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QUASI-HODGE METRICS AND CANONICAL SINGULARITIES

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1. Introduction and Statement

Throughout this paper we work over the base field $\mathbb{C}$. The basic problem in algebraic geometry treated here is the *filling-in problem* (or *degeneration problem*). Given a smooth projective family $\mathcal{X}^\times \to \Delta^\times$, we would like to know when we can fill in a reasonably nice special fiber $\mathcal{X}_0$ to form a projective family $\mathcal{X} \to \Delta$, perhaps up to a finite base change on $\Delta^\times$. For example, when can $\mathcal{X}_0$ be smooth? When can it be irreducible? Or when can it be irreducible with at most certain type of mild singularities? We would like to search for conditions depending only on the punctured family.

In this paper, for any smooth projective family $\mathcal{X} \to S$ over a smooth base S such that $\mathcal{X}_s$ has semi-ample canonical bundle, we shall define for each large $m \in \mathbb{N}$ a Kähler metric g_m on S, called the m-th quasi-Hodge metric. When $S = \Delta^\times$, we propose that the incompleteness of g_m near 0 for suitable m's provides a necessary and sufficient condition for the existence of $\mathcal{X}_0$ to be irreducible and with at most *canonical singularities* (c.f. Remark 2.5). Notice that the metric incompleteness condition is insensitive to base changes.

More precisely, for a smooth projective family $\pi : \mathcal{X} \to S$ with $p_g(\mathcal{X}_s) \neq 0$, the (possibly degenerate) quasi-Hodge metric $g_H = g_1$ on S is given by the semi-positive first Chern form of the rank p_g Hodge bundle $F^n = \pi_* K_{\mathcal{X}/S}$. When $S = \Delta^\times$, let T be the monodromy operator acting on $H^n(\mathcal{X}_s, \mathbb{C})$ where $n = \dim \mathcal{X}_s$, $s \neq 0$.

Theorem 1.1. *For any smooth projective family $\pi : \mathcal{X} \to \Delta^\times$,*

(1) *The quasi-Hodge metric g_H is incomplete if and only if that $H^{n,0}(\mathcal{X}_s)$ consists of T^r-invariant cycles for some $r \in \mathbb{N}$.*
(2) *In terms of a semi-stable model with $\mathcal{X}_0 = \bigcup_{i=0}^N X_i$, g_H is incomplete if and only if that $p_g(\mathcal{X}_s) = \sum_{i=0}^N p_g(X_i)$.*
(3) *In particular, degenerations with Gorenstein canonical singularities have finite g_H distance.*

This generalizes an earlier result on Calabi-Yau families in [10] concerning Weil-Petersson metrics (c.f. §7). In the Calabi-Yau case, since $p_g = 1$, there is exactly one component X_i with $p_g \neq 0$. Starting from this we may deduce

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Proposition 1.2. *Let $\mathcal{X} \to \Delta$ be a degeneration of Calabi-Yau n-folds such that $0 \in \Delta$ has finite g_H distance. Then the minimal model conjecture (MMC) in dimension $n+1$ implies that, up to a finite base change, $\mathcal{X} \to \Delta$ is birational to $\mathcal{X}' \to \Delta$ such that $\mathcal{X}_s \cong \mathcal{X}'_s$ for $s \neq 0$ and $\mathcal{X}'_0$ is a Calabi-Yau variety with at most canonical singularities.*

This had been solved in the literature for abelian varieties and K3 surfaces (type I degenerations in Kulikov's work [5]). Indeed in both cases one ends up with smooth families. Here we show that the relative minimal model is the filling-in we seek for. The primary reason is that all other components in $\mathcal{X}_0$ are uni-ruled hence should be contractible. The MMC is known in dimensions up to three by the Mori theory. Recently Shokurov announced the existence of log-flips in dimension four. Its validity would imply the result for degenerations of Calabi-Yau threefolds.

For the general cases when $p_g(\mathcal{X}_s) > 1$, there could be more than one *essential components* with $p_g(X_i) \neq 0$, so g_H is insufficient for our purpose. In order to concentrate all the pluri-genera P_m in one component, we need a finer metric on the punctured disk. The actual construction in §3 is to consider the case that $\mathcal{X}_s$ has *semi-ample canonical bundle* and to take an m-cyclic cover $\mathcal{Y} \to S$ of $\mathcal{X} \to S$ along a smooth divisor $\mathcal{D} \in |K^m_{\mathcal{X}/S}|$ for suitable m's. We then define the m-th quasi-Hodge metrics g_m to be g_H with respect to $\mathcal{Y} \to S$. The semi-ample assumption is to guarantee the invariance of pluri-genera in projective families, which is not needed in Theorem 1.1 by the well-known invariance of Hodge numbers.

Proposition 1.3. *Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of smooth projective manifolds $\mathcal{X}_s$ with semi-ample canonical bundle. If $\mathcal{X}_0$ is irreducible with only canonical singularities then $0 \in \Delta$ is at finite g_m distance for all m such that g_m is defined.*

We conjecture that the converse statement holds true. To support some evidence, we verify it in §4 for degenerations of curves.

Theorem 1.4. *Let $\pi : \mathcal{X}^\times \to \Delta^\times$ be a projective family of smooth curves of genus ≥ 2, then g_m is defined for all $m \in \mathbb{N}$ and the incompleteness of g_m for any three values of m's implies that up to a finite base change π can be completed into a smooth family.*

In §6 we discuss the notion of *essential incomplete boundary points*. Namely a finite distance degeneration without any smooth filling-in up to any finite base change. Notice that there is no such essential degenerations for curves (Theorem 1.4), abelian varieties and K3 surfaces (as mentioned above).

Theorem 1.5. *There exist essential finite g_m distance degenerations for threefolds:*

(1) *Nodal degenerations of Calabi-Yau threefolds with $h^1(\mathcal{O}) = 0$: these provide the simplest type of examples with nontrivial monodromy.*

(2) *Terminal degenerations of smooth minimal threefolds: some of them have trivial C^∞ monodromy hence provide more subtle examples.*

In §7 we make a few remarks on the Weil-Petersson metric on Calabi-Yau moduli spaces and its relation to Viehweg's theory on moduli spaces.

2. Quasi-Hodge Metrics g_H

Let $\pi : \mathcal{X} \to S$ be a family of polarized algebraic manifolds with $H^{n,0}(\mathcal{X}_s) = H^0(\mathcal{X}_s, K_{\mathcal{X}_s}) \neq 0$. It is well known that these spaces have constant rank p_g (the geometric genus) in $s \in S$ and they form a holomorphic vector bundle $F^n = \pi_* K_{\mathcal{X}/S}$. It is the last piece of the Hodge filtration $F : F^0 \supset F^1 \supset \cdots \supset F^n$ with a natural Hermitian metric induced from the topological cup product on each fiber:

$$Q(u,v)|_{F_s} = \sqrt{-1}^n \int_{\mathcal{X}_s} u \cup v.$$

Griffiths has shown that the first Chern form of (F^n, Q) (for a local frame $\{u_i\}$):

$$\omega = \frac{\sqrt{-1}}{2} \operatorname{Ric}_Q(H^{n,0}) = -\frac{\sqrt{-1}}{2} \partial\bar\partial \log \det \Big[Q(u_i, \bar u_j)\Big]_{i,j=1}^{p_g}$$

is semi-positive, hence defines a (possibly degenerate) Kähler metric on S. We call it the quasi-Hodge metric. For Calabi-Yau families this agrees with the Weil-Petersson metric defined via the Ricci flat metric on each fiber, but not in other cases (c.f. §7).

When $\pi : \mathcal{X} \to \Delta$ is a degeneration, i.e. π is smooth outside the puncture, we are interested in relating the differential-geometric properties of Δ under the quasi-Hodge metric and the singularities occur in $\mathcal{X}_0$. Applying Mumford's semi-stable reduction theorem we may and will first assume that $\mathcal{X}$ is smooth and $\mathcal{X}_0$ is a simple normal crossing divisor in it. In this case the monodromy T is unipotent. Let $N = \log T$ be the nilpotent monodromy associated to it. We need Schmid's theory of limiting mixed Hodge structure [7] to analyze this situation.

Fix a reference fiber $X = \mathcal{X}_s$ with $s \neq 0$ and let V to be the primitive cohomology of $H^n(X, \mathbb{C})$. Recall that π induces a variation of Hodge structures (VHS) of weight n on V over $\Delta^\times$ and gives rise to the period map $\phi : \Delta^\times \to \langle T \rangle \backslash D$, with D the period domain. The map ϕ lifts to the upper half plane $\Phi : \mathbb{H} \to D$ with the coordinates $s \in \Delta^\times$ and $z \in \mathbb{H}$ related by $s = e^{2\pi\sqrt{-1}z}$. Set

$$A(z) = e^{-zN}\Phi(z) : \mathbb{H} \to \check{D}.$$

($\check{D}$ is the compact dual of D.) Since $A(z+1) = A(z)$, A descends to a function $\alpha(s)$ on $\Delta^\times$. Schmid's Nilpotent Orbit Theorem implies that $\alpha(s)$ extends holomorphically over $s = 0$. The special value $F_\infty := \alpha(0)$ is called the limiting filtration.

N uniquely determines the monodromy weight filtration on V: $0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2n-1} \subset W_{2n} = V$ such that $N(W_k) \subset W_{k-2}$ and induces an isomorphism

$$N^\ell : G^W_{n+\ell} \cong G^W_{n-\ell},$$

on graded pieces. F^p_∞ and W_k together define a polarized mixed Hodge structure (MHS) on V: namely the induced Hodge filtration

$$F^p_\infty G^W_k := F^p_\infty \cap W_k / F^p_\infty \cap W_{k-1}, \quad p = 0, \ldots, n$$

is a pure Hodge structure of weight k on G^W_k. N acts on them as a morphism of type $(-1,-1)$ — $N(F^p_\infty G^W_k) \subset F^{p-1}_\infty G^W_{k-2}$. Moreover, for $\ell \geq 0$, the primitive part $P^W_{n+\ell} := \ker N^{\ell+1} \subset G^W_{n+\ell}$ is polarized by $Q(\cdot, N^\ell \bar{\cdot})$. By adding the non-primitive part, the total cohomology $H^n(X, \mathbb{C})$ admits non-polarized MHS.

In the rest of this section we prove Theorem 1.1 by dividing it into steps. We start with the following theorem which generalizes the one in [10]:

Theorem 2.1. *The center $0 \in \Delta$ is at finite geodesic distance from the generic point $s \neq 0$ under the quasi-Hodge metric if and only if $NF^n_\infty = 0$.*

Proof. Let $\Phi : \mathbb{H} \to D$ be the lifting of the period map. To start the computation, we need to choose a good holomorphic frame Ω_j, $j = 1, \ldots, p_g$ of F^n. Let $p^n : D \to G(p_g, V)$ be the projection to the F^n part. we have $\Phi^n(z) = (e^{zN}\alpha(s))^n = e^{zN}\alpha^n(s)$. Here $*^n := p^n(*) \in G(p_g, V)$ is the n-th flag. Near $t = 0$, we can represent α^n through local homogeneous coordinates as p_g vectors $\mathbf{a}^j$, $j = 1, \ldots, p_g$ in V. Then $\mathbf{a}^j(s) = a^j_0 + a^j_1 s + \cdots$ is holomorphic in s. We have correspondingly

$$\mathbf{A}_j(z) = a^j_0 + a^j_1 e^{2\pi\sqrt{-1}z} + a^j_2 e^{4\pi\sqrt{-1}z} + \cdots.$$

As in [10], the function $e^{2\pi\sqrt{-1}z} = e^{2\pi\sqrt{-1}x}e^{-2\pi y}$ has the property that all the partial derivatives in x and y decay to 0 exponentially as $y \to \infty$, with rate of decay independent of x. Let h be the class of functions with this property and $\mathbf{h}$ the corresponding class of functions with value in V.

Let $\Omega_j(z) = e^{zN}\mathbf{A}_j(z)$ for $j = 1, \ldots, p_g$. This is the desired frame because frame representations correspond to sections of the universal rank p_g subbudle of $G(p_g, V)$ which pulls back to F^n by Φ. So the Kähler form ω of the induced quasi-Hodge metric on $\mathbb{H}$ is given by

$$\omega = -\frac{\sqrt{-1}}{2}\partial\bar{\partial} \log \det \Big[Q\big(e^{zN}\mathbf{A}_i(z), e^{\bar{z}N}\overline{\mathbf{A}_j(z)}\big)\Big]^n_{i,j=1}.$$

Since the base is one dimensional, if we write the metric as $G|dz|^2$ then $G = -(1/4)\triangle \log \det Q$. From $Q(Tu, Tv) = Q(u, v)$, it follows easily that $Q(Nu, v) = -Q(u, Nv)$ and $Q(e^{zN}u, v) = Q(u, e^{-zN}v)$. Since $\mathbf{A}_j = a^j_0 + \mathbf{h}$, we have

$$\begin{aligned}\det\left[Q(e^{zN}\mathbf{A}_i, e^{\bar{z}N}\bar{\mathbf{A}}_j)\right] &= \det\left[Q(e^{zN}a_0^i, e^{\bar{z}N}\bar{a}_0^j) + h\right]\\ &= \det\left[Q(e^{2\sqrt{-1}yN}a_0^i, \bar{a}_0^j)\right] + h\\ &= p(y) + h,\end{aligned}$$

where $p(y)$ is a polynomial in y. Let $d = \deg\, p(y)$. It has the property that $d = 0$ if and only if $NF_\infty^n = 0$. This is a consequence of the polarization condition for the mixed Hodge structure.

To see this, we may choose the basis $\mathbf{a}^j$ in such a way that for $a_0^j \in G_{n+\ell_j}$ in the limiting MHS, $i < j$ implies $\ell_i \geq \ell_j$. Let $\ell = \ell_1 = \cdots = \ell_q > \ell_{q+1}$, that is $\ell = \max\{\, i \in \mathbb{N} \cup \{0\} \mid N^i F_\infty^n \neq 0\,\}$. Then the determinant of the $q \times q$ matrix corresponds to those entries with index between 1 and q is a positive polynomial in y of degree $q\ell$ by the polarization condition on $G_{n+\ell}$. Inductively we may find numbers q_k and $\ell_{(k)}$ with $\sum_k q_k = p_g$ and $\ell_{(k)}$ decreases such that the determinant of the corresponding $q_k \times q_k$ matrix (as a block on the diagonal) is a positive polynomial in y of degree $q_k\ell_{(k)}$. Now we conclude that the original determinant is a positive polynomial in y of degree $d = \sum_k \ell_{(k)}$ dominant by the product of these diagonal blocks — because all other elements have smaller degree in y. It is clear that $d = 0$ if and only if $\ell = \ell_1 = 0$ if and only if $NF_\infty^n = 0$.

Then

$$\begin{aligned}4G &= \frac{(p'+h)^2 - (p+h)(p''+h)}{(p+h)^2} = \frac{({p'}^2 - pp'') + h}{p^2 + h}\\ &\sim \frac{{p'}^2 - pp''}{p^2} + h \sim \frac{d^2 - d(d-1)}{y^2} + h = \frac{d}{y^2} + h.\end{aligned}$$

Here we have used the fact that $p^{-2}h \in h$. Obviously, if $NF_\infty^n = 0$ then $d = 0$ and $G = h$, so $\int_{z_0}^\infty \sqrt{G}\,|dz| < \infty$ for some curve (e.g. $x = c$). When $NF_\infty^n \neq 0$ we have $d \geq 1$ and for y large enough we can make $h < 1/y^3$ uniformly in x, then clearly $\int_{z_0}^\infty \sqrt{G}\,|dz| \sim 2\log y\,|_{y_0}^\infty = \infty$ for any path with $y \to \infty$. $\square$

Remark 2.2. From the proof, we know that in the case of infinite distance, the quasi-Hodge metric is exponentially asymptotic to a scaling of the Poincaré metric.

So far it is purely Hodge-theoretic and makes perfect sense in the framework of abstract variation of Hodge structures. Now we plug in the geometric data:

Theorem 2.3. *For a semi-stable degeneration $\pi : \mathfrak{X} \to \Delta$ of varieties with $p_g > 0$, if $\mathfrak{X}_0 = \bigcup_{i=0}^N X_i$ then the induced quasi-Hodge metric is incomplete at* 0 *if and only if for $s \neq 0$, $p_g(\mathfrak{X}_s) = \sum_{i=0}^N p_g(X_i)$.*

Proof. The proof is essentially the same as in [10], so we only sketch it briefly.

Deligne has shown that the cohomologies of the normal crossing divisor $\mathfrak{X}_0$ admit mixed Hodge structures. Let $i : \mathfrak{X}_s \to \mathfrak{X}$ be the inclusion map and

$i^{\#} : H^n(\mathcal{X}_0) \cong H^n(\mathcal{X}) \to H^n(\mathcal{X}_s)$ the induced map. The Clemens-Schmid exact sequence

$$\cdots \to H^n(\mathcal{X}_0) \xrightarrow{i^{\#}} H^n(\mathcal{X}_s) \xrightarrow{N} H^n(\mathcal{X}_s) \to \cdots$$

is an exact sequence of MHS's. The is also known as the Invariant Cycle Theorem which implies that $p_g := p_g(\mathcal{X}_s) \geq \sum_{i=1}^N p_g(X_i)$ with $\sum_{i=1}^N p_g(X_i)$ corresponds to T-invariant cycles in F_s^n. Now the condition $NF_\infty^n = 0$ says that all those p_g cycles are T-invariant. □

Notice that in the statement of Theorem 1.1, part (1), r is the degree of the base change $t \mapsto t^r$ performed on Δ in obtaining the semi-stable model.

Corollary 2.4. *Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of smooth projective manifolds $\mathcal{X}_s$, $s \in \Delta^\times$ with geometric genus $p_g > 0$. If $\mathcal{X}_0$ is an irreducible variety with only Gorenstein canonical singularities then $0 \in \Delta$ is at finite distance to $s \neq 0$ with respect to the quasi-Hodge metric.*

Proof. By elementary commutative algebra $\mathcal{X}_0$ is Gorenstein implies that $\mathcal{X}$ is Gorenstein. Moreover $K_{\mathcal{X}_s} = (K_{\mathcal{X}} + \mathcal{X}_s)|_{\mathcal{X}_s} = K_{\mathcal{X}}|_{\mathcal{X}_s}$ for all $s \in \Delta$, hence the theorem on semi-continuity implies that $h^0(\mathcal{X}_0, K_{\mathcal{X}_0}) \geq p_g$.

Now for $\mathcal{X}' \to \mathcal{X}$ a resolution of singularities such that $\mathcal{X}'_0 = \bigcup_{i=0}^N X'_i$ is a (not necessarily reduced) normal crossing divisor, there exists a component X'_0 such that $\phi : X'_0 \to \mathcal{X}_0$ is a resolution of singularities. By definition of canonical singularities we thus have $K_{X'_0} = \phi^* K_{\mathcal{X}_0} + E$ for E an effective exceptional divisor. By pulling back canonical sections via ϕ^* we conclude that $h^0(X'_0, K_{X'_0}) \geq h^0(\mathcal{X}_0, K_{\mathcal{X}_0}) \geq p_g$.

Clearly this inequality holds true for $\mathcal{X}' \to \Delta$ a semi-stable reduction of $\mathcal{X} \to \Delta$. So by the Invariant Cycle Theorem we must have that $p_g(\mathcal{X}_s) = \sum_{i=0}^N p_g(X_i)$. The above theorem then implies that $0 \in \Delta$ is at finite quasi-Hodge distance. □

Remark 2.5. A normal $\mathbb{Q}$-Gorenstein variety X is said to have (at most) canonical singularities if for a (hence for any) resolution of singularities $\phi : Y \to X$, $K_Y =_{\mathbb{Q}} K_X + \sum_i e_i E_i$ with $e_i \geq 0$, where the sum is over all exceptional divisors. In dimension two, canonical singularities are precisely $\mathbb{C}^2/G$ for a finite sub-group $G \subset \mathrm{SL}(2, \mathbb{C})$, the so called ADE singularities. In dimension three there is a classification theory due to Reid and Mori [6]. The generic hyperplane section of a canonical singularities is again canonical, so the generic surface slices of canonical singularities are ADE singularities.

3. The m-th Quasi-Hodge Metrics g_m

Following Viehweg [9], we will assume that $\mathcal{X}_s$ has semi-ample canonical bundle for the smooth family $\pi : \mathcal{X} \to S$. This implies that $\pi_* K^m_{\mathcal{X}/S}$ is a locally free sheaf for each $m \in \mathbb{N}$ (see Theorem 8.16 in [9], for this one needs only that $\mathcal{X}_s$ has Gorenstein canonical singularities). Unlike the case $m = 1$, we do not have an immediate natural hermitian metric on the underlying vector bundle. Let $m \in \mathbb{N}$ be sufficiently large so that there exists a divisor $\mathcal{D} \in |K^m_{\mathcal{X}/S}|$ smooth over S. By taking an m-cyclic cover along $\mathcal{D}$ (c.f. [9] Lemma 2.3), we get a family

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\phi"] \arrow[dr, "\tau"'] & & \mathcal{X} \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with eigenspace decomposition $\phi_*\mathcal{O}_\mathcal{Y} = \bigoplus_{k=0}^{m-1}(K_{\mathcal{X}/S}^{-k})$ and $R^i\phi_* = 0$ for $i > 0$ (since ϕ is finite). Also from the generalized Hurwitz formula $K_{\mathcal{Y}/S} = \phi^*K_{\mathcal{X}/S} + (m-1)\mathcal{H}$ with $m\mathcal{H} = \phi^*\mathcal{D} = \phi^*(mK_{\mathcal{X}/S})$, we see that $\phi^*K_{\mathcal{X}/S}^m = K_{\mathcal{Y}/S}$. Simple spectral sequence argument and projection formula then show that

$$\tau_*K_{\mathcal{Y}/S} = \pi_*\phi_*K_{\mathcal{Y}/S} = \pi_*\phi_*(\phi^*K_{\mathcal{X}/S}^m) = \pi_*\left(\bigoplus_{k=0}^{m-1} K_{\mathcal{X}/S}^{m-k}\right) = \bigoplus_{k=1}^{m}\pi_*K_{\mathcal{X}/S}^k.$$

Definition 3.1. *The m-th quasi-Hodge metric g_m on S is defined to be the quasi-Hodge metric attached to $\tau_*K_{\mathcal{Y}/S}$ as defined in §2.*

Let $S = \Delta^\times$. By Theorem 2.1 (or 2.3), the Kähler metric g_m is incomplete at 0 if and only if all $H^{n,0}(\mathcal{Y}_s)$ $(s \neq 0)$ are T^r-invariant. In view of the above splitting, this should indicate certain extension properties of pluri-canonical forms in $H^0(\mathcal{X}_s, K_{\mathcal{X}_s}^k)$ $(s \neq 0)$ to the central fiber. Here is our proposal:

I. Take a semi-stable model $\mathcal{X} \to \Delta$ with $\mathcal{X}_0 = \bigcup_{i=0}^N X_i$. The incompleteness of g_m should be equivalent to certain twisted pluri-genera equalities $P_k(\mathcal{X}_s) = \sum_{i=0}^N \tilde{P}_k(X_i)$ for $1 \leq k \leq m$ (c.f. Lemma 4.1).

II. The twisted pluri-genera equalities for all $m \in \mathbb{N}$ should force that there is only one component, say X_0, to have non-zero pluri-genera. Mori theory implies that all the other components are uni-ruled. It should be enough to verify the equality up to a sufficiently large m which depends only on $\dim \mathcal{X}_s$.

III. Finally we expect to contract all X_i with $i \neq 0$ using Mori's extremal contractions. If we apply directly the MMC, we should then arrive at a model $\mathcal{X}' \to \Delta$ such that $\mathcal{X}'_0$ has only canonical singularities.

To make sense of this, we first show that the incompleteness of g_m is a necessary condition for degenerations with canonical singularities.

Proof of Proposition 1.3. The only key point is that the m-th cyclic covering construction along $D \in |K_X^m|$ can be carried over to families $\pi : \mathcal{X} \to \Delta$ when $\mathcal{X}_0$ has canonical singularities of index dividing m, and the resulting covering families $\mathcal{Y} \to \Delta$ has the property that $\mathcal{Y}_0$ has only Gorenstein canonical singularities. Hence by Theorem 1.1 (or rather Corollary 2.4) that g_m is incomplete. □

Remark 3.2. Indeed one has the constancy of P_m in $s \in \Delta$ for canonical degenerations (with semi-ample K) and hence the P_m equalities for any semi-stable model. It is thus natural to conjecture that the P_m equalities for a finite number of m's are equivalent to the existence of canonical degenerations. Notice that the P_k equality for all $k \le m$ is a stronger assumption than the incompleteness of g_m.

It is long conjectured that for smooth projective families $\pi : \mathcal{X} \to S$ the plurigenera is locally constant in $s \in S$. Recently Siu [8] proved this for smooth varieties of general type. This has then been extended to families with canonical singularities [4]. We have chosen to work on families with semi-ample K to avoid the technicality involved in Siu's theorem.

In the following two sections, we justify our proposal by verifying it for curves and by showing that MMC implies the case of Calabi-Yau manifolds.

4. The Case of Curves

We will consider curves of genus ≥ 2. The case of elliptic curves is easier, which is also a special case of Calabi-Yau manifolds to be discussed later.

The natural algebro-geometric way to treat this problem is to start with a semistable degeneration $\mathcal{X} \to \Delta$ with $\mathcal{X}_0 = \bigcup_{i\in I} X_i$ and then try to simplify it through birational modifications. Since the total space $\mathcal{X}$ is a surface, the modifications needed are simply contraction of (-1) curves. So we may assume that $\mathcal{X} \to S$ is relative minimal and semi-stable and $g(\mathcal{X}_s) \ge 2$. Let $g_i = g(X_i)$ and $d_i = \sum_{j\ne i} X_j.X_i = (\sum X_j - X_i).X_i = -X_i^2$. The famous stable reduction theorem for curves states that we may further contract (-2) curves so that every component X_i is stable in the sense that if $g_i = 0$ then $d_i \ge 3$ (and if $g_i = 1$ then $d_i \ge 1$, which is always true here since $\mathcal{X}_0$ is connected). The only subtle point is that $\mathcal{X}$ may have A_n type singularities. Since this will not affect our later discussion, for the sake of simplicity we will assume we are already in a stable reduction.

For $m \in \mathbb{N}$ such that $K^m_{\mathcal{X}/S}$ is S-very ample, let $\mathcal{D} \in |K^m_{\mathcal{X}/S}|$ be a smooth member which does not contain any special point in any $X_i \cap X_j$ and the singular points of $\mathcal{X}$. We also assume that S is small enough so that $S \cong \Delta$ and $K_S \cong \mathcal{O}_S$. By Theorem 1.1, the quasi-Hodge metric g_m constructed from the m-th cyclic cover $\mathcal{Y} \to \mathcal{X}$ along $\mathcal{D}$ is incomplete if and only if $p_g(\mathcal{Y}_s) = \sum_{i\in I} p_g(Y_i)$. This is because $\mathcal{Y} \to S$ is a stable degeneration by construction and the presence of A_n singularities does not affect the result — further blowing-ups gives (-2) curves which have no contribution to the geometric genus. By the generalized Hurwitz formula, we have seen in §3 that $p_g(\mathcal{Y}_s) = \sum_{k=1}^m P_k(\mathcal{X}_s)$. Moreover, we have

Lemma 4.1. *$p_g(Y_i) = \sum_{k=1}^m h^0(X_i, \tilde{K}^k_{X_i})$ where $\tilde{K}^k_{X_i}$ is the twisted (or logarithmic) pluri-canonical sheaf defined as $K^k_{X_i} \otimes \mathcal{O}_{X_i}(\sum_{j\ne i} X_j \cap X_i)^{k-1}$.*

Proof. Let $\tau : \mathcal{Y} \to S$ be the new families with $\mathcal{Y}_0 = \bigcup_{i\in I} Y_i$. By the construction, $\phi_i : Y_i \to X_i$ is an m-cyclic cover along

$$\mathcal{D}|_{X_i} = mK_{\mathcal{X}}|_{X_i} = m(K_{X_i} - X_i|_{X_i}) = m(K_{X_i} + \sum\nolimits_{j\ne i} X_j \cap X_i) =: mD_i.$$

Then we have eigenspace decomposition $\phi_{i*}\mathcal{O}_{Y_i} = \bigoplus_{k=0}^{m-1} \mathcal{O}(D_i)^{-k}$. The same proof as in §3 gives $\phi_{i*}K_{Y_i} = K_{X_i} \otimes \bigoplus_{k=0}^{m-1} \mathcal{O}(D_i)^k$. □

Proof of Theorem 1.4. We will show that if g_m is incomplete for three m's then in a stable reduction there is only one component X_0 in $\mathcal{X}_0$, which is then smooth of genus $g(X_0) = g(\mathcal{X}_s)$.

From the Riemann-Roch formula,

$$\begin{aligned} h^0(\tilde{K}^k_{X_i}) - h^1(\tilde{K}^k_{X_i}) &= k(2g_i - 2) + (k-1)d_i + (1 - g_i) \\ &= (2k-1)(g_i - 1) + (k-1)d_i \end{aligned}$$

It is clear that $h^1(\tilde{K}^k_{X_i}) = 0$ for $k \geq 2$ by the stability condition. Also $h^1(\tilde{K}_{X_i}) = h^0(\mathcal{O}_{X_i}) = 1$. So the equality

$$\sum\nolimits_{k=1}^{m} h^0(K^k_{\mathcal{X}_s}) = \sum\nolimits_{i\in I} \sum\nolimits_{k=1}^{m} h^0(\tilde{K}^k_{X_i})$$

becomes

$$1 + \sum\nolimits_{k=1}^{m} (2k-1)(g-1) = |I| + \sum\nolimits_{i\in I} \sum\nolimits_{k=1}^{m} \big[(2k-1)(g_i - 1) + (k-1)d_i\big]$$

which is

$$m^2(g - \sum\nolimits_{i\in I} g_i) = -(|I| - 1)m^2 + \frac{m(m-1)}{2} \sum\nolimits_{i\in I} d_i + (|I| - 1).$$

If this is true for any three values of m's then we get $|I| = 1$, say $I = \{0\}$ and $g = \sum_{i\in I} g_i = g(X_0)$ as desired. □

5. The Case of Calabi-Yau Manifolds

We start with the following corollary of Theorem 1.1.

Corollary 5.1. *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of Calabi-Yau manifolds. Then g_H is incomplete at $0 \in \Delta$ if and only if there is an irreducible component $X_i \subset \mathcal{X}_0$ such that $h^{n,0} \neq 0$. This is equivalent to that there is exact one component with $h^{n,0} = 1$.*

Lemma 5.2. *For any relative minimal model $\mathcal{X} \to \Delta$ of a degeneration of Calabi-Yau manifolds $\mathcal{X}' \to \Delta$, if $\mathcal{X}_0 = \sum_{i=0}^{N} X_i$ has more than one component then each X_i has $-K_{X_{i,\mathrm{red}}}$ a nontrivial effective divisor on $X_{i,\mathrm{red}}$.*

Proof. For divisors we use the notation "$\sim$" to denote $\mathbb{Q}$-linear equivalence. Since $\pi : \mathcal{X} \to \Delta$ is a holomorphic function, we have that $\mathcal{X}_s \sim 0$ for any $s \in \Delta$. Also if $D \subset \mathcal{X}$ is a divisor such that $\pi(D)$ is not a point, then $D|_{\mathcal{X}_s}$ will be a nontrivial divisor of $\mathcal{X}_s$.

Since $\mathcal{X}_s$ is birational to $\mathcal{X}'_s$, it is a terminal Calabi-Yau variety for $s \neq 0$, that is $K_{\mathcal{X}_s} = 0$. By adjunction formula on such $\mathcal{X}_s$, we conclude that $K_{\mathcal{X}}$ is supported on the central fiber and so is of the form $K_{\mathcal{X}} = \sum a_i X_i$ with $a_i \in \mathbb{Q}$.

Since $\sum X_i = \mathcal{X}_0 \sim 0$, we may adjust a_i so that $\max a_i = 0$. Let $I = \{i \,|\, a_i = 0\}$. For $i \in I$ and a curve $\ell \subset X_i$ one has

$$K_{\mathcal{X}}.\ell = \sum\nolimits_{j\notin I} a_j (X_j.\ell) \leq 0.$$

If $I \neq \{0, \dots, N\}$, we may choose ℓ such that $X_j.\ell > 0$ and $a_j < 0$ by the connectedness of $\bigcup X_i$. But then $K_{\mathfrak{X}}.\ell < 0$, contradicts to the nefness of $K_{\mathfrak{X}}$.

So we must have $a_i = 0$ for all i. That is, $K_{\mathfrak{X}} = 0$. Now take any component X_i, one has

$$K_{X_i} = (K_{\mathfrak{X}} + X_i)\big|_{X_i} = X_i|_{X_i} = -\sum\nolimits_{j \neq i} X_j|_{X_i},$$

which by the connectedness again is a nontrivial negative effective divisor if there are more than one components. The case for $X_{i,\mathrm{red}}$ is entirely similar. □

Proof of Proposition 1.2. We first take a semi-stable reduction $\mathfrak{X}' \to \Delta$ of our original degenerations. This may require a finite base change on the base, but this will not change the finite distance condition on the metric. We want to conclude that any relative minimal model (if there exists any) of it, denoted by $\mathfrak{X} \to \Delta$ has only one component in the central fiber. Let $\mathfrak{X}'_0 = \bigcup_{i=0}^N X'_i$ with X'_0 the unique component with a canonical section $\Omega \in \Gamma(X'_0, K_{X'_0})$.

By the conjectural construction of relative minimal models, one uses only divisorial contractions and flips of relative extremal rays, hence only the central fiber will be modified since the general fibers are already smooth Calabi-Yau. Notice that during the process the proper transform of X'_0 is never contracted. Indeed, if a component W in the central fiber is contracted in some step, then W is covered by (extremal) rational curves [3]. This will continue to hold true for any smooth model Y of W hence $\kappa(Y) = -\infty$ and so Y can not be X'_0.

By Lemma 5.2, if there are more than one components in $\mathfrak{X}_0$ then any component X_i has $K_{X_i} = -D \neq 0$. If X_i is not normal, passing to normalization $\psi : W \to X_i$ with conductor $C \subset W$ can only make $K_W = \psi^* K_{X_i} - C$ more non-effective. Hence passing to any smooth model $\phi = \phi' \circ \psi : Y \to X_i$ with $\phi' : Y \to W$, $K_Y = \phi'^* K_W + E$ shows that $K_Y = -\phi^* D - \phi'^* C + E$. This is never an effective divisor because E is ϕ'-exceptional. By the birational invariance of p_g among smooth models, this contradicts to the existence of Ω on X'_0. Hence $\mathfrak{X}_0$ has only one component and $K_{\mathfrak{X}_0} = (K_{\mathfrak{X}} + \mathfrak{X}_0)|_{\mathfrak{X}_0} = \mathfrak{X}_0|_{\mathfrak{X}_0} = 0$.

To see that $X := \mathfrak{X}_0$ has at most canonical singularities, let $\phi : Y \to X$ be a resolution of singularities. Clearly $\Gamma(Y, K_Y) \cong \mathbb{C}$ since X has a smooth model with $p_g = 1$. In particular, K_Y is effective. Also $\Gamma(X, K_X) \cong \mathbb{C}$ since $K_X = 0$. If X is normal, since it is Gorenstein we have $K_Y = \phi^* K_X + E = E$ which shows that E is effective and so X has at most canonical singularities. If X is not normal, let $\psi : W \to X$ be the normalization with non-zero conductor divisor $C \subset W$ and let ϕ factors through ψ as $\phi = \phi' \circ \psi$ with $\phi' : Y \to W$. Then $K_W = \psi^* K_X - C = -C$ and $K_Y = \phi'^* K_W + E' = -\phi'^* C + E'$, which is never effective, a contradiction. Hence X is normal and is in fact a Calabi-Yau variety with at most canonical singularities. The proof is thus completed. □

6. Essential Incomplete Boundary Points

A degeneration over the unit disk is called *non-essential* if it admits a finite base change so that the punctured family can be completed into a smooth family.

Otherwise it is called *essential*. Clearly a necessary condition of a degeneration to be non-essential is that the monodromy is of finite order, or $N = 0$. In particular it is of finite distance. In fact for K3 surfaces any finite distance degeneration is non-essential [5]. It is thus interesting to see whether there exists essential finite distance degenerations in higher dimensions.

The classical Picard-Lefschetz theory states that for a Lefschetz pencil (i.e. nodal degenerations) each node, or ODP, p_i will correspond to a vanishing cycle $\sigma_i \in H_n(\mathcal{X}_s, \mathbb{Z})$ and all these σ_i's generate the space of vanishing cycles V which is the kernel of the map $H_n(\mathcal{X}_s) \to H_n(\mathcal{X}_0) = H_n(\mathcal{X})$. Moreover the monodromy $N \neq 0$ if and only if $V \neq 0$. However, there is no general local criterion to test whether $\sigma_i \neq 0$ for a given ODP p_i. The σ_i is always trivial for n even and must be trivial even for n odd if $H_n(\mathcal{X}_s) = 0$ — for example, nodal degenerations for cubic or quartic threefolds.

Here we show that $V \neq 0$ for any nodal degenerations of Calabi-Yau 3-folds.[1]

Proof of Theorem 1.5, part (1). First of all, a nodal threefold $\mathcal{X}_0$ always admits (not necessarily projective) small resolutions $X \to \mathcal{X}_0$ with smooth rational curves $X \supset C_i \to p_i \in \mathcal{X}_0$ contracted to ODP's. In the case of Calabi-Yau threefolds (Gorenstein threefolds with trivial canonical bundle) with $h^1(\mathcal{O}) = 0$, the existence of global smoothing $\mathcal{X} \to \Delta$ of $\mathcal{X}_0$ forces that there are nontrivial relations of $[C_i] \in H_2(X)$ by Friedman's result [1], [2]. That is, the canonical map $e : \bigoplus_i \mathbb{Z}[C_i] \to H_2(X, \mathbb{Z})$ has nontrivial kernel dimension $\rho > 0$. Consider the resulting surgery diagram:

$$\begin{array}{l} X \\ \downarrow \\ \mathcal{X}_0 \subset \mathcal{X} \supset \mathcal{X}_s \end{array}$$

It has the following local description: let $V_i \ni p_i$ be a contractible neighborhood of an ODP, $V_i' \subset \mathcal{X}_s$ be the smoothing of V_i and $U_i \subset X$ be the inverse image of V_i. Then

I. U_i is a deformation retract neighborhood of C_i and so has the homotopy type of $S^2 \sim D^4 \times S^2$.
II. V_i' has the homotopy type of $S^3 \times D^3$. Where the sections $\sigma_i \sim S^3$ are the vanishing cycles.
III. The surgery from X to $\mathcal{X}_s$ is induced from $\partial(D^4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times D^3)$.

Let us assume that there are k ODP's. An immediate consequence is the Euler number formula:

$$\chi(X) - k\chi(\mathbb{P}^1) = \chi(\mathcal{X}_0) - k\chi(\mathrm{pt}) = \chi(\mathcal{X}_s) - k\chi(S^3).$$

[1]The problem on essential finite distance boundaries is discussed in length in [10], §3-§4. However, the argument in [10], §3 concerning nodal degenerations is not complete. The author is grateful to M. Gross for discussions on this issue.

Let W be the "common open set" of X, $\mathcal{X}_0$ and $\mathcal{X}_s$ away from all points p_i's such that W and V_i's cover $\mathcal{X}_s$ etc. A portion of the Mayer-Vietoris sequence of the covering $\{W, V_i'\}$ of $\mathcal{X}_s$ gives

$$0 \to H_3(W) \to H_3(\mathcal{X}_s) \to \bigoplus_i \mathbb{Z}[C_i] \to H_2(X) \to H_2(\mathcal{X}_s) \to 0.$$

Hence that $b_2(X) = b_2(\mathcal{X}_s) + (k - \rho)$.

Take into account of $b_2(\mathcal{X}_0) = b_2(\mathcal{X}_s)$ and $b_4(\mathcal{X}_0) = b_4(X)$ (which also follows from suitable Mayer-Vietoris sequences), simple manipulations show that $b_3(\mathcal{X}_s) = b_3(\mathcal{X}_0) + \rho$. Comparing with the (Mayer-Vietoris) sequence defining the vanishing cycles:

$$\bigoplus_i \mathbb{Z}[\sigma_i] \to H_3(\mathcal{X}_s) \to H_3(\mathcal{X}_0) \to 0,$$

we conclude that ρ, which is non-zero, is the dimension of V. $\square$

It is possible for an essential degeneration to have trivial monodromy. In [2], Friedman remarked that Clemens has constructed families of quintic hypersurfaces in $\mathbb{P}^4$ acquiring an A_2 singularity and have monodromy of finite order. He then asked whether this family can be filled in smoothly up to a base change. This has been answered negatively in [10]. We recall the statement here for the reader's convenience.

Theorem 6.1. *Let $\mathcal{X} \to \Delta$ be a projective smoothing of a nontrivial Gorenstein terminal minimal threefold $\mathcal{X}_0$ over the unit disk. Then, up to any finite base change, $\mathcal{X} \to \Delta$ is not Δ-birational to a projective smooth family $\mathcal{X}' \to \Delta$ of minimal threefolds.*

In view of Proposition 1.3, this implies that terminal degenerations of smooth minimal threefolds provide essential incomplete boundary points of the moduli spaces with respect to the quasi-Hodge metrics g_m for all m. This proves Theorem 1.5, part (2).

7. Remarks on Moduli and Weil-Petersson Metrics

Canonical singularity naturally occurs in minimal and canonical models in algebraic geometry. It also plays a significant role in string theory through the connection with Calabi-Yau manifolds. On the other hand, in various situations it behaves just like smooth points. A nice example is Viehweg's program on constructing quasi-projective moduli spaces of polarized manifolds [9]. He showed that it is essentially the same proof to include varieties with canonical singularities as long as the deformation invariance can be verified. This was recently proved by Kawamata [4] based on [8].

Along different lines, the author had attempted to understand the boundary of moduli spaces of Calabi-Yau manifolds from the differential geometric point of view [10]. It was found that the natural Weil-Petersson metric on the moduli space is incomplete, therefore the metric completion of moduli spaces becomes an important problem. It was proved that degenerations of Calabi-Yau manifolds with at most canonical singularities are at finite Weil-Petersson distance. It was also conjectured there that the converse holds. Its truth would imply that Viehweg's enlarged moduli spaces coincide with the metric completion of the moduli spaces, hence a perfect match between viewpoints in algebraic geometry and differential geometry. Now this follows from the minimal model conjectures by Proposition 1.2.

The Weil-Petersson metric on the moduli space of Calabi-Yau manifolds is defined as the variation of the underlying Ricci flat metrics. For a given polarized Calabi-Yau family $\mathfrak{X} \to S$ with Ricci flat metrics $g(s)$ on $\mathfrak{X}_s$, under the Kodaira-Spencer map $\rho : T_{S,s} \to H^1(\mathfrak{X}_s, T_{\mathfrak{X}_s}) \cong \mathbb{H}^{0,1}_{\bar\partial}(T_{\mathfrak{X}_s})$ (harmonic forms with respect to $g(s)$), we have for $v,\ w \in T_s(S)$,

$$g_{WP}(v,w) := \int_{\mathfrak{X}_s} \langle \rho(v), \rho(w) \rangle_{g(s)}.$$

It is a quite surprising fact that g_{WP} admits a Hodge theoretic description. Indeed $g_{WP} = g_H$. This follows from the fact that the holomorphic volume form $\Omega(s)$ is parallel with respect to $g(s)$, which again is equivalent to the Ricci flat condition

$$\Omega(s) \wedge \bar\Omega(s) = f(s)\, \omega^n_{g(s)}$$

for a constant $f(s)$ depending only on $s \in S$. Indeed $f(s)$ is the point-wise length square of $\Omega(s)$ if we normalize the volume to be 1. This viewpoint provides an alternative differential geometric way to look at the above canonical singularity conjecture without using the minimal model theory.

With the Ω chosen as in §2, the incompleteness of $g_H = g_{WP}$ of a punctured Calabi-Yau family $\pi : \mathfrak{X} \to \Delta^\times$ is equivalent to the continuity of $f(s)$ over $0 \in \Delta$. We attempt to show from this the uniform boundedness of diameter of $\mathfrak{X}_s$ for all $s \in \Delta^\times$. With this done, we may then proceed by using the theory of Hausdorff convergence. The details of this differential geometric approach will appear in a separate work.

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