{ij} = 0$ unless $(ij) = (22)$.

Now we can use these equations and the facts that the off-diagonal blocks of \bar{D} have the specific form to perform the induction. It is enough to show that H_l^{11} is in powers of $x = u^d$ only. Note that \bar{E} has vanishing off-diagonal blocks and P has vanishing diagonal blocks (except for $P_0 = I$). We see from the above that, for $l \geq 1$,

$$H_l^{11} = -\bar{D}_l^{11} - \sum_{s=0}^{l-1} \bar{D}_{l-s}^{12} P_s^{21} = -(\bar{D}^{11} + \bar{D}^{12} P^{21})_l,$$

since $E^{21} = 0 = \bar{D}_0^{12}$ and $P_{>0}^{11} = 0$. Now note that

$$\begin{aligned} u^{-1} \bar{D}^{12} &= d \cdot C_1^{12}(x) \cdot \text{diag}(1, u^1, \dots, u^{d-1}), \\ u^{-1} \bar{D}^{21} &= d \cdot \text{diag}(1, u^{-1}, \dots, u^{-(d-1)}) \cdot C_1^{21}(x). \end{aligned}$$

Inductively, it can be shown that

$$P^{21} = \text{diag}(1, u^{-1}, \dots, u^{-(d-1)}). (\text{matrix function in } x).$$

Thus, $\text{diag}(1, u^1, \dots, u^{d-1})$ is always cancelled by its inverse in the (11) block. Since \bar{D}^{11}/u depends only in x , we conclude that \tilde{C}_1^{11} is a power series in x (and y, z).

The proof for C_2 is simpler and hence omitted. \square

3.2. Decomposition of small quantum rings. The quantum product $*$ in this subsection is assumed to be the small quantum product on $H(X)$.

The Dubrovin connection is flat and the connection matrices C_1 and C_2 are simultaneously block-diagonalized to \tilde{C}_1 and \tilde{C}_2 respectively. Since h^* and $(\xi - h)^*$ generate the quantum ring, which is commutative, we conclude that C_1, C_2 generate the matrix C_μ for T_μ^* and \tilde{C}_1, \tilde{C}_2 induce block-diagonalization of all C_μ 's, i.e. the entire small quantum ring, to \tilde{C}_μ 's.

Indeed for $a, b \in H(X)$ we have $ab = a * b + \sum_\beta q^\beta c_\beta$ for some $c_\beta \in H(X)$. Hence by induction on the Mori cone we conclude that h^* and ξ^* generate the small quantum algebra over the Novikov ring. Namely

$$(3.11) \quad T_\mu^* = \sum_{\beta \in NE(X)} q^\beta P_\beta(h^*, \xi^*)$$

where P_β is a polynomial. Since X is Fano (cf. Lemma 2.4), (3.11) is actually a finite sum. The top degree term P_0 is the cup product expression for T_μ .

In particular the block diagonalization under variables $u = x^{1/d}, y, z$ extends to all T_μ^* . Moreover, it follows from Proposition 3.5 that all the corresponding $(1, 1)$ blocks are still expressible in terms of x, y and z .

Nevertheless, two issues needs to be taken care in details:

- (i) Remove the z -dependence introduced in the block-diagonalization to interpret the product structure correctly.
- (ii) Identify the ground ring where the construction works. Since T_μ^* is generated by h^* and ξ^* over $NE(X)$ instead of over $NE(X')$, the $(1, 1)$ block of \tilde{C}_μ might contains negative powers in x even if T_μ is in the image of Ψ .

Denote the frame leading to the block diagonalization by

$$\mathcal{F} = \{\tilde{T}_0, \dots, \tilde{T}_{R'-1}, \tilde{K}_0, \dots, \tilde{K}_{d-1}\}$$

which further diagonalizes the $(2, 2)$ blocks. Let \mathcal{K} be the sub-bundle generated by $\{\tilde{K}_0, \dots, \tilde{K}_{d-1}\}$. The frame $\{\tilde{T}_0, \dots, \tilde{T}_{R'-1}\}$ is also a frame of

$$\mathcal{T} = \mathcal{K}^\perp,$$

the orthogonal sub-bundle with respect to the polarized pairing (cf. Lemma 2.1). By §3.1, \mathcal{F} is defined in variables $u = x^{1/d}$ and y, z . For convenience we denote the corresponding cyclic extension of the Novikov ring \mathcal{R}' by

$$\tilde{\mathcal{R}}' := \mathcal{R}'[u]/(u^d - x).$$

Our constructions above are over the ring $\tilde{\mathcal{R}}'[[z]]$.

Denoted by \mathcal{T}_0 and \mathcal{K}_0 the restriction of \mathcal{T} and \mathcal{K} at $z = 0$ respectively. As in §2.1, for an element $f \in \tilde{\mathcal{R}}'[[z]]$ we write $f(0) = f(u, y, z = 0)$.

By (2.5), a simple solution to issue (i) is to restrict to the $z = 0$ slice which we will take in this subsection. A more sophisticated and complete solution needs the machinery of BF/GMT which will be done in the next section.

Issue (ii) is more subtle: let $\{T_\mu\}_{\mu=0}^{R-1} = \{T_e, k_i\}$ be the constant frame constructed in Lemma 1.1. For a class T_m and a divisor D , we have

$$(3.12) \quad D * T_m = D.T_m + \sum_{\mu=0}^{R-1} \langle D, T_m, T^\mu \rangle_+ T_\mu,$$

where $+$ stands for the invariants with non-trivial curve classes. By Lemma 2.9 and 2.10, $\langle D, T_m, T^\mu \rangle_+ \neq 0$ only in the following two cases:

- (1) If $\mu \leq R' - 1$ then $\deg T_\mu < \deg T_m$ (cf. Lemma 1.8). In this case the invariant is a scalar multiple of xy or y .
- (2) If $\mu \in [R', R - 1]$ then $T_\mu = T_{R'} = k_0 = (\zeta - h)^{r'+1}$ and $T_m = T_{R-1} = k_{d-1} = h^{d-1}(\zeta - h)^{r'+1}$. The invariant is $(D.\ell)(-1)^{r'+1}/x$.

With (1) and (2), (3.12) becomes

$$(3.13) \quad \begin{aligned} D * T_m &= D.T_m \\ &+ \sum_{\mu=0}^{R'-1} \langle D, T_m, T^\mu \rangle_+ T_\mu + \delta_{m,R-1} (-1)^{r'+1} \frac{(D.\ell)}{x} k_0, \end{aligned}$$

where the sum can be restricted to the range $\deg T_\mu < \deg T_m$.

Equation (3.13) leads to a recursive formula for $(D.T_m)^*$, hence the polynomial expression of T_μ^* in h^* and ζ^* as in (3.11). For example, we have

Lemma 3.6. *For $j \geq 1$, in the polynomial expression of $(h^j k_{d-1})^*$ in h^* and ζ^* , the terms with singular coefficient arise from*

$$\frac{(-1)^{r'+1}}{x} (h^*)^{j-1} * k_0^* = \frac{(-1)^{r'+1}}{x} (h^*)^{j-1} * ((\zeta - h)^*)^{r'+1} + O(y).$$

For $k_i \in K$, no singular coefficients occur for k_i^* .

Lemma 3.7. *For any $\alpha \in H(X)$, the matrix for $(\zeta.\alpha)^*$ has no singular entries in x, y . Also the $(1, 1)$ block of the matrix \tilde{C}_μ for T_μ^* , $\mu \in [0, R' - 1]$, has no singular entries in x, y . This resolves issue (ii).*

Proof. The first statement follows from (3.13) and induction since $(\zeta.\ell) = 0$. For the second statement, notice that the constant frame T_μ in Lemma 1.1 has the property that whenever there is a correction term by $k_{j-(r'+1)}$ given in (1.5), then T_μ contains the factor ζ . The result follows. \square

Now we may derive the splitting of small quantum rings:

Proposition 3.8. *Let \mathcal{K}_0 be the sub-bundle generated by $\{\tilde{K}_0(0), \dots, \tilde{K}_{d-1}(0)\}$. Then both \mathcal{K}_0 and $\mathcal{T}_0 = \mathcal{K}_0^\perp$ are ideals of $Q_0H(X) \otimes \mathcal{R}'$ and*

$$(3.14) \quad Q_0H(X) \otimes \mathcal{R}' \cong \mathcal{T}_0 \times \mathcal{K}_0 \cong_{\tilde{\mathcal{R}}'} \langle \tilde{T}_0(0), \dots, \tilde{T}_{R'-1}(0) \rangle \times \mathbf{C}^d,$$

where \mathbf{C}^d is the trivial ring consisting of d idempotents.

The second isomorphism is valid only over the extension $\tilde{\mathcal{R}}'$ of \mathcal{R}' .

Proof. There exist $C_{\mu\nu}^\rho$ and distinct eigenvalues $\Lambda_{\mu i}$ and $\Lambda_i \neq 0$ such that

$$(3.15) \quad \begin{aligned} \tilde{T}_\mu(0) * \tilde{K}_i(0) &= \Lambda_{\mu i} \tilde{K}_i(0), & \forall \mu = 0, \dots, R' - 1, \\ \tilde{K}_i(0) * \tilde{K}_j(0) &= \delta_{ij} \Lambda_i \tilde{K}_i(0), & \forall i = 0, \dots, d - 1, \\ \tilde{T}_\mu(0) * \tilde{T}_\nu(0) &= \sum_{\rho=1}^{R'} C_{\mu\nu}^\rho \tilde{T}_\rho(0). \end{aligned}$$

In fact, $\Lambda_{\mu i} = 0$ due to the self-duality of \mathcal{K} and the Frobenius property

$$0 = (\tilde{T}_\mu, \tilde{K}_i * \tilde{K}_j)(0) = (\tilde{T}_\mu * \tilde{K}_i, \tilde{K}_j)(0) = \Lambda_{\mu i} (\tilde{K}_i, \tilde{K}_j)(0)$$

for all i, j, μ . The second equality in (3.15) follows from

$$\Lambda_{ij} \tilde{K}_j(0) = \tilde{K}_i(0) * \tilde{K}_j(0) = \tilde{K}_j(0) * \tilde{K}_i(0) = \Lambda_{ji} \tilde{K}_i(0)$$

and hence $\Lambda_{ij} = \delta_{ij} \Lambda_i$. It also follows that $\epsilon_i := \tilde{K}_i(0) / \Lambda_i$ is an idempotent for each i since $\epsilon_i * \epsilon_j = \delta_{ij} \epsilon_j$.

We need to show that $C_{\mu\nu}^\rho \in \tilde{\mathcal{R}}'$: the block diagonalization gives

$$\tilde{T}_\mu(0)* = T_\mu * + \sum f_\mu^i(u, y) \tilde{K}_i(0)*$$

for some $f_\mu^i \in \tilde{\mathcal{R}}'$. By Lemma 3.6 and (3.7) in Remark 3.7, the matrix for the last term has entries in $\tilde{\mathcal{R}}'$. And by Lemma 3.7, the same holds for $T_\mu*$.

We also need to show that $\epsilon_j \in \mathcal{K}_0$. By Lemma 2.6, the eigenvalue function for $(\tilde{\zeta} - h)*$ on $\tilde{K}_j(0)$, with $K_j = \sum_{l=0}^{d-1} \omega^{-jl} u^l k_l$ being given by (3.7), has its leading terms being

$$\mu_j - \lambda_j = \omega^{-j(r+1)} u^{r+1} y - \frac{\omega^j}{u} = -\frac{\omega^j}{u} (1 - \omega^{-j(r+2)} u^{r+2} y).$$

By (3.15) and (3.7) again, the leading terms of $\Lambda_j(u, y)$ is then given by

$$\left(\sum_{l=0}^{d-1} \omega^{-jl} u^l \frac{\omega^{jl}}{u^l} \right) (-1)^{r'+1} \frac{\omega^{j(r'+1)}}{u^{r'+1}} = d (-1)^{r'+1} \frac{\omega^{j(r'+1)}}{u^{r'+1}}.$$

Hence $1/\Lambda_j \in \tilde{\mathcal{R}}'$ and $\epsilon_j = \tilde{K}_j(0) / \Lambda_j$ is a regular vector field over $\tilde{\mathcal{R}}'$. This shows the splitting of quantum product property at $z = 0$:

$$Q_0 H(X) \otimes \tilde{\mathcal{R}}' \cong \langle \tilde{T}_0(0), \dots, \tilde{T}_{R'-1}(0) \rangle \times \mathbf{C}^{r-r'}.$$

It remains to observe that while the frame $\{\tilde{T}_\mu, \tilde{K}_i\}$ is defined over $\tilde{\mathcal{R}}'$, the bundles \mathcal{T} and \mathcal{K} are actually defined over \mathcal{R}' . the proof is complete. \square

We note that ϵ_j vanishes along the divisor $u = 0$.

4. EXISTENCE OF $\hat{\Psi}$ AS AN F -EMBEDDING OF QH

In the above the quantum product $*$ is performed in $H(X)$. To get the quantum product $*'$ on $H(X')$ we apply the BF/GMT procedure on the $(1, 1)$ blocks \tilde{C}_μ^{11} 's. By Proposition 4.1 below, this produces $QH(X')$ along certain locus $\sigma(\hat{\mathfrak{s}})$ which is a non-linear map over the small parameter space $\hat{\mathfrak{s}} \in H^{\leq 2}(X')$. In particular we get an isomorphism

$$(4.1) \quad \langle \tilde{T}_0(0), \dots, \tilde{T}_{R'-1}(0) \rangle \cong \sigma^* QH(X')$$

in a suitable sense—it is not a ring isomorphism since $\tilde{T}_0(0)$ is not the identity element in \mathcal{T} . Efforts will be paid to modify this isomorphism, or rather the frame \tilde{T}_i 's, to achieve a “ring isomorphism” (cf. Theorem 4.12).

It turns out that the correct category to state this isomorphism for the full (big) quantum cohomology is the category of F -manifolds. This is worked out in §4.3.2. (See in particular Proposition 4.15 and equation (4.35).)

4.1. Birkhoff factorization and generalized mirror transform.

Proposition 4.1. *After the Birkhoff factorization and generalized mirror transformation $\sigma(\hat{\mathfrak{s}}) \in H(X') \otimes \mathcal{R}'$ with $\hat{\mathfrak{s}} \in H^{\leq 2}(X')$, \tilde{C}_1^{11} and \tilde{C}_2^{11} become the corresponding connection matrices for the quantum cohomology on X' along $\sigma(\hat{\mathfrak{s}})$.*

Proof. The Picard–Fuchs (higher order) equations on X have coefficients as polynomials in q_1 and q_2 , and similarly for X' . When restricting the variables to $P_{q_1}^1 \setminus \{0, \infty\}$, by Lemma 1.2, these two systems are equivalent.

The Picard–Fuchs (first order) system on X is entire, with irregular singularity of order 1 at $q_1 = \infty$. What we have done to the Picard–Fuchs system of X is to perform gauge transformations and then block diagonalization to remove the irregular part at $q_1 = \infty$. The regular singular part still satisfies the same PF equations up to gauge transformation.

Since the flat connection of the quantum cohomology of X' is equivalent to the above PF system along the small parameter space $\hat{\mathfrak{s}} \in H^{\leq 2}(X')$ up to Birkhoff factorization (gauge transformation) and generalized mirror transformation $\hat{\mathfrak{s}} \mapsto \sigma(\hat{\mathfrak{s}}) \in H(X')$ matching the initial conditions (due to Iritani and Coates–Givental [1]), the resulting system must be equivalent up to BF and GMT.

After the BF (and GMT), the connection matrices are independent of z . However, the *frame* (in terms of constant vectors in $H(X)$) might still have apparent z dependence. The new frame is to be identified with the constant frames in $H(X')$, which establishes the desired correspondence. \square

Below we review the process of BF/GMT in the current situation aiming at a better understanding of Proposition 4.1 (and the isomorphism (4.1)).

Since the original $C_0 = \text{Id}$ (corresponding to T_0*) on $H(X)$, we have also $\tilde{C}_0^{11} = \text{Id}$ on $H(X')$. So in practice it is sufficient to perform the BF/GMT only on \tilde{C}_a^{11} for $a = 1, 2$.

Let $B_1 = B_1(x, y, z)$ be the BF matrix and set $B_1(0) = B_1(x, y, 0)$. Consider the small parameter

$$\hat{\mathbf{s}} = s^0 T'_0 + s^1 h' + s^2 \zeta' \in H^0(X') \oplus H^2(X').$$

From $\hat{\mathbf{s}} = \Phi \hat{\mathbf{t}} = t^0 + t^1(\zeta' - h') + t^2 \zeta'$ we have identifications

$$s^0 = t^0, \quad s^1 = -t^1, \quad s^2 = t^1 + t^2.$$

Then under the substitution $x = q^{\ell'} e^{s^1}$, $y = q^{\gamma'} e^{s^2}$, the “z-free” matrix

$$(4.2) \quad \begin{aligned} C'_a(\hat{\mathbf{s}}) &= -(z \partial_a B_1) B_1^{-1} + B_1 \tilde{C}_a^{11} B_1^{-1} \\ &= B_1(0) \tilde{C}_{a,0}^{11} B_1(0)^{-1}(x, y), \quad a \in \{0, 1, 2\} \end{aligned}$$

is related to the matrices $A'_\mu(\sigma)$ for $T'_\mu \ast'$ at the generalized mirror point $\sigma = \sigma(\hat{\mathbf{s}}) \in H(X')[[x, y]]$ via

$$(4.3) \quad C'_a(\hat{\mathbf{s}}) = \sum_\mu A'_\mu(\sigma(\hat{\mathbf{s}})) \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}), \quad a = 0, 1, 2.$$

In terms of the connection one form $A' = \sum_\mu A'_\mu d\sigma^\mu$, this is simply $\sigma^* A'$.

To proceed, it is convenient to consider the weight zero variables

$$(4.4) \quad s := zu, \quad t := u^{r+2}y.$$

This is not to be confused with the above flat coordinates s^i and t^i .

Lemma 4.2. *For the \tilde{C}_a^{22} diagonalized block, the “Birkhoff factorization B_2 ” can be found for each 1×1 block by elementary integrations.*

More precisely, there is a weight zero power series $\phi_j(zu, u^{r+2}y)$ for each $j \in [0, d-1]$ such that the frame $\mathbf{K}_j := \phi_j \tilde{\mathbf{K}}_j$ satisfies

$$(4.5) \quad zu \frac{\partial}{\partial u} \hat{\mathbf{K}}_j I = \frac{a_j^\circ(t)}{u} \hat{\mathbf{K}}_j I, \quad zy \frac{\partial}{\partial y} \hat{\mathbf{K}}_j I = \frac{b_j^\circ(t)}{u} \hat{\mathbf{K}}_j I,$$

where a_j° and b_j° are analytic in t with $a_j^\circ(0) = \omega^j$ and $b_j^\circ(t) = \omega^{-j(r+1)}t + \dots$

Indeed, $\lambda_j = a_j^\circ/u$ (resp. $\mu_j = b_j^\circ/u$) is the eigenvalue function of $h_{\ast, \text{small}}$ (resp. $\xi_{\ast, \text{small}}$) with eigenvector $\mathbf{K}_j(u, y, z=0)$.

Proof. Each 1×1 equation is irregular of the form

$$(4.6) \quad zu \frac{\partial}{\partial u} \hat{\mathbf{K}}_j I = \frac{1}{u} a_j(zu, u^{r+2}y) \hat{\mathbf{K}}_j I, \quad zy \frac{\partial}{\partial y} \hat{\mathbf{K}}_j I = \frac{1}{u} b_j(zu, u^{r+2}y) \hat{\mathbf{K}}_j I.$$

In the (s, t) coordinates we write $a_j(s, t) = a_j(0, t) + s \alpha_j(s, t)$ and it is elementary to see that there is a series $\phi_j(s, t)$ such that the equation for $\mathbf{K}_j := \phi_j \tilde{\mathbf{K}}_j$ eliminates $sr_j(s, t)$. Indeed the equation becomes

$$s \frac{\partial}{\partial s} \hat{\mathbf{K}}_j I = \frac{1}{s} a_j(0, t) \hat{\mathbf{K}}_j I + \left(s \frac{\partial}{\partial s} \phi_j + \alpha_j(s, t) \phi_j \right) \hat{\mathbf{K}}_j I = 0$$

and $\phi_j(s, t)$ is solved from the regular equation

$$s \frac{\partial}{\partial s} \phi_j + \alpha_j(s, t) \phi_j = 0.$$

The initial condition $\phi_j(0, t)$ is selected so that

$$(4.7) \quad t \frac{\partial}{\partial t} \phi_j(0, t) + \beta_j(0, t) \phi_j(0, t) = 0,$$

where $b_j(s, t) = b_j(0, t) + s\beta_j(s, t)$.

The compatibility of the system then implies that equation (4.7) holds without setting $s = 0$, which is what we want to prove. The last statement is a general statement about the small quantum product. \square

Remark 4.3. As in the proof of Proposition 3.8, we have $\mathbf{K}_i(0) * \mathbf{K}_j(0) = \delta_{ij} \Lambda_j \mathbf{K}_j(0)$ at $\hat{\mathbf{s}}$ for $\Lambda_j(u, y) = \phi_j \Lambda_j$. The idempotents are

$$(4.8) \quad \epsilon_j(u, y) = \mathbf{K}_j(0) / \Lambda_j(u, y),$$

hence the additional information provided by \mathbf{K}_j lies in (4.5).

Denote $T_i = \Psi T'_i$ as before. We combine the block diagonalization P and Birkhoff factorization $B = B_1 \oplus B_2$ into a single gauge transformation

$$(4.9) \quad G = PB^{-1} = [\tilde{\mathbf{T}}_0, \dots, \tilde{\mathbf{T}}_{R'-1}, \mathbf{K}_0, \dots, \mathbf{K}_{d-1}]$$

with $\tilde{\mathbf{T}}_i$ (resp. \mathbf{K}_j) being the resulting frames on \mathcal{T} (resp. \mathcal{K}) such that

$$\tilde{\mathbf{T}}_i \cong T_i \pmod{NE(X')}.$$

Let $\tilde{\mathbf{T}}^i = \sum_l \mathbf{g}^{il} \tilde{\mathbf{T}}_l$ be the dual frame with respect to the pairing in (2.1):

$$\mathbf{g}_{i\bar{j}} = (\langle \tilde{\mathbf{T}}_i, \tilde{\mathbf{T}}_{\bar{j}} \rangle).$$

Since the connection matrices $C'_a(\hat{\mathbf{s}})$'s in (4.2) are z -free, the (i, j) -th entry is precisely the GW invariant in the frame at $z = 0$:

$$(4.10) \quad \mathbf{T}_i := \tilde{\mathbf{T}}_i(0).$$

Hence

$$(4.11) \quad (C'_a)_j^i(\hat{\mathbf{s}}) = (T_a * \mathbf{T}_j, \mathbf{T}^i)^X(\hat{\mathbf{s}}) \equiv \langle \langle T_a, \mathbf{T}_j, \mathbf{T}^i \rangle \rangle^X(\hat{\mathbf{s}}).$$

Since $H^2(X)$ also generates $H(X)$ via small quantum product, we thus have (by WDVV equations) a slightly stronger vanishing result:

Lemma 4.4. *For any $a \in H(X)$, $\langle \langle a, \mathbf{T}^i, \mathbf{K}_j \rangle \rangle^X(\Psi \hat{\mathbf{s}}) = 0 = \langle \langle a, \mathbf{K}^j, \mathbf{T}_i \rangle \rangle^X(\Psi \hat{\mathbf{s}})$.*

In terms of their (i, j) -th entries, equation (4.2) becomes

$$(4.12) \quad \langle \langle T_a, \mathbf{T}_j, \mathbf{T}^i \rangle \rangle^X(\hat{\mathbf{s}}) = \sum_{\mu} \frac{\partial \sigma^{\mu}}{\partial s^a}(\hat{\mathbf{s}}) \langle \langle T'_\mu, T'_j, T'^i \rangle \rangle^{X'}(\sigma(\hat{\mathbf{s}})).$$

Since $(A'_\mu)_0^i = \delta_{\mu}^i$, by comparing the first column we find

$$(4.13) \quad (C'_a)_0^\mu(\hat{\mathbf{s}}) = \langle \langle T_a, \mathbf{T}_0, \mathbf{T}^\mu \rangle \rangle^X(\hat{\mathbf{s}}) = \frac{\partial \sigma^\mu}{\partial s^a}(\hat{\mathbf{s}}).$$

Also we have $\sigma \equiv \hat{\mathbf{s}}$ modulo $q^{\ell'}$, $q^{\gamma'}$, equation (4.13) then determines $\sigma(\hat{\mathbf{s}})$.

Notice that $\partial \sigma^\mu / \partial s^0 = \delta_0^\mu$, but $\sigma^0(\hat{\mathbf{s}})$ depends on $\hat{\mathbf{s}}$ non-trivially and $\sigma(0) \neq 0$. (See Corollary 6.9 for an explicit example on $\sigma(\hat{\mathbf{s}})$.)

We may also rewrite (4.12) (or rather (4.2) and (4.3)) into its intrinsic form in Dubrovin connections.

Proposition 4.5. *Along $\hat{\mathbf{s}} \in H^{\leq 2}(X')$ we have a canonical isomorphism*

$$(4.14) \quad (\mathcal{T}, \nabla^X |_{\mathcal{T}}) \cong (H(X') \otimes \mathcal{R}', \sigma^* \nabla^{X'})$$

of connections, where $\sigma : H^{\leq 2}(X') \rightarrow H(X') \otimes \mathcal{R}'$ is uniquely determined by (4.14) and the constraint that $\sigma(\hat{\mathbf{s}}) \equiv \hat{\mathbf{s}} \pmod{NE(X')}$.

4.2. Special quantum invariance under the normalized frame.

4.2.1. *Compatibility on quantum products via WDVV.* In order to deduce consequences on quantum products from Proposition 4.5, the following lemma is the starting point.

Lemma 4.6. *The isomorphism in (4.14) is compatible with the small quantum D -module structures. Equivalently the quantum products of divisor classes $T_a := \Psi T'_a$ on X and of classes $\sigma_* T'_a$ on X' are compatible along the small parameter $\hat{\mathbf{s}} \in H^{\leq 2}(X')$.*

Proof. We first notice that

$$(4.15) \quad \sigma_* T'_0 = \sum_{\mu} \partial \sigma^{\mu} / \partial s^0 T'_{\mu} = \sum_{\mu} \delta_0^{\mu} T'_{\mu} = T'_0.$$

Take two divisor classes T'_a, T'_b . Then from the WDVV equations,

$$(4.16) \quad \langle\langle T_a * T_b, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X = \sum_{\lambda} \langle\langle T_a, \mathbf{T}^i, \mathbf{T}_{\lambda} \rangle\rangle^X \langle\langle T_b, \mathbf{T}^{\lambda}, \mathbf{T}_j \rangle\rangle^X.$$

Along the small parameters $\Psi \hat{\mathbf{s}}$, by Lemma 4.4, the sum is non-zero only in the non-kernel indices ($T_{\lambda} \notin K$). By (4.12), the sum then becomes

$$\sum_{\lambda} \left(\langle\langle \sigma_* T'_a, T'^i, T'_{\lambda} \rangle\rangle^{X'} \langle\langle \sigma_* T'_b, T'^{\lambda}, T'_j \rangle\rangle^{X'} \right) (\sigma(\hat{\mathbf{s}})).$$

By the WDVV equations on the X' side we then conclude

$$(4.17) \quad \langle\langle T_a * T_b, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X (\Psi \hat{\mathbf{s}}) = \langle\langle \sigma_* T'_a *' \sigma_* T'_b, T'^i, T'_j \rangle\rangle^{X'} (\sigma(\hat{\mathbf{s}})),$$

where the tangent map σ_* is performed at $\hat{\mathbf{s}}$ and the quantum product on the right-hand-side is on X' at $\sigma(\hat{\mathbf{s}})$.

By induction on $r \in \mathbb{N}$, the equation (4.17) holds for $T_{a_1} * \dots * T_{a_r}$ and $\sigma_* T'_{a_1} * \dots * \sigma_* T'_{a_r}$. The proof is complete. \square

4.2.2. *Pseudo identity and the normalized frame.* Recall that $\mathbf{T}_0 \equiv T_0 = \text{id} \pmod{NE(X')}$. The next step is to transform \mathbf{T}_0 to the identity element (section) e in \mathcal{T} and normalized \mathbf{T}_i 's accordingly.

For $T_k \in K^{\perp}$, by Lemma 3.7 we may represent

$$(4.18) \quad T_k *_{\Psi \hat{\mathbf{s}}} = P_k(h *_{\Psi \hat{\mathbf{s}}}, \zeta *_{\Psi \hat{\mathbf{s}}})$$

where P_k is a polynomial with coefficient in x, y .

Definition 4.7. We define the $\mathbb{C}[[x, y]]$ -valued $R' \times R'$ matrix (\mathcal{J}_k^μ) by

$$(4.19) \quad \sum_{\mu=0}^{R'-1} \mathcal{J}_k^\mu T'_\mu := P_k(\sigma_*(\zeta' - h') *', \sigma_* \zeta' *') *' T'_0,$$

where the quantum product $*'$ is taken at $\sigma(\hat{\mathbf{s}})$. Note that $\mathcal{J}_a^\mu = \partial \sigma^\mu / \partial s^a$ for $a \in \{0, 1, 2\}$ and $\mathcal{J}_k^\mu \cong \delta_k^\mu \pmod{NE(X')}$ for all k . Hence (\mathcal{J}_k^μ) is invertible.

Then by Lemma 4.6, or rather equation (4.17), we have

$$(4.20) \quad \begin{aligned} \langle\langle T_k, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X(\Psi \hat{\mathbf{s}}) &= \langle\langle P_k(h*, \zeta*) * T_0, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X(\Psi \hat{\mathbf{s}}) \\ &= \langle\langle P_k(\sigma_*(\zeta' - h') *', \sigma_* \zeta' *') *' \sigma_* T'_0, T'^i, T'_j \rangle\rangle^{X'}(\sigma(\hat{\mathbf{s}})) \\ &= \sum_{\mu} \mathcal{J}_k^\mu \langle\langle T'_\mu, T'^i, T'_j \rangle\rangle^{X'}(\sigma(\hat{\mathbf{s}})). \end{aligned}$$

Lemma 4.8. *There is a unique element $\mathbf{S}_0 \in \mathcal{T}$ such that $\mathbf{S}_0 * \mathbf{T}_0$ is the identity element (section) e in \mathcal{T} (and so e acts as zero on \mathcal{K}).*

Proof. By our constructions, the structure constants $c_{kj}^i(u, y, z)$ defined by

$$\mathbf{T}_k * \mathbf{T}_j = \sum_i c_{kj}^i \mathbf{T}_i$$

are series in u, y, z . In particular, by writing $\mathbf{S}_0 = \sum_i w^i \mathbf{T}_i$, then \mathbf{S}_0 can be solved explicitly using the relation $\mathbf{T}_0 * \mathbf{T}_j = \sum_i c_{0j}^i \mathbf{T}_i$. Indeed, from (4.8), the identity e in \mathcal{T} is given by

$$e = T_0 - \sum_{j=0}^{d-1} e_j = \sum_i \varphi^i \mathbf{T}_i$$

for some series $\varphi^i(u, y, z)$ in u, y, z . So we need to solve the $R' \times R'$ linear system of equations

$$\sum_{j=0}^{R'-1} c_{0j}^i w^j = \varphi^i, \quad i = 0, 1, \dots, R' - 1.$$

Notice that, by Lemma 3.7 and the property that $B_1 \equiv \text{Id} \pmod{x, y}$,

$$c_{0j}^i = \langle\langle \mathbf{T}_0, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X = \langle\langle T_0, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X + \sum_k f_0^k(u, y) \langle\langle T_k, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X$$

is a series in u, y with $f_0^k(0, 0) = 0$. Also $\langle\langle T_0, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X = \langle\langle \mathbf{T}_j, \mathbf{T}^i \rangle\rangle = \delta_j^i$. This shows that (c_{0j}^i) is invertible and the lemma is proved. \square

Definition 4.9. We call \mathbf{S}_0 the pseudo-inverse of \mathbf{T}_0 , which is the inverse of \mathbf{T}_0 in \mathcal{T} , and we define the *normalized frame*

$$\check{\mathbf{T}}_\mu := \mathbf{T}_\mu * \mathbf{S}_0$$

on \mathcal{T} .

Along $\Psi\hat{s}$, by setting $j = 0$ in (4.20) we find

$$(4.21) \quad \mathbf{T}_0 * T_k = \sum_{\mu} \langle\langle \mathbf{T}_0, T_k, \mathbf{T}^{\mu} \rangle\rangle^X \mathbf{T}_{\mu} = \sum_{\mu} \delta_k^{\mu} \mathbf{T}_{\mu}.$$

Applying \mathbf{S}_0^* to (4.21), Lemma 4.8 then leads to the important

Proposition 4.10 (Basic transformation rule). *For $k \in [0, R' - 1]$, we have*

$$(4.22) \quad T_k = \sum_{\mu=0}^{R'-1} \delta_k^{\mu} \check{\mathbf{T}}_{\mu} \pmod{\mathcal{K}}.$$

In particular, the normalized frame $\check{\mathbf{T}}_{\mu}$ is defined over x, y .

4.2.3. *Special quantum invariance.* With (4.22), then equation (4.20) becomes

$$(4.23) \quad \langle\langle \check{\mathbf{T}}_{\mu}, \mathbf{T}^i, \mathbf{T}_j \rangle\rangle^X(\Psi\hat{s}) = \langle\langle T'_{\mu}, T^i, T'_j \rangle\rangle^{X'}(\sigma(\hat{s})).$$

Lemma 4.11. *With respect to the pairing $\check{\mathbf{g}}_{ij} = (\check{\mathbf{T}}_i, \check{\mathbf{T}}_j)$, the dual frame $\check{\mathbf{T}}^i := \sum_j \check{\mathbf{g}}^{ij} \check{\mathbf{T}}_j$ is given by $\check{\mathbf{T}}^i = \mathbf{T}^i * \mathbf{T}_0$.*

Proof. Indeed,

$$(\mathbf{T}_j * \mathbf{S}_0, \mathbf{T}^i * \mathbf{T}_0) = (\mathbf{T}_j * \mathbf{S}_0 * \mathbf{T}_0, \mathbf{T}^i) = (\mathbf{T}_j, \mathbf{T}^i) = \delta_j^i.$$

Here, the Frobenius property on the pairing is used. \square

Hence for any class a we have

$$(4.24) \quad \begin{aligned} \langle\langle a, \check{\mathbf{T}}_j, \check{\mathbf{T}}^i \rangle\rangle &= (a * \check{\mathbf{T}}_j, \check{\mathbf{T}}^i) = (a * \mathbf{T}_j * \mathbf{S}_0, \mathbf{T}^i * \mathbf{T}_0) \\ &= (a * \mathbf{T}_j * \mathbf{S}_0 * \mathbf{T}_0, \mathbf{T}^i) = (a * \mathbf{T}_j, \mathbf{T}^i) \\ &= \langle\langle a, \mathbf{T}_j, \mathbf{T}^i \rangle\rangle. \end{aligned}$$

Together with (4.23), we arrive at a simple statement:

Theorem 4.12. *Under the $\mathbb{C}[[NE(X')]]$ -linear map*

$$\sum a^i \check{\mathbf{T}}_i \mapsto \sum a^i T'_i,$$

the quantum product on \mathcal{T} at $\Psi\hat{s} \in H^{\leq 2}(X)$ is isomorphic to the quantum product on $H(X')$ at $\sigma(\hat{s}) \in H(X') \otimes \mathbb{C}[[NE(X')]]$. Namely

$$(4.25) \quad \langle\langle \check{\mathbf{T}}_{\mu}, \check{\mathbf{T}}^i, \check{\mathbf{T}}_j \rangle\rangle^X(\Psi\hat{s}) = \langle\langle T'_{\mu}, T^i, T'_j \rangle\rangle^{X'}(\sigma(\hat{s}))$$

for all $0 \leq i, j, \mu \leq R' - 1$.

The “subring”, or rather “ideal”, $(\mathcal{T}, *)$ of $Q_0H(X)$ is not isomorphic to $Q_0H(X')$ since $\sigma(0) \neq 0$ (cf. Corollary 6.9 for contributions from the extremal ray). Nevertheless a standard induction on Mori cone implies that

Corollary 4.13. *The big quantum cohomology $QH(X')$ can be effectively computed from $QH(X)$ through equation (4.25).*

4.3. Non-linear reconstructions. In this subsection we will complete the proof of Theorem 0.1 by constructing the embedding $\widehat{\Psi}$ with the imposed properties.

4.3.1. *Remarks on reconstructions over the big parameter spaces.* The complication in dealing with the GMT $\widehat{\mathbf{s}} \mapsto \sigma(\widehat{\mathbf{s}})$ lies on the fact that it is a graph over the small parameters instead of an invertible transformation. The basic idea to resolve the problem is to apply suitable reconstruction theorems on X and X' respectively and to study the compatibility between them.

When the total cohomology H is generated by H^2 under cup product, the reconstruction from 3-point genus zero GW invariants to all n -point genus zero invariants follows from the WDVV equations as done by Kontsevich–Manin. Under the same condition, a version in the setup of abstract quantum \mathcal{D} -modules was formulated and carried out by Iritani in [6, Theorem 4.9], which says that the “abstract big QDM” is naturally determined by the “abstract small QDM”. The abstract version is suitable in our current context since it does not require inductions on the Mori cone.

To trace the reconstruction procedure in all directions of H' consistently, we set $\widehat{\mathbf{s}} = 0$ and keep only the Novikov variables $\{q^{l1}, q^{l2}\} = \{q^{\ell'}, q^{\gamma'}\}$ in equation (4.25) as the starting point. Namely we *decouple* the roles played by $\widehat{\mathbf{s}}$ and $q^{a'}$'s back to the status they are in the definition in (0.1).

Denote the resulting frames by $\mathbb{T}_i(q')$ and let $\sigma_0 = \sigma_0(q')$ be the generalized mirror point at $\widehat{\mathbf{s}} = 0$. Equation (4.12), via (4.24), takes the form

$$(4.26) \quad \langle T_a, \mathbb{T}_j, \mathbb{T}^i \rangle^X = \sum_{\mu} \left(\delta_a^{\mu} + q^{a'} \frac{\partial \sigma_0^{\mu}}{\partial q^{a'}} \right) \langle \mathbb{T}'_{\mu}, T'_j, T'^i \rangle^{X'}(\sigma_0).$$

Here δ_a^{μ} is inserted since $\partial \sigma^{\mu} / \partial s^a \equiv \delta_a^{\mu} \pmod{NE(X')}$. Also (4.25) becomes

$$(4.27) \quad \langle \mathbb{T}_{\mu}, \mathbb{T}_j, \mathbb{T}^i \rangle^X = \langle \mathbb{T}'_{\mu}, T'_j, T'^i \rangle^{X'}(\sigma_0).$$

We regard (4.26) as the connection matrix $A_a(q', \mathbf{s})$ at $\mathbf{s} = 0$ for

$$z \nabla_a = z \partial_a - A_a \equiv z q^{a'} \frac{\partial}{\partial q^{a'}} - A_a,$$

and (4.27) as the connection matrix $\Omega_{\mu}(q', \mathbf{s})$ at $\mathbf{s} = 0$ for

$$z \nabla_{\mu} = z \partial_{\mu} - \Omega_{\mu} \equiv z \frac{\partial}{\partial s^{\mu}} - \Omega_{\mu}.$$

Notice that the coordinates $s^0, \dots, s^{R'-1}$ are centered at σ_0 when viewing on the $H(X')$ side and centered at 0 on the $H(X)$ side. Also while the matrices A_a and Ω_{μ} are *identically the same* for X and X' , their meaning in quantum product are taken in completely different manners.

The flatness of ∇ is equivalent to the WDVV equations

$$(4.28) \quad \begin{aligned} [A_a, A_b] &= [A_a, \Omega_{\mu}] = [\Omega_{\mu}, \Omega_{\nu}] = 0, \\ \partial_a A_b &= \partial_b A_a, \quad \partial_a \Omega_{\mu} = \partial_{\mu} A_a, \quad \partial_{\mu} \Omega_{\nu} = \partial_{\nu} \Omega_{\mu}. \end{aligned}$$

Consider the ideal $\mathfrak{m} = (s^0, s^1, \dots, s^{R'-1})$. By induction on $k \in \mathbb{N}$, we may

- (i) solve $A_a(q', \mathbf{s}) \pmod{\mathfrak{m}^k}$ from $\partial_\mu A_a = \partial_a \Omega_\mu \pmod{\mathfrak{m}^{k-1}}$, and then
- (ii) solve $\Omega_\mu(q', \mathbf{s}) \pmod{\mathfrak{m}^k}$ as a polynomial in $\Omega_j(q', \mathbf{s})$'s $\pmod{\mathfrak{m}^{k-1}}$ and A_a 's $\pmod{\mathfrak{m}^k}$.

The starting case $k = 1$ for (ii) is essentially Theorem 4.12. The relevant formulas are (4.19) and (4.20) used in the proof of Lemma 4.8. Indeed, let

$$\mathcal{J}(q') = (\mathcal{J}_\mu^k) := \mathcal{J}^{-1}$$

be the inverse matrix of (\mathcal{J}_μ^k) which depends only on q' 's. Then at σ_0 ,

$$\begin{aligned} T'_\mu *' &= \sum_k \mathcal{J}_\mu^k P_k(\sigma_*(\zeta' - h')*' , \sigma_* \zeta' *') \\ (4.29) \quad &= \sum_k \mathcal{J}_\mu^k P_k(A_2 - A_1, A_2), \end{aligned}$$

and $\Omega_\mu(q', 0)$ is given by (4.25) via (4.22).

Thus it remains to understand the geometric meanings on both sides under the WDVV reconstruction. On X' this is standard and it leads to

$$(\Omega_\mu)_j^i(q', \mathbf{s}) = \langle\langle T'_\mu, T_j, T^i \rangle\rangle^{X'}(\sigma_0 + \mathbf{s}).$$

In particular $(\Omega_\mu)_0^i(q', \mathbf{s}) = \delta_\mu^i$ since T'_0 is the identity.

On X the reconstruction is *not linear*—in each step of (ii) the identity section $\mathbb{T}_0(q', \mathbf{s}) \pmod{\mathfrak{m}^k}$ receives new correction terms. With this modification been done for each k , which is hard, the resulting structure should then lead to deformations of the embedding $\Psi : H(X') \hookrightarrow H(X)$ to certain $\hat{\Psi}(q', \mathbf{s})$ which relates quantum products of X and X' .

When the GW theory under consideration is analytic, alternatively we may view WDVV as a Frobenius integrability condition in the context of integrable distributions and to construct $\hat{\Psi}$ through certain “canonical coordinates”. We will take this approach in the next section, and it is best described in terms of the notion of F -manifolds.

4.3.2. Integrable distribution and the canonical coordinates. Recall that an F -manifold M is a complex manifold equipped with a commutative and associative product structure on each tangent space $T_p M$, such that a WDVV-type integrability condition is forced when p varies. In the context of quantum cohomology, this is simply the structure which remembers the quantum product but forgets the metric g_{ij} , and with a coordinate-free form of the WDVV (integrability) equations.

Indeed, viewing the quantum product $*$ as a $(2, 1)$ tensor, Hertling and Manin (cf. [5, Definition 2.8, Theorem 2.14, 2.15]) had shown that the WDVV equations can be rewritten as

$$(4.30) \quad L_{X*Y}* = X * L_Y * + Y * L_X *$$

for any local vector fields X and Y , where L denotes the Lie derivatives. In explicit terms this means that for any local vector fields X, Y, Z, W we have

$$(4.31) \quad \begin{aligned} & [X * Y, Z * W] - [X * Y, Z] * W - [X * Y, W] * Z \\ &= X * [Y, Z * W] - X * [Y, Z] * W - X * [Y, W] * Z \\ & \quad + Y * [X, Z * W] - Y * [X, Z] * W - Y * [X, W] * Z. \end{aligned}$$

To apply it to our flip situation, we denote by \mathcal{K} the irregular eigenbundle and its orthogonal complement $\mathcal{T} = \mathcal{K}^\perp$ the regular eigenbundle which extend the corresponding \mathcal{K} and \mathcal{T} from $\mathfrak{s} = 0$ to the big parameter space.

Lemma 4.14. *Both \mathcal{K} and \mathcal{T} and the irregular/regular decomposition of the big quantum product on $TH_{\mathcal{R}'}$ are defined over the big parameter space $H_{\mathcal{R}'}$ over a punctured neighborhood of $q^{\ell'} = 0$.*

Proposition 4.15. *The regular eigenbundle \mathcal{T} is an integrable distribution of the relative tangent bundle $TH_{\mathcal{R}'}$.*

In particular, the image of $\widehat{\Psi}$ is the integrable submanifold \mathcal{M} (over \mathcal{R}') containing the slice $(q^{\ell'} \neq 0, \mathfrak{t} = 0)$ which contains the classical correspondence when modulo \mathcal{R}' .

Proof. Let X, Z be any two local vector fields valued in $\mathcal{T} = \mathcal{K}^\perp$. Let $Y = e_i$ and $W = e_j$ be two idempotents valued in \mathcal{K} . Since $a * b = 0$ for any $a \in \mathcal{K}$ and $b \in \mathcal{K}^\perp$, (4.31) becomes

$$(4.32) \quad 0 = -X * Z * [e_i, e_j] - \delta_{ij} e_j * [X, Z].$$

Let $i = j$ we conclude that $e_j * [X, Z] = 0$ for all j . Hence $[X, Z] \in \mathcal{K}^\perp$. \square

Remark 4.16. The above proof requires only that \mathcal{K} contains no nilpotent sections, i.e. generically semi-simple. Hence Proposition 4.15 works in the global case as well, though in the formal setting. In the local case all the local models are toric and the analyticity is known (by Iritani), thus the Frobenius theorem needed is the classical one. In the global case we need to invoke the Frobenius theorem in the formal setting.

Now we use the full strength of the local model structure. The quantum product on the Frobenius manifold $H(X') \otimes \mathcal{R}'$ is semi-simple. Deonte by the idempotent vector fields on $H(X') \otimes \mathcal{R}'$ by $v'_0, \dots, v'_{R'-1}$. A well-known result of Dubrovin [2, Main Lemma (3.47)] says that canonical coordinates exist. In our setting, we apply it in a family in q' with center at $\sigma_0(q')$:

Lemma 4.17. *We have $[v'_i, v'_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the corresponding canonical coordinates $u^0, \dots, u^{R'-1}$ satisfying*

$$(u^i(q', \mathfrak{s} = 0)) = \sigma_0(q')$$

and $v'_i = \partial / \partial u^i$ exist.

Dubrovin's result was extended to F -manifolds by Hertling [5, Theorem 2.11]. In our setting, the F -manifold \mathcal{M} is semi-simple (or massive) in the sense that the quantum product on $T_p\mathcal{M}$ for $p \in \mathcal{M}$ is semi-simple. Denote the idempotent vector field be $v_1, \dots, v_{R'}$.

Lemma 4.18. *We have $[v_i, v_j] = 0$ for all $0 \leq i, j \leq R' - 1$. Hence the canonical coordinates $u^0, \dots, u^{R'-1}$ on \mathcal{M} exist in the sense that $v_i = \partial/\partial u^i$.*

We emphasize that we have constructed an analytic family of coordinate systems $(u^0(q', p), \dots, u^{R'-1}(q', p))$ parametrized by $q' \in \mathcal{R}'$. Write

$$(4.33) \quad \mathbb{T}_i(q') = \sum_{j=0}^{R'-1} a_i^j(q') v_j(q', \mathbf{s} = 0)$$

for an invertible $R' \times R'$ matrix $(a_i^j(q'))$. From Theorem 4.12 (or (4.27)), we see easily that the same linear combination passes to the X' side:

Lemma 4.19. *After a possible reordering, we have*

$$(4.34) \quad T'_i = \sum_{j=0}^{R'-1} a_i^j(q') v'_j(\sigma_0(q')),$$

for all $i = 0, \dots, R' - 1$.

Now we may define the map $\hat{\Psi}$ by matching the canonical coordinates. Namely, $\hat{\Psi}(q', \mathbf{s}) \in \mathcal{M}$ is the unique point on \mathcal{M} so that

$$(4.35) \quad u^i(\hat{\Psi}(q', \mathbf{s})) = u^{ii}(q', \mathbf{s}) = u^{ii}(\sigma_0(q')) + \mathbf{s}$$

for $i = 0, \dots, R' - 1$. Since the tangent map $\hat{\Psi}_*$ matches the idempotents

$$\hat{\Psi}_* \frac{\partial}{\partial u^{ii}} = \frac{\partial}{\partial u^i},$$

it induces a product structure isomorphism, and hence an F -structure isomorphism by (4.31). Also along $\mathbf{s} = 0$, by (4.33) and (4.34) we have

$$\hat{\Psi}_* T'_i = \mathbb{T}_i$$

which matches the initial condition along the \mathcal{R}' -axis.

At the beginning $\hat{\Psi}$ exists only locally. But since $H(X')$ is contractible, it exists globally by gluing the local maps. This completes the proof of Theorem 0.1.

5. EXACT DETERMINATION OF THE DUBROVIN CONNECTION

The main purpose in this section is to observe the extremely nice phenomenon that we are able to modify the basis given by the "quantized version" of the basis given in Lemma 1.1 "in a canonical manner" to get the Gromov–Witten system on X directly, without going through the BF/GMT process!

5.1. Dubrovin connection in the Ψ -corrected quantum frame. Now we are going to rewrite the higher order PDEs (PF on X) in terms of systems of first order PDE's

$$(5.1) \quad (z\partial_i - C_i(q_1, q_2))S = 0, \quad i = 1, 2,$$

such that C_i 's are polynomials in q_1, q_2 and are independent of z . Here we think of S as the $R \times R$ fundamental solution matrix. Also we keep the notation C_i 's though we eventually will show that they are precisely A_i 's.

Remark 5.1. Before we perform the calculations, it is important to point out that in reducing the order of differentiations via the PF system, variables q_1, q_2 are created in the middle of a formula. It is clear that $z\partial_i q_j = q_j z\partial_i$ if $i \neq j$. However, for a term like $(z\partial_i)^j q_i$, after commutation we get

$$(z\partial_i)^j q_i = q_i(z\partial_i + z)^j.$$

In this way non-trivial z -dependence occurs, which is not allowed in the matrices C_i 's. The trick to avoid such a situation is to perform commutations only for terms of the form

$$(z\partial_1 - z\partial_2)^j q_1 q_2 = q_1 q_2 (z\partial_1 - z\partial_2)^j.$$

We will see that this is always possible for our choice of quantum basis.

5.1.1. The matrix for $z\partial_1$. We will complete the proof of Lemma 2.9.

For $e_1 + e_2 \in [0, r' - 1]$,

$$z\partial_1 v_{(e_1, e_2)} = v_{(e_1+1, e_2)}.$$

And for the boundary case $e_1 + e_2 = r'$,

$$(5.2) \quad z\partial_1 v_{(e_1, e_2)} = v_{(e_1+1, e_2)} - (-1)^{r'-e_2} K_1.$$

We emphasize here that $r > r'$ is essential for (5.2) to be valid.

Also, if either $e_1 + e_2 \in [r' + 1, r]$, or $e_1 + e_2 \in [r + 1, r + r']$ and $e_1 \leq r$, then we have

$$z\partial_1 v_{(e_1, e_2)} = v_{(e_1+1, e_2)}.$$

For the remaining part, we find

$$\begin{aligned} & (z\partial_1)^{r-r'+i+j+1} z\partial_2 (z\partial_2 - z\partial_1)^{r'-j} I \\ &= (z\partial_1)^{r-r'+i+j+1} (z\partial_2 - z\partial_1)^{r'-j+1} I + (z\partial_1)^{r-r'+i+j+2} (z\partial_2 - z\partial_1)^{r'-j} I \\ &= v_{(r-r'+i+j+1, r'-j+1)} + v_{(r-r'+i+j+2, r'-j)}, \end{aligned}$$

where $1 \leq i, j \leq r' - 1$ and $i + j \leq r' - 1$. Also,

$$\begin{aligned} & (z\partial_1)^{r-r'+i+1} z\partial_2 (z\partial_2 - z\partial_1)^{r'} I \\ &= (z\partial_1)^{r-r'+i+1} (z\partial_2 - z\partial_1)^{r'+1} I + (z\partial_1)^{r-r'+i+2} (z\partial_2 - z\partial_1)^{r'} I \\ &= v_{(r-r'+i+2, r')} \end{aligned}$$

and

$$\begin{aligned}
(z\partial_1)^{r+1}z\partial_2I &= z\partial_2q_1(z\partial_2 - z\partial_1)^{r'+1}I \\
&= q_1q_2I, \\
(z\partial_1)^{r+1}z\partial_2(z\partial_2 - z\partial_1)^iI &= (z\partial_2 - z\partial_1)^i z\partial_2q_1(z\partial_2 - z\partial_1)^{r'+1}I \\
&= (z\partial_2 - z\partial_1)^i q_1q_2I \\
&= q_1q_2(z\partial_2 - z\partial_1)^iI \\
&= q_1q_2v_{(0,i)}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&z\partial_1v_{(r+1,0)} \\
&= (z\partial_1)^{r+2}I + (-1)^{r'}(z\partial_1)^{r-r'+1}(z\partial_2 - z\partial_1)^{r'+1}I \\
&= (z\partial_1)^{r-r'+1}((z\partial_1)^{r'+1} + (-1)^{r'}(z\partial_2 - z\partial_1)^{r'+1})I \\
&= (-1)^{r'}(z\partial_1)^{r-r'+1}z\partial_2((z\partial_2 - z\partial_1)^{r'} - z\partial_1(z\partial_2 - z\partial_1)^{r'-1} + \cdots + (-1)^{r'}(z\partial_1)^{r'})I \\
&= (-1)^{r'}(v_{(r-r'+2,r')} - (v_{(r-r'+2,r')} + v_{(r-r'+3,r'-1)}) + \cdots + (-1)^{r'}q_1q_2I) \\
&= (-1)^{r'}((-1)^{r'-1}v_{(r+1,1)} + (-1)^{r'}q_1q_2I) \\
&= q_1q_2I - v_{(r+1,1)},
\end{aligned}$$

where the elementary formula $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$ is used in deriving the third equality.

Similarly, for $1 \leq i \leq r' - 1$,

$$\begin{aligned}
&z\partial_1v_{(r+1,i)} \\
&= (z\partial_1)^{r+2}(z\partial_2 - z\partial_1)^iI + (-1)^{r'-i}(z\partial_1)^{r-r'+1+i}(z\partial_2 - z\partial_1)^{r'+1}I \\
&= (z\partial_1)^{r-r'+1+i}(z\partial_2 - z\partial_1)^i((z\partial_1)^{r'+1-i} + (-1)^{r'-i}(z\partial_2 - z\partial_1)^{r'+1-i})I \\
&= (-1)^{r'}(z\partial_1)^{r-r'+1+i}z\partial_2(z\partial_2 - z\partial_1)^i((z\partial_2 - z\partial_1)^{r'-i} - \cdots + (-1)^{r'-i}(z\partial_1)^{r'-i})I \\
&= (-1)^{r'}(v_{(r-r'+i+2,r')} - (v_{(r-r'+i+2,r')} + v_{(r-r'+i+3,r'-1)}) + \cdots + (-1)^{r'-i}q_1q_2v_{(0,i)}) \\
&= (-1)^{r'}((-1)^{r'-i-1}v_{(r+i+1,1)} + (-1)^{r'-i}q_1q_2v_{(0,i)}) \\
&= (-1)^i(q_1q_2v_{(0,i)} - v_{(r+i+1,1)}).
\end{aligned}$$

For the last one with $i = r'$, we have

$$\begin{aligned}
&z\partial_1v_{(r+1,r')} \\
&= (z\partial_1)^{r+2}(z\partial_2 - z\partial_1)^{r'}I + (z\partial_1)^{r+1}(z\partial_2 - z\partial_1)^{r'+1}I \\
&= (z\partial_1)^{r+1}(z\partial_2 - z\partial_1)^{r'}(z\partial_1 + z\partial_2 - z\partial_1)I \\
&= q_1q_2(z\partial_2 - z\partial_1)^{r'}I \\
&= q_1q_2v_{(0,r')}.
\end{aligned}$$

Together with the calculations on the K part in (2.14), the proof of Lemma 2.9 is thus complete.

5.1.2. *The matrix for $z\partial_2$.*

It suffices to determine the matrix $C_2(q_1, q_2) - C_1(q_1, q_2)$.

By definition, for $e_1 + e_2 \leq r' - 1$,

$$(z\partial_2 - z\partial_1)v_{(e_1, e_2)} = v_{(e_1, e_2+1)},$$

and for $e_1 + e_2 = r'$,

$$(5.3) \quad (z\partial_2 - z\partial_1)v_{(e_1, e_2)} = v_{(e_1, e_2+1)} - (-1)^{r'-e_2-1}K_1.$$

Note that

$$\begin{aligned} (z\partial_2 - z\partial_1)^{r'+2}I &= z\partial_2(z\partial_2 - z\partial_1)^{r'+1}I - z\partial_1(z\partial_2 - z\partial_1)^{r'+1}I \\ &= q_2 I - K_2. \end{aligned}$$

Then for $e_1 + e_2 \in [r' + 1, 2r' + 1]$ and $e_2 \leq r' - 1$, we have

$$\begin{aligned} &(z\partial_2 - z\partial_1)v_{(e_1, e_2)} \\ &= (z\partial_1)^{e_1}(z\partial_2 - z\partial_1)^{e_2+1}I + (-1)^{r'-e_2}(z\partial_1)^{e_1+e_2-r'-1}(z\partial_2 - z\partial_1)^{r'+2}I \\ &= (z\partial_1)^{e_1}(z\partial_2 - z\partial_1)^{e_2+1}I + (-1)^{r'-e_2-1}(z\partial_1)^{e_1+e_2-r'}(z\partial_2 - z\partial_1)^{r'+1}I \\ &\quad + (-1)^{r'-e_2}q_2(z\partial_1)^{e_1+e_2-r'-1}I \\ &= v_{(e_1, e_2+1)} + (-1)^{r'-e_2}q_2 v_{(e_1+e_2-r'-1, 0)}. \end{aligned}$$

And, for $e_1 + e_2 \in [2r' + 2, r' + r]$ and $e_2 \leq r' - 1$ we have

$$\begin{aligned} &(z\partial_2 - z\partial_1)v_{(e_1, e_2)} \\ &= (z\partial_1)^{e_1}(z\partial_2 - z\partial_1)^{e_2+1}I + (-1)^{r'-e_2}(z\partial_1)^{e_1+e_2-r'-1}(z\partial_2 - z\partial_1)^{r'+2}I \\ &= v_{(e_1, e_2+1)} + (-1)^{r'-e_2}q_2(v_{(e_1+e_2-r'-1, 0)} - (-1)^{r'}K_{e_1+e_2-2r'-1}) \\ &= v_{(e_1, e_2+1)} + (-1)^{r'-e_2}q_2 v_{(e_1+e_2-r'-1, 0)} + (-1)^{e_2-1}q_2 K_{e_1+e_2-2r'-1}. \end{aligned}$$

For the remaining part, if $e_1 \in [1, r' + 1]$ then

$$\begin{aligned} &(z\partial_2 - z\partial_1)v_{(e_1, r')} \\ &= (z\partial_1)^{e_1}(z\partial_2 - z\partial_1)^{r'+1}I + (z\partial_1)^{e_1-1}(z\partial_2 - z\partial_1)^{r'+2}I \\ &= q_2(z\partial_1)^{e_1-1}I \\ &= q_2 v_{(e_1-1, 0)}. \end{aligned}$$

And if $e_1 \in [r' + 2, r + 1]$ then

$$\begin{aligned} (z\partial_2 - z\partial_1)v_{(e_1, r')} &= q_2(z\partial_1)^{e_1-1}I \\ &= q_2 v_{(e_1-1, 0)} - (-1)^{r'}q_2 K_{e_1-1-r'}. \end{aligned}$$

Finally, for $i \in [0, r - r' - 2]$,

$$\begin{aligned} (z\partial_2 - z\partial_1)K_{i+1} &= (z\partial_1)^i(z\partial_2 - z\partial_1)^{r'+2}I \\ &= q_2(z\partial_1)^i I - (z\partial_1)^{i+1}(z\partial_2 - z\partial_1)^{r'+1}I \\ &= q_2 v_{(i, 0)} - \delta_{(i, 0)}(-1)^{r'}q_2 K_{i-r'} - K_{i+2}, \end{aligned}$$

and

$$\begin{aligned}
 (z\partial_2 - z\partial_1)K_{r-r'} &= q_2(z\partial_1)^{r-r'-1}I - (z\partial_1)^{r-r'}(z\partial_2 - z\partial_1)^{r'+1}I \\
 (5.4) \qquad \qquad \qquad &= q_2 v_{(r-r'-1,0)} - \delta_{(r-r'-1,0)}(-1)^{r'}q_2 K_{r-2r'-1} \\
 &\quad + (-1)^{r'}q_1 K_1 - (-1)^{r'}v_{(r+1,0)}.
 \end{aligned}$$

The above calculations determine the matrix $C_2(q_1, q_2) - C_1(q_1, q_2)$, and hence $C_2(q_1, q_2)$, completely. Notice that the only appearance of the monomial q_1 is in (5.4), with all the other entries being either 0, 1, q_2 or q_1q_2 up to sign. In particular, by combining with (2.15) we conclude the proof of Lemma 2.10.

The first calculation in §5.1.1 (resp. §5.1.2) shows that the first column of C_1 (resp. C_2) has the same behavior as $h \cup$ (resp. $\zeta \cup$). Thus no mirror transform is needed and we conclude

Theorem 5.2. *For the local model of simple (r, r') flip $f : X \dashrightarrow X'$, under the Ψ -corrected frame, the Drobrovin connection for $QH(X)$ in the small parameter space are determined by $A_k = C_k$ given in Lemma 2.9 and Lemma 2.10.*

5.2. The degenerate cases: simple flops and simple blow-ups.

5.2.1. *The simple flops: $r = r'$.* The case of flops can be considered as a degenerate case of flips. Most of the discussions for flips will be valid except that some boundary cases need to be taken care carefully.

First of all, Φ induces a group isomorphism $H(X) \cong H(X')$ with inverse $\Phi^{-1} = \Psi$, and with Poincaré pairing preserved. In particular $K = \ker \Phi = 0$. Nevertheless, the Ψ -corrected frame v_e in (2.13) is still well defined with $r' = r$ understood. In particular, using \square_ℓ in (1.9) we find

$$\begin{aligned}
 v_{(r+1,0)} &= (z\partial_1)^{r+1}I + (-1)^r(z\partial_2 - z\partial_1)^{r+1}I \\
 (5.5) \qquad \qquad \qquad &= (1 + (-1)^r q_1^{-1})(z\partial_1)^{r+1}I \\
 &= (-1)^r \mathbf{f}(q_1)^{-1} K_1
 \end{aligned}$$

where the fundamental rational function

$$\mathbf{f}(q) = \frac{q}{1 - (-1)^{r+1}q}$$

appears naturally and $K_1 = (z\partial_1)^{r'+1}I$ is defined in (2.12). The point is that, when $r > r'$ the kernel of Φ exists nontrivially and K_1 is independent of $v_{(r+1,0)}$. The relation (5.5) exists only in the case of flops!

All the calculations done in §5.1.1 about $z\partial_1 v_e$ are valid except that the term K_1 in (5.2) needs to be further substituted by (5.5). Namely for the boundary case $e_1 + e_2 = r$, (5.2) becomes

$$(5.6) \qquad \qquad \qquad z\partial_1 v_{(e_1, e_2)} = v_{(e_1+1, e_2)} - (-1)^{e_2} \mathbf{f}(q_1) v_{(r+1, 0)}.$$

All the calculations done in §5.1.2 about $(z\partial_2 - z\partial_1)v_e$ are also valid as long as we notice that, under the assumption $r = r'$, all the cases with

appearance of K_j actually does not exist except for the K_1 in (5.3). Since K_1 is just treated by (5.5), we thus conclude that

Theorem 5.3. *For the projective local model of simple P^r flops $f : X \dashrightarrow X'$, the Ψ -corrected frame $\{v_e\}$ on X leads to the connection matrices $C_1(q_1, q_2, \mathbf{f})$ and $C_2(q_1, q_2, \mathbf{f})$ such that they are independent of z .*

Indeed they are precisely the Dubrovin connection matrices over the small parameters: $C_1 = A_1$ and $C_2 = A_2$. Moreover, all the monomials in q_1 and q_2 are either 1, q_2 or q_1q_2 up to sign.

The reason that no (generalized) mirror transformation is needed comes from the simple fact that the first columns of C_1 and C_2 has the correct form as the classical product. Since X and X' have isomorphic Picard–Fuchs ideal for small $I = J$ function, and the above construction of A_1, A_2 depends only on the Picard–Fuchs ideal, we get the analytic continuation of Dubrovin connection, along the small parameters, under simple P^r flops.

Example 5.4 (Atiyah flop). For $r = 1$, we get the Ψ -corrected frame

$$\begin{aligned} v_1 &= I, & v_2 &= \hat{h}I, & v_3 &= (\hat{\xi} - \hat{h})I, \\ v_4 &= \hat{h}^2I - (\hat{\xi} - \hat{h})^2I, & v_5 &= \hat{h}(\hat{\xi} - \hat{h})I + (\hat{\xi} - \hat{h})^2I, \\ v_6 &= \hat{h}^2(\hat{\xi} - \hat{h})I + \hat{h}(\hat{\xi} - \hat{h})^2I. \end{aligned}$$

The relation between v_4 and $K_1 = (z\partial_2 - z\partial_1)^2I$ is given by

$$v_4 = -\mathbf{f}^{-1}(z\partial_1)^2I = -q_1\mathbf{f}^{-1}(z\partial_2 - z\partial_1)^2I,$$

where $\mathbf{f} = \mathbf{f}(q_1)$. Then the connection matrices for $z\partial_1$ and $z\partial_2$ are

$$A_1 = \begin{bmatrix} 1 & & q_1q_2 & & & \\ & -\mathbf{f} & q_1^{-1}\mathbf{f} & & q_1q_2 & \\ & & 1 & & & \\ & & & -1 & 1 & \\ & & & & & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & & q_2(1 - q_1) & q_2 & & \\ 1 & & & & q_2 & \\ & 1 & & & & q_1q_2 \\ & 1 & 1 & & & \\ & & & & & 1 \end{bmatrix}.$$

Notice that $v_6 = \hat{h}\hat{\xi}(\hat{\xi} - \hat{h})I = \hat{h}\hat{\xi}^2I - q_1q_2I$ does not come from a naive quantization. The z -independence of C_k 's fails if v_6 is not Φ' -corrected.

5.2.2. Simple blow-ups: $r' = 0$. For the other extreme of degenerate cases $r' = 0$, we actually get a blow-up $f : X = \text{Bl}_p X' \rightarrow X'$ at a point $p \in X$. The structure of simple blow-ups is particularly simple. Notice that now $\xi - h = [Z] \cong P^r$ is the f -exceptional divisor.

Our discussion of Φ' -corrected frame on flips is valid by simply setting $r' = e_2 = 0$ in Definition 2.8. Namely $K = \ker \Phi$ is spanned by

$$(5.7) \quad K_{i+1} := \hat{\kappa}_i I = \hat{h}^i(\hat{\xi} - \hat{h})I, \quad i \in [0, r - 1].$$

And the basis elements corresponding to $\Psi H(X')$ are

$$(5.8) \quad \begin{aligned} v_e &:= \hat{h}^e I + (1 - \delta_{e,0}) \hat{h}^{e-1} (\hat{\xi} - \hat{h}) I, \quad e \in [0, r+1], \\ &= \begin{cases} \hat{h}^0 I = I, & e = 0; \\ \hat{h}^{e-1} \hat{\xi} I, & e \in [1, r+1]. \end{cases} \end{aligned}$$

Example 5.5 (Hirzebruch surface F_1). For $(r, r') = (1, 0)$, we get the Hirzebruch surface $X = F_1 = P_{p1}(\mathcal{O}(-1) \oplus \mathcal{O})$ as the blow-up of $X' = P^2$ at a point $p \in X'$. The Ψ -corrected frame is

$$v_0 = I, \quad v_1 = \hat{\xi} I, \quad v_2 = \hat{h} \hat{\xi} I, \quad K_1 = (\hat{\xi} - \hat{h}) I$$

with Picard–Fuchs operators $\square_\ell = \hat{h}^2 - q_1(\hat{\xi} - \hat{h})$, $\square_\gamma = \hat{\xi}(\hat{\xi} - \hat{h}) - q_2$.

The connection matrices for $z\partial_1$ and $z\partial_2$ are then obtained easily:

$$(5.9) \quad A_1 = \left[\begin{array}{cc|c} & q_1 q_2 & \\ 1 & & \\ \hline & 1 & 1 \\ -1 & & -q_1 \end{array} \right], \quad A_2 = \left[\begin{array}{cc|c} & q_2 & q_1 q_2 \\ 1 & & q_2 \\ \hline & 1 & \\ -q_2 & & \end{array} \right].$$

6. AN EXAMPLE ON SIMPLE $(2, 1)$ FLIP FOR FOURFOLDS

6.1. Dubrovin connection on $H(X)$. Let

$$\hat{t} = t^0 T_0 + t^1 h + t^2 \zeta \in H^0(X) \oplus H^2(X)$$

be the small parameters, $q_1 = q^\ell e^{t^1}$, and $q_2 = q^\gamma e^{t^2}$.

With respect to the basis given in Definition 2.8, namely

$$(6.1) \quad \begin{aligned} v_1 &= I, \\ v_2 &= \hat{h} I, \quad v_3 = (\hat{\xi} - \hat{h}) I, \\ v_4 &= \hat{h}^2 I - (\hat{\xi} - \hat{h})^2 I, \quad v_5 = \hat{h}(\hat{\xi} - \hat{h}) I + (\hat{\xi} - \hat{h})^2 I, \\ v_6 &= \hat{h}^3 I - \hat{h}(\hat{\xi} - \hat{h})^2 I, \quad v_7 = \hat{h}^2(\hat{\xi} - \hat{h}) I + \hat{h}(\hat{\xi} - \hat{h})^2 I, \\ v_8 &= \hat{h}^3(\hat{\xi} - \hat{h}) I + \hat{h}^2(\hat{\xi} - \hat{h})^2 I, \\ v_9 &= K_1 = (\hat{\xi} - \hat{h})^2 I, \end{aligned}$$

we get the Dubrovin connection matrices A_k^v 's on $H(X)$:

$$(6.2) \quad A_1^v(\hat{t}) = h_{*_{t=\hat{t}}} = \left[\begin{array}{cccc|cc} & & & & q_1 q_2 & \\ & & & & & q_1 q_2 \\ 1 & & & & & \\ \hline & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \\ 1 & -1 & & -1 & 1 & \\ \hline & & & & & q_1 \end{array} \right],$$

$$(6.3) \quad A_2^v(\hat{t}) = \zeta_{*t=\hat{t}} = \begin{bmatrix} & & & -q_2 & q_2 & q_1 q_2 & & & & q_2 \\ & & & & & -q_2 & q_2 & & & \\ & & & & & & & & q_1 q_2 & \\ & & & & & & & & & q_2 \\ & & & & 1 & & & & & \\ & & & 1 & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & & 1 & \\ & & & & & & & & & q_2 \end{bmatrix}.$$

Notice that $\ker \Phi$ is spanned by the one dimensional direction

$$K_1 = (z\partial_2 - z\partial_1)^2 I.$$

It is precisely the location where A_1 has a pole at $q_1 = \infty$, i.e. $q_1' = 0$.

Next we study the analytic structure of the Dubrovin connection along the infinity divisor $q_1 = \infty$ on the Hopf–Möbius stripe \mathcal{M} and its relation to the Dubrovin connection on $H(X')$. For this purpose, we use coordinates

$$x := q_1' = 1/q_1, \quad y := q_2' = q_1 q_2.$$

The chain rule implies that

$$y \partial_y = xy \partial_{q_2},$$

and

$$x \partial_x = x(-x^{-2} \partial_{q_1} + y \partial_{q_2}) = \partial_{\zeta-h}.$$

Further simplifications are possible by choosing the basis to be orthogonal. Although this simplification is not strictly necessary, it will however make the structure of the connection more transparent.

Indeed, let $w_i = \sum_j v_j T_{ji}$ with

$$T = \left[\begin{array}{cccccccc|c} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & \frac{1}{2} & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & \frac{1}{2} & & & & \\ & & & & & & 1 & & & \\ & & & & & & & \frac{1}{2} & & \\ & & & & & & & & 1 & \\ \hline & & & & & & & & & 1 \end{array} \right],$$

then we have the underlying topological Poincaré pairing on $H(X)$:

$$(6.4) \quad g_{ij} := (w_i, w_j)^X = \delta_{9, i+j}, \quad 1 \leq i, j \leq 8,$$

and $w_9 = v_9 = K_1$ satisfies $(w_9, w_i)^X = \delta_{9, i}$.

Under this frame w_i 's, we compute the QDE (here $A_k = A_k^w$):

$$(6.5) \quad z(x \partial_x)S = A_1 S = \left[\begin{array}{cccccc|c} & -\frac{1}{2}xy & xy & & & & xy \\ 1 & & & -\frac{1}{2}xy & xy & & \\ & 1 & & \frac{1}{4}xy & -\frac{1}{2}xy & & \\ & & 1 & & & xy & \\ & & & & 1 & -\frac{1}{2}xy & \\ & & & & & & 1 \\ & & & & & & -\frac{1}{2} \\ \hline -\frac{1}{2} & 1 & & & & & xy \\ & & & & & & -1/x \end{array} \right] S,$$

$$(6.6) \quad z(y \partial_y)S = A_2 S = \left[\begin{array}{cccccc|c} & -\frac{1}{2}xy & xy & y & & & xy \\ \frac{1}{2} & & & -\frac{1}{2}xy & xy & & \\ & 1 & & \frac{1}{4}xy & -\frac{1}{2}xy & y & \\ & 1 & 1 & & & xy & \\ & & & 1 & & -\frac{1}{2}xy & \\ & & & 1 & 1 & & \\ & & & & & & \\ & & & & \frac{1}{2} & 1 & \\ \hline & & & & & & xy \end{array} \right] S.$$

The symmetry pattern respects the Poincaré pairing (6.4) due to the Frobenius property $(T_i * T_j, T_k) = (T_j, T_i * T_k)$: for both matrices, the first 8×8 block is symmetric under $(i, j) \mapsto (9 - j, 9 - i)$. For the remaining 9-th column and row, it is symmetric under $(9, j) \mapsto (9 - j, 9)$.

The symmetry patterns can be unified to $(i, j) \mapsto (10 - j, 10 - i)$ if we insert the basis element K_1 in its original natural ordering w_5 instead of w_9 . The reason for moving it to the end of the matrix is merely for the convenience of presentation on the block decomposition of the Dubrovin connection we shall perform.

Notice that the system is irregular along $x = 0$ with Poincaré rank one. Let $S = PZ$ be the formal gauge transformation leading to the block decomposition with respect to the basis w_i 's:

$$z(x \partial_x)Z = E_1 Z,$$

$$z(y \partial_y)Z = E_2 Z.$$

Then a recursive algorithm with respect to the power of x determines P and E_1 as matrices in $\mathbb{Q}[y, z][[x]]$. And then the block decomposition on E_2 follows automatically since the connection is integrable (flat).

The matrix P has the form

$$P(x, y, z) = \begin{bmatrix} 1 & & & g_1 \\ & \ddots & & \vdots \\ & & 1 & g_8 \\ f_1 & \cdots & f_8 & 1 \end{bmatrix},$$

where $P^{11} = I_8$, $P^{22} = I_1$, and we have the new (non-constant) frame

$$(6.7) \quad \begin{aligned} \tilde{w}_i &= w_i + f_i K_1, \quad 1 \leq i \leq 8, \\ \tilde{K}_1 &= K_1 + \sum_{i=1}^8 g_i \tilde{w}_i. \end{aligned}$$

Corollary 6.1. For $1 \leq i \leq 8$,

$$(6.8) \quad f_i(x, y, z) = -g_{9-i}(x, y, -z).$$

Proof. Notice that $((\tilde{w}_i, \tilde{K}_1)) = g_{9-i}(-z) + f_i(z)$. Also the block decomposition leads to decomposition of bundles and connections. By Lemma 2.1,

$$\partial_j((\tilde{w}_i, \tilde{K}_1)) = ((\nabla_j^z \tilde{w}_i, \tilde{K}_1)) + ((\tilde{w}_i, \nabla_j^z \tilde{K}_1)) = 0 = ((\tilde{w}_i, \tilde{K}_1))$$

at $x = 0 = y$, $j = 1, 2$. This implies $((\tilde{w}_i, \tilde{K}_1)) = 0$ as power series in x, y . \square

The polarized Hermitian pairing under the frame \tilde{w}_i 's takes the form

$$\tilde{g}_{i\bar{j}} = ((\tilde{w}_i, \tilde{w}_j)) = \delta_{ij} + f_i \bar{g}_j.$$

Hence the dual frame $\tilde{w}^i = \sum \tilde{g}^{i\bar{j}} \overline{\tilde{w}_j}$ can be determined in explicit terms. For ease of notations, we denote by $i' = 9 - i$ for $i \in [1, 8]$. Hence $w^i = w_{i'}$ and

$$f_i = -\bar{g}_{i'}.$$

Lemma 6.2. We have $D := \det P = 1 - \sum_{i=1}^8 f_i g_i = ((\tilde{K}_1, \tilde{K}_1))$, and the dual frame of \tilde{w}_i 's and \tilde{K}_1 are

$$\tilde{K}^1 = \frac{\overline{\tilde{K}_1}}{D}, \quad \tilde{w}^i = w^i + \frac{f_{i'}}{D} \tilde{K}_1 = w_{i'} + f_{i'} \overline{\tilde{K}_1}.$$

Proof. Only the dual frame property needs to be verified:

$$\begin{aligned} ((\tilde{w}^i, \tilde{K}_1)) &= ((w^i, \tilde{K}_1)) + f_{i'} = \bar{g}_i + f_{i'} = 0, \\ ((\tilde{K}^1, \tilde{w}_j)) &= D^{-1}((\tilde{K}_1, \tilde{w}_j)) = D^{-1}(f_{j'} + \bar{g}_j) = 0, \\ ((\tilde{w}^i, \tilde{w}_j)) &= ((w^i, \tilde{w}_j)) + f_{i'}((\overline{\tilde{K}_1}, \tilde{w}_j)) = \delta_j^i. \end{aligned}$$

Thus $\tilde{w}^{i'}$'s and \tilde{K}^1 form the dual frame. \square

Set

$$f_\bullet = (f_1, \dots, f_8), \quad g^\bullet = (g_1, \dots, g_8)^T.$$

From

$$-z \partial_k P + A_k P = P E_k,$$

the block decomposition is equivalent to

$$(6.9) \quad \begin{aligned} &\begin{bmatrix} A_k^{11} + A_k^{12} f_\bullet & -z \partial_k g^\bullet + A_k^{11} g^\bullet + A_k^{12} \\ -z \partial_k f_\bullet + A_k^{21} + A_k^{22} f_\bullet & A_k^{21} g^\bullet + A_k^{22} \end{bmatrix} \\ &= \begin{bmatrix} E_k^{11} & g^\bullet E_k^{22} \\ f_\bullet E_k^{11} & E_k^{22} \end{bmatrix}. \end{aligned}$$

Here we are using the notations $\partial_1 = x \partial_x$ and $\partial_2 = y \partial_y$.

In particular we get the equation for f_i :

$$\begin{aligned}
 (6.10) \quad z\partial_k f_i &= A_k^{22} f_i + (A_k^{21})_i - \sum_{j=1}^8 f_j (E_k^{11})_{ji} \\
 &= -\frac{\delta_{k1}}{x} f_i + (A_k)_{9i} - \sum_{j=1}^8 \left(f_j (A_k)_{ji} + f_j (A_k)_{j9} f_i \right).
 \end{aligned}$$

Since P, E can be solved recursively, we may reinterpret (6.10) as an inhomogeneous perturbation of the irregular ODE $zx\partial_x h = -x^{-1}h$. Similar observation applies to g_i too.

The first few terms of the formal power series g_i 's are listed below:

$$\begin{aligned}
 g_1 &= -x^2 y (1 + 2zx + 6x^2 z^2 + 24z^3 x^3 + (120z^4 + 5y)x^4 \\
 &\quad + (720z^5 + 63yz)x^5 + (5040z^6 + 642yz^2)x^6 + \dots), \\
 g_2 &= -x^3 y (1 + 4zx + 18z^2 x^2 + 96z^3 x^3 + (600z^4 + 7y)x^4 \\
 &\quad + (4230z^5 + 115yz)x^5 + (35280z^6 + 1448yz^2)x^6 + \dots), \\
 g_3 &= \frac{1}{2} x^3 y (3 + 14zx + 70z^2 x^2 + 404z^3 x^3 + (2688z^4 + 23y)x^4 \\
 &\quad + (20376z^5 + 407yz)x^5 + (173808z^6 + 5454yz^2)x^6 + \dots), \\
 g_4 &= -x^4 y (1 + 7zx + 46z^2 x^2 + 326z^3 x^3 + (2556z^4 + 9y)x^4 \\
 &\quad + (22212z^5 + 192yz)x^5 + \dots), \\
 g_5 &= \frac{1}{2} x^4 y (3 + 23zx + 162z^2 x^2 + 1214z^3 x^3 + (9972z^4 + 29y)x^4 \\
 &\quad + (90180z^5 + 654yz)x^5 + \dots), \\
 g_6 &= -x(1 + zx + 2z^2 x^2 + 6z^3 x^3 + (24z^4 + 3y)x^4 + (120z^5 + 30yz)x^5 \\
 &\quad + (720z^6 + 253yz^2)x^6 + (5040z^7 + 2168yz^3)x^7 + \dots), \\
 g_7 &= \frac{1}{2} x(1 + zx + 2z^2 x^2 + 6z^3 x^3 + (24z^4 + 5y)x^4 + (120z^5 + 54yz)x^5 \\
 &\quad + (720z^6 + 489yz^2)x^6 + (5040z^7 + 4472yz^3)x^7 + \dots), \\
 g_8 &= x^2(1 + 3zx + 11z^2 x^2 + 50z^3 x^3 + (274z^4 + 6y)x^4 + (1764z^5 + 87yz)x^5 \\
 &\quad + (13068z^6 + 986yz^2)x^6 + (109584z^7 + 10803yz^3)x^7 + \dots).
 \end{aligned}$$

All of them are indeed named special generating series. In the following we give explanations for the main sub-series g_i° with the lowest y degree.

The main sub-series in the three series g_1, g_6, g_7 , denoted by $g_i^\circ(s)$ with $s = zx$, are multiple of the factorial series $g(s) = \sum_{n=0}^{\infty} n! s^n$, and $g_2^\circ(s)$ is essentially its derivative $g'(s) = \sum_{n=1}^{\infty} n \cdot n! s^{n-1}$.

The coefficients of g_8° is known as *Stirling numbers of first kind*, which counts the number of $\sigma \in S_{n+1}$ with exactly two cycles. It satisfies $a_0 = 1$,

$$(6.11) \quad a_n = (n+1)a_{n-1} + n!, \quad n \geq 2.$$

Its closed form is simply given by

$$(6.12) \quad a_n = (n+1)! H_{n+1},$$

where $H_n = \sum_{k=1}^n 1/k$ is the harmonic series.

The coefficients of g_4° , treated as $a_1 = 0, a_2 = 1, a_3 = 7$, etc., satisfy

$$a_n = n!(n - H_n)$$

Recursively, $a_n = na_{n-1} + (n-1)(n-1)!$ for $n \geq 1$.

For g_3° , we consider the series

$$g_3 + \frac{1}{2}g_2 = x^3y(1 + 5zx + 26z^2x^2 + 154z^3x^3 + 1044z^4x^4 + 8028z^5x^5 + \dots) \\ + \frac{1}{2}x^7y^2(9 + 177zx + 2558z^2x^2 + \dots) + \dots.$$

The coefficients $1, 5, 26, \dots$ satisfy

$$a_n = (n+1)!(H_{n+1} - 1).$$

Recursively, $a_0 = 0$ and $a_n = (n+1)a_{n-1} + n!$ for $n \geq 1$.

For g_5° , similarly, we consider the series

$$g_5 + \frac{1}{2}g_4 = x^4y(1 + 8zx + 58z^2x^2 + 444z^3x^3 + 3708z^4x^4 + 33984z^5x^5 + \dots) \\ + \frac{1}{2}x^8y^2(11 + 270zx + \dots) + \dots.$$

The coefficients $1, 8, 58, \dots$ satisfy

$$a_n = (n+2)!(H_{n+2} - 2) + (n+1)!.$$

Recursively, $a_0 = 0$ and $a_n = (n+2)a_{n-1} + n \cdot n!$ for $n \geq 1$.

Remark 6.3. The calculations suggest that the only essential power series in zx to be considered is the factorial series g . All the other series arise from standard algebraic operations and/or differentiations on the exponents (Frobenius method) which produces the harmonic series naturally.

It is easy to see that

$$h := xg(-zx) = \sum_{n=0}^{\infty} (-1)^n n! z^n x^{n+1}$$

satisfies the irregular ODE

$$zxh' = -\frac{1}{x}h + 1.$$

This is the simplest inhomogeneous perturbation of the equation

$$(6.13) \quad zxh'_0 = -\frac{1}{x}h_0,$$

whose solution is $h_0 = Ae^{1/(zx)}$. Equation (6.13) is precisely the equation which appears in the kernel part of (6.5), thus we call it the *kernel equation*.

All the series f_i 's and hence g_i 's are all determined through certain perturbations of the kernel equation. It is important to locate the topological or geometric data which determines the perturbation.

For quantum cohomology, the most important part of the frame $\tilde{w}_i = w_i + f_i K_1$ in (6.7) is however its restriction to the slice $z = 0$. This can be determined in a purely algebraic manner:

Theorem 6.4. Denote $f_1(x, y, 0), \dots, f_8(x, y, 0)$ by

$$x^2h_1, xh_2, xh_3, h_4, h_5, x^{-1}h_6, x^{-1}h_7, x^{-2}h_8$$

respectively, and let $t := x^4y$ be the Calabi–Yau variable. Then all h_i 's are power series in t . $X = h_1(t)$ satisfies the 9-th degree polynomial

$$(6.14) \quad F(X) = 1 + X + 6tX^2 + 3t^2X^3 - 2t^3X^5 + 3t^4X^6 + t^6X^9$$

with explicit analytic formula given by Lambert's generalized binomial series

$$(6.15) \quad h_1(t) = -\mathcal{B}_9(t)^6 = -\sum_{n=0}^{\infty} \frac{2}{3n+2} \binom{9n+6}{n} t^n$$

(cf. [4, §5.4, p.201]), which converges in $|t| < 8^8/9^9$.

Moreover, all h_j 's, $j = 2, \dots, 8$, can be explicitly expressed as polynomials in h_1 with degree ≤ 8 , and with coefficients in $\mathbb{Q}(t)$.

Proof. We apply the second equation in (6.10) to the case $k = 2$ and restrict to the case $z = 0$. The derivative term disappears and we arrive at the following non-linear system

$$(6.16) \quad \begin{aligned} h_2 + \frac{1}{2}h_3 + th_1^2 &= 0, \\ h_4 + h_5 + th_1h_2 &= 0, \\ h_5 + th_1h_3 &= 0, \\ h_6 + h_7 - \frac{1}{2}th_1 + th_1h_4 &= 0, \\ h_7 + th_1 + th_1h_5 &= 0, \\ \frac{1}{2}h_8 + th_1 - \frac{1}{2}th_2 + \frac{1}{4}th_3 + th_1h_6 &= 0, \\ h_8 + th_2 - \frac{1}{2}th_3 + th_1h_7 &= 0, \\ -1 + h_3 + h_4 - \frac{1}{2}h_5 + h_1h_8 &= 0. \end{aligned}$$

By viewing t, h_1 as parameters, we may regard (6.16) as a linear system in the 7 unknowns h_2, \dots, h_8 . Since the 8 equations in (6.16) are consistent, we must have

$$\begin{vmatrix} th_1^2 & 1 & \frac{1}{2} & & & & & \\ & th_1 & & 1 & 1 & & & \\ & & th_1 & & 1 & & & \\ -\frac{1}{2}th_1 & & & th_1 & & 1 & 1 & \\ th_1 & & & & th_1 & & 1 & \\ th_1 & -\frac{1}{2}t & \frac{1}{4}t & & & th_1 & & \frac{1}{2} \\ & t & -\frac{1}{2}t & & & & th_1 & 1 \\ -1 & & 1 & 1 & -\frac{1}{2} & & & h_1 \end{vmatrix} = 0.$$

where the constant terms are put in the first column. It is straightforward to compute the determinant to get the polynomial (6.14):

$$t(1 + h_1 + 6th_1^2 + 3t^2h_1^3 - 2t^3h_1^5 + 3t^4h_1^6 + t^6h_1^9) = tF(h_1).$$

Now by Carmer's rule, all h_2, \dots, h_8 can be solved in terms of rational expressions in h_1 (and t). Since $F(X)$ is irreducible, the elementary field extension theory shows that all those rational expressions can be written as polynomials in h_1 with degree ≤ 8 , with coefficients in $\mathbb{Q}(t)$.

It remains to prove (6.15). Once we know the expected expression in the generalized binomial series, the proof becomes a direct substitution as to be shown below. In reality, the expression (6.15) is found by calculating the first few terms from the recursive relations deducing from (6.14) and by an internet search on integer sequences.

We start with the definition and properties of Lambert's generalized binomial series. The general reference on this is [4, §5.4]. For any $s \in \mathbb{R}_{\geq 0}$,

$$\mathcal{B}_s(t) := \sum_{n \geq 0} \binom{sn+1}{n} \frac{1}{sn+1} t^n.$$

Moreover, for all $l \in \mathbb{R}$, taking powers corresponds to *twists*:

$$(6.17) \quad \mathcal{B}_s(t)^l = \sum_{n \geq 0} \binom{sn+l}{n} \frac{l}{sn+l} t^n.$$

It is then easily seen that $\mathcal{B}_s(t)$ satisfies a simple algebraic equation:

$$(6.18) \quad t\mathcal{B}_s(t)^s = \mathcal{B}_s(t) - 1.$$

In our current situation we need only the case $s = 9$. Namely for $b := \mathcal{B}_9(t)$ we have an equation

$$(6.19) \quad tb^9 = b - 1,$$

and we would like to show that $h_1 := -b^6$ satisfies the equation $F(X) = 0$ in (6.14). This is then of course just a simple algebra. Indeed by substituting $X = -b^6$ in (6.14) and making use of (6.19), we get

$$-(b-1)^6 + 3(b-1)^4 + 2b^3(b-1)^3 - 3(b-1)^2 + 6b^3(b-1) - b^6 + 1$$

which reduced to zero as expected. The proof is complete. \square

Proposition 6.5. *All h_2, \dots, h_8 are polynomials in $b := \mathcal{B}_9(t)$ with degree ≤ 8 with coefficients in $\mathbb{Q}[t]$. Explicit formulae are given below:*

$$(6.20) \quad \begin{aligned} h_1 &= -b^6, \\ h_2 &= \frac{1}{2}b^3 - b^4, & h_3 &= b^3, \\ h_4 &= \frac{1}{2}(1+b) - b^2, & h_5 &= -1 + b, \\ h_6 &= -\frac{1}{2}b^7t - b^8t, & h_7 &= b^7t, \\ h_8 &= b^5t. \end{aligned}$$

Proof. The formal part follows from Theorem 6.4, (6.17) and (6.19).

The explicit formulae can be obtained by straightforward yet lengthy manipulations. The table is obtained with help from Mathematica. \square

Remark 6.6. The space spanned by $1, b, b^2$ is not covered by the span of h_4 and h_5 . This *missing direction* deserves further investigation.

Remark 6.7. In terms of generalized hypergeometric series, we have

$$b = \mathcal{B}_9(t) = F\left(\frac{1}{9}, \dots, \frac{8}{9}; \frac{2}{8}, \dots, \frac{8}{8}, \frac{9}{8}; \frac{9}{88}t\right),$$

and $b^l = F\left(\frac{l}{9}, \dots; \frac{l+1}{8}, \dots; \frac{9}{88}t\right)$ is the $(l-1)$ -th shift with $\frac{9}{9}$ and $\frac{8}{8}$ skipped.

It is also possible to determine $f_i(x, y, z)$ without setting $z = 0$. The main idea is to make use of the homogeneity of the Gromov–Witten theory. The natural coordinates system is given by the weight zero variables

$$(6.21) \quad s = zx, \quad t = x^4y.$$

Here t is the Calabi–Yau variable. It follows that

$$t\partial_t = y\partial_y, \quad s\partial_s = x\partial_x - 4y\partial_y.$$

The weight zero normalizations \tilde{h}_i 's of f_i 's are given similarly by

$$x^2\tilde{h}_1, x\tilde{h}_2, x\tilde{h}_3, \tilde{h}_4, \tilde{h}_5, x^{-1}\tilde{h}_6, x^{-1}\tilde{h}_7, x^{-2}\tilde{h}_8.$$

After some elementary manipulations, the the PDE system (6.10) in (s, t) coordinates then reads as, in the t direction:

$$(6.22) \quad \begin{aligned} t\partial_t\tilde{h}_1 &= -s^{-1}(\tilde{h}_2 + \frac{1}{2}\tilde{h}_3 + t\tilde{h}_1^2), \\ t\partial_t\tilde{h}_2 &= -s^{-1}(\tilde{h}_4 + \tilde{h}_5 + t\tilde{h}_2\tilde{h}_1), \\ t\partial_t\tilde{h}_3 &= -s^{-1}(\tilde{h}_5 + t\tilde{h}_3\tilde{h}_1), \\ t\partial_t\tilde{h}_4 &= -s^{-1}(-\frac{1}{2}t\tilde{h}_1 + \tilde{h}_6 + \tilde{h}_7 + t\tilde{h}_4\tilde{h}_1), \\ t\partial_t\tilde{h}_5 &= -s^{-1}(t\tilde{h}_1 + \tilde{h}_7 + t\tilde{h}_5\tilde{h}_1), \\ t\partial_t\tilde{h}_6 &= -s^{-1}(t\tilde{h}_1 - \frac{1}{2}t\tilde{h}_2 + \frac{1}{4}t\tilde{h}_3 + \frac{1}{2}\tilde{h}_8 + t\tilde{h}_6\tilde{h}_1), \\ t\partial_t\tilde{h}_7 &= -s^{-1}(t\tilde{h}_2 - \frac{1}{2}t\tilde{h}_3 + \tilde{h}_8 + t\tilde{h}_7\tilde{h}_1), \\ t\partial_t\tilde{h}_8 &= -s^{-1}(-t + t\tilde{h}_3 + t\tilde{h}_4 - \frac{1}{2}t\tilde{h}_5 + t\tilde{h}_8\tilde{h}_1); \end{aligned}$$

and in the s direction:

$$(6.23) \quad \begin{aligned} s\partial_s\tilde{h}_1 &= -2\tilde{h}_1 + s^{-1}(4\tilde{h}_2 + \tilde{h}_3 + \Phi\tilde{h}_1), \\ s\partial_s\tilde{h}_2 &= -\tilde{h}_2 + s^{-1}(-\frac{1}{2} + 4\tilde{h}_4 + 3\tilde{h}_5 + \Phi\tilde{h}_2), \\ s\partial_s\tilde{h}_3 &= -\tilde{h}_3 + s^{-1}(1 + 4\tilde{h}_5 + \Phi\tilde{h}_3), \\ s\partial_s\tilde{h}_4 &= s^{-1}(-\frac{3}{2}t\tilde{h}_1 + 4\tilde{h}_6 + 3\tilde{h}_7 + \Phi\tilde{h}_4), \\ s\partial_s\tilde{h}_5 &= s^{-1}(3t\tilde{h}_1 + 4\tilde{h}_7 + \Phi\tilde{h}_5), \\ s\partial_s\tilde{h}_6 &= \tilde{h}_6 + s^{-1}(4t\tilde{h}_1 - \frac{3}{2}t\tilde{h}_2 + \frac{3}{4}t\tilde{h}_3 + \tilde{h}_8 + \Phi\tilde{h}_6), \\ s\partial_s\tilde{h}_7 &= \tilde{h}_7 + s^{-1}(3t\tilde{h}_2 - \frac{3}{2}t\tilde{h}_3 + 4\tilde{h}_8 + \Phi\tilde{h}_7), \\ s\partial_s\tilde{h}_8 &= 2\tilde{h}_8 + s^{-1}(-3t + 4t\tilde{h}_3 + 3t\tilde{h}_4 - \frac{3}{2}t\tilde{h}_5 + \Phi\tilde{h}_8), \end{aligned}$$

where $\Phi := 3t\tilde{h}_1 - \tilde{h}_6 + \frac{1}{2}\tilde{h}_7 - 1$.

By induction on the degree of s , it is evident that (6.22), with initial values $\tilde{h}_i(0, t) = h_i(t)$ in Proposition 6.5, determines \tilde{h}_i 's completely. However, this does not give information on the structures. Instead, we make one more normalization with $\tilde{h}_i = L_i$ for $1 \leq i \leq 3$ and

$$(6.24) \quad \tilde{h}_i = tL_i, \quad 4 \leq i \leq 8,$$

and rewrite (6.23) as

$$(6.25) \quad \begin{aligned} s\partial_s L_1 &= -2L_1 + s^{-1}(4L_2 + L_3 + \Phi L_1), \\ s\partial_s L_2 &= -L_2 + s^{-1}\left(-\frac{1}{2} + 4tL_4 + 3tL_5 + \Phi L_2\right), \\ s\partial_s L_3 &= -L_3 + s^{-1}(1 + 4tL_5 + \Phi L_3), \\ s\partial_s L_4 &= s^{-1}\left(-\frac{3}{2}L_1 + 4L_6 + 3L_7 + \Phi L_4\right), \\ s\partial_s L_5 &= s^{-1}(3L_1 + 4L_7 + \Phi L_5), \\ s\partial_s L_6 &= L_6 + s^{-1}\left(4L_1 - \frac{3}{2}L_2 + \frac{3}{4}L_3 + L_8 + \Phi L_6\right), \\ s\partial_s L_7 &= L_7 + s^{-1}\left(3L_2 - \frac{3}{2}L_3 + 4L_8 + \Phi L_7\right), \\ s\partial_s L_8 &= 2L_8 + s^{-1}\left(-3 + 4L_3 + 3L_4 - \frac{3}{2}L_5 + \Phi L_8\right), \end{aligned}$$

where now $\Phi = t(3L_1 - L_6 + \frac{1}{2}L_7) - 1 \equiv -1 \pmod{t}$.

The non-linear system (6.25) reduces to a linear system when we set $t = 0$. In fact the resulting system is a perturbation of the irregular ODE (6.13), and it is elementary to check that the the solution is given by those special power series related to $g_i^\circ(x, 0, -z)$ written down before.

6.2. Dubrovin connection on $H(X')$. Under the frame (6.7), the connection matrix E_i , $i = 1, 2$ are decomposed into two diagonal blocks

$$E_i^{11} = A_i^{11} + A_2^{21} f.$$

and E_i^{22} . From (6.9) and (6.5), they are determined by A_i and P as

$$(6.26) \quad E_1^{11} = \begin{bmatrix} xyf_1 & xyf_2 & xyf_3 & xy(-\frac{1}{2} + f_4) & xy(1 + f_5) & xyf_6 & xyf_7 & xyf_8 \\ & & & & & -\frac{1}{2}xy & xy & \\ & & & & & \frac{1}{4}xy & -\frac{1}{2}xy & \\ & & & & & & & xy \\ & & & & & & & -\frac{1}{2}xy \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ -\frac{1}{2}f_1 & -\frac{1}{2}f_2 & -\frac{1}{2}f_3 & 1 - \frac{1}{2}f_4 & -\frac{1}{2}f_5 & -\frac{1}{2}f_6 & -\frac{1}{2}f_7 & -\frac{1}{2}f_8 \\ & & & & & & & 1 \end{bmatrix},$$

$$\begin{aligned}
 E_1^{22} &= -\frac{1}{x} + \left(-\frac{1}{2}g_2 + g_3 + xy g_8 \right) \\
 (6.27) \quad &= -\frac{1}{x} \left(1 - yx^4(3 + 12zx + 55z^2x^2 + 300z^3x^3 \right. \\
 &\quad \left. + 1918z^4x^4 + 14112z^5x^5 + \dots) \right. \\
 &\quad \left. - y^2x^8(21 + 348zx + \dots) + \dots \right).
 \end{aligned}$$

Notice that the i -th row is affected by $f_\bullet = (f_1, \dots, f_8)$ with a multiple given by the $(i, 9)$ -th entry of $z(x \partial_x)$ in (6.5).

Exactly the same pattern applies to $E_2^{11} = A_2^{11} + A_2^{21} f_\bullet$ too, where now only the first row is added by $xy f_\bullet$:

$$(6.28) \quad E_2^{11} = \begin{bmatrix} xyf_1 & xyf_2 & xyf_3 & xy(-\frac{1}{2} + f_4) & xy(1 + f_5) & y + xyf_6 & xyf_7 & xyf_8 \\ 1 & & & & & -\frac{1}{2}xy & xy & \\ \frac{1}{2} & & & & & \frac{1}{4}xy & -\frac{1}{2}xy & y \\ & 1 & & & & & & xy \\ & 1 & 1 & & & & & -\frac{1}{2}xy \\ & & & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & & \frac{1}{2} & 1 & \end{bmatrix},$$

$$(6.29) \quad E_2^{22} = \frac{1}{x} \left(yx^4(1 + 3zx + 11z^2x^2 + \dots) \right) = yx g_8.$$

Now the Dubrovin connection on $H(X')$ follows from the BF/GMT procedure applied to E_1^{11} and E_2^{11} . Namely to solve B with

$$(6.30) \quad z\partial_a B = BE_a^{11} - B_0 E_{a,0}^{11} B_0^{-1} B, \quad a = 1, 2,$$

where 0 means its value at $z = 0$.

This is still complicated. But in contrast to the direct computation based on the Picard–Fuchs equations on the X' side, we have a better structure on the connection matrices before the BF/GMT. This allows us to determine the BF/GMT in explicit terms at least for the extremal ray directions, which is given below.

We denote $\delta = zx \partial_x$ and define its (pseudo) inverse \mathcal{J} by

$$(6.31) \quad \mathcal{J}\phi = \mathcal{J}(\phi - \phi(z=0)) = \int \frac{\phi - \phi(z=0)}{zx} dx.$$

For example, in terms of formal series expansion, we have

$$\begin{aligned}
 \mathcal{J}f_1 &= \frac{3}{3}x^3 - \frac{11}{4}zx^4 + \frac{50}{5}z^2x^5 + \dots \pmod{y}, \\
 \mathcal{J}(f_1/x) &= \frac{3}{2}x^2 - \frac{11}{3}zx^3 + \frac{50}{4}z^2x^4 + \dots \pmod{y}.
 \end{aligned}$$

6.3. Quantum invariance along the extremal variable. Consider the local $(2, 1)$ flip. For $a \in H(X)$, we denote by $[a] = \tilde{a}(0)$ the z -constant part of the deformed vector under the block diagonalization.

The procedure (6.9) gives, for $k = 1, 2$,

$$(6.37) \quad A_k^{11} + A_k^{12} f_\bullet = E_k^{11}.$$

Let $\hat{T}_i I = w_i$. The (i, j) -th component of the left hand side gives

$$\langle T_k, T^i, T_j \rangle^X + \langle T_k, T^i, \kappa_0 \rangle^X f_j = \langle T_k, T^i, \tilde{T}_j \rangle^X.$$

Remark 6.11. Under gauge transform, the $z = 0$ part of the LHS in (6.37) should be $\langle T_k, \tilde{T}^i(0), \tilde{T}_j(0) \rangle^X$. By Lemma 6.2 and the block diagonalization, the $\tilde{\kappa}_0(0)$ part in $\tilde{T}^i(0)$ has no contribution in the quantum product. The simple effect on z in (6.37) is due to the PDE on f_i 's in (6.10).

If no more BF/GMT is needed on E_k^{11} , which is the case only if we modulo (x^2, y) (by (6.26) and $f_1 = -x^2 + \dots$), then its (i, j) -th component equals

$$\langle T'_k, T'^i, T'_j \rangle^{X'}$$

where $T_j = \Psi T'_j$. The only non-trivial case says that

$$(6.38) \quad \langle \tilde{\xi} - h, \tilde{\xi} - h, \tilde{\xi} - \tilde{h} \rangle^X \equiv \langle h', h', h' \rangle^{X'} \pmod{x^2, y}.$$

We will see that this follows easily by a direct comparison.

From the previous calculations, especially (6.8) and the table on $g_i(x, y, z)$'s following it, we know that $\Psi h' = \tilde{\xi} - h$ and

$$(6.39) \quad \begin{aligned} [\tilde{\xi} - h] &= \tilde{w}_3(0) = (\tilde{\xi} - h) + f_3(x, y, 0) K_1 \\ &= (\tilde{\xi} - h) + x \kappa_0. \end{aligned}$$

Then the LHS of (6.38) is simply a topological term

$$x \langle \tilde{\xi} - h, \tilde{\xi} - h, \kappa_0 \rangle_{\beta=0}^X = x.$$

And the only extremal ray invariant of the RHS of (6.38) is easily seen, from the I function of X' , to be $1 \cdot q^{\ell'} = x$. Hence (6.38) holds.

Theorem 6.12 (Linear invariance along extremal rays). *For extremal primary Gromov–Witten invariants of at least $n \geq 3$ insertions, we have*

$$\langle [\tilde{\xi} - h]^{\otimes n} \rangle^X = \langle (h')^{\otimes n} \rangle^{X'}.$$

In fact this is equivalent to the quantum interpretation of Cayley's formula

$$a_d := \langle \kappa_0^{\otimes (d+1)} \rangle_{d\ell}^X = d^{d-2}, \quad d \geq 1.$$

Proof. We start with the one point invariant. For $d'_2 = 0$ we have

$$(6.40) \quad I_{d'\ell'}^{X'} = (-1)^{3(d'-1)} \frac{(\tilde{\xi}' - h')^3 \prod_{m=1}^{d'-1} (h' + mz)}{(h' + d'z)^2},$$

while the virtual dimension is $-d' + (4 - 3) + 1 = 2 - d'$. Thus $d' = 1$ and we have a divisor insertion. Indeed from the above I expression $I = J$ for $\beta' = \ell'$ and we have $\langle h' \rangle^{X''} = q^{\ell'} I_{\varrho'}^{X'}. h' = q^{\ell'}$. Hence for all $n \geq 0$:

$$\langle (h')^{\otimes n} \rangle^{X'} = q^{\ell'} = x.$$

On the X side, we compute the corresponding term. The virtual dimension for $\beta = d\ell$ is now $d + 1 + n$. So there are exactly $d + 1$ insertions which support the class κ_0 . In the following we assume that $n \geq 2$.

We expand the homogeneous expression

$$\begin{aligned} \langle [\zeta - h]^{\otimes n} \rangle^X &= \langle ((\zeta - h) + x\kappa_0)^{\otimes n} \rangle^X \\ (6.41) \quad &= \sum_{j=2}^n C_j^n x^j q^{(j-1)\ell} \langle (\zeta - h)^{\otimes n-j} \otimes \kappa_0^{\otimes j} \rangle_{(j-1)\ell}^X + 3x\delta_{n,3} \\ &= x \sum_{j=2}^n (-1)^{n-j} C_j^n (j-1)^{n-j} \langle \kappa_0^{\otimes j} \rangle_{(j-1)\ell}^X + 3x\delta_{n,3} \end{aligned}$$

where the divisor axiom is used, and in the case $n = 3$ there are 3 more terms coming from the classical product $(\zeta - h, \zeta - h, \kappa_0) = 1$.

From (6.41), the theorem amounts to the assertion that

$$(6.42) \quad \sum_{j=0}^n (-1)^{n-j} C_j^n (j-1)^{n-j} a_{j-1} = -3\delta_{n,3}.$$

Here we use the convention $a_{-1} = -1 = (-1)^{-1-2}$ and a_0 can be assigned arbitrarily since it always comes with the coefficient 0.

For $n = 2$, (6.42) requires that $a_1 = 1$. This can be proved by direct divisorial reconstruction or by looking at the 3 point invariant $\langle h, \kappa_0, \kappa_0 \rangle = q_1$ as shown in the explicit calculation (6.2).

For $n = 3$, (6.42) is then equivalent to $-1 + 0 - 3a_1 + a_2 = -3$, that is $a_2 = 1$. For $n = 4$, (6.42) is then equivalent to

$$-1 - 0 + 6a_1 - 4 \times 2^1 a_2 + a_3 = 0,$$

that is $a_3 = 3$. For $n = 5$ this gives $a_4 = 16 = 4^2$. For $n = 6$ this gives $a_5 = 125 = 5^3$. Thus it is tempted to guess if $a_d = d^{d-2}$ holds? We will see that this is indeed the case.

Recall that the Strling number of second kind $S(m, n)$ for two integers $m, n \geq 0$ is the number of partitions of m elements into n disjoint subsets. It is defined to be zero if $m < n$. It admits a nice relation with the combinatorial number C_j^n , namely for any $m, n \geq 0$,

$$(6.43) \quad n!S(m, n) = \sum_{j=0}^n (-1)^{n-j} C_j^n j^m.$$

(cf. [4, p.265 (6.19)]). It follows from (6.43) easily that

$$(6.44) \quad \sum_{j=0}^n (-1)^{n-j} C_j^n (j-1)^{n-3} = 0.$$

Now we may continue the proof of Theorem 6.12 to establish $a_d = d^{d-2}$ under the assumption on quantum invariance relation (6.42). Indeed, by induction on $n = d + 1$, the validity of (6.42) is equivalent to equation (6.44) for all $n \geq 4$ by substituting $a_{j-1} = (j-1)^{j-3}$ into it and notice that the power $n - 3$ is then uniform for all j .

Conversely, we will now prove the quantum invariance (6.42) by establishing $a_d = d^{d-2}$, for all $d \geq 1$ directly.

Let $a_0 = 0$, $a_1 = 1$ and $d \geq 2$. Since $\kappa_0 = (\xi - h)^2$, by applying the divisorial reconstruction we get the following recursive formula

$$(6.45) \quad a_d = \sum_{d'=1}^{d-1} d' a_{d'} a_{d-d'} C_{d'-1}^{d-2}.$$

While it is possible to show that (6.45) is equivalent to (6.42), which must be the case after the theorem is proved, we will proceed differently. Since

$$d(d-1) \frac{d' C_{d'-1}^{d-2}}{d!} = \frac{(d')^2}{(d')!} \frac{d-d'}{(d-d')!}$$

the exponential generating function $g = \sum_{d \geq 0} a_d t^d / d!$ then satisfies

$$t^2 g'' = t(tg')' \cdot (tg').$$

Let $\mathcal{E} = g'$. Then $\mathcal{E}' = (t\mathcal{E})'\mathcal{E}$. That is, $\mathcal{E}'/\mathcal{E} = (t\mathcal{E})'$. Since $\mathcal{E}(0) = g'(0) = 1$, after integration we get $\log \mathcal{E} = t\mathcal{E}$. That is we arrive at the famous functional equation of Euler (see e.g. [4, p.369]):

$$(6.46) \quad \mathcal{E} = e^{t\mathcal{E}}.$$

Equation (6.46) has explicit solution given by (cf. [4, p.369 (7.85)])

$$\mathcal{E}(t) = \sum_{d \geq 1} d^{d-2} \frac{t^{d-1}}{(d-1)!},$$

hence $a_d = d^{d-2}$ as expected. The proof is completed. \square

Remark 6.13. The number $a_d = d^{d-2}$ is traditionally known as the Cayley number in combinatorics. It is the number of *spanning trees* in the complete graph on d vertexes (and hence with $d - 1$ edges). It is interesting to see if the localization techniques in evaluating $\langle \kappa_0^{\otimes (d+1)} \rangle_{d\ell}$ leads to the graph sum corresponding to these spanning trees.

Remark 6.14. Theorem 6.12 implies that for $(2, 1)$ -flips, the embedding

$$QH(X') \hookrightarrow QH(X)$$

is linear when restricting to the extremal ray variable. However, a lengthy yet straightforward calculation shows that

$$\langle \tilde{K}_1, \tilde{w}_3, \tilde{w}^6, \tilde{w}_3 \rangle \equiv -x^4y \pmod{y^2},$$

which implies that the embedding is necessarily non-linear. It is an interesting question whether the embedding over the extremal ray variable is always linear for (r, r') -flips.

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