Curvature Properties of the Calabi-Yau Moduli

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Received: September 1, 2003

Communicated by Thomas Peternell

Abstract. A curvature formula for the Weil-Petersson metric on the Calabi-Yau moduli spaces is given. Its relations to the Hodge metrics and the Bryant-Griffiths cubic form are obtained in the threefold case. Asymptotic behavior of the curvature near the boundary of moduli is also discussed via the theory of variations of Hodge structures.

2000 Mathematics Subject Classification: 32G20, 32Q25.
Keywords and Phrases: Weil-Petersson metric, Calabi-Yau manifold, variations of Hodge structures.

Introduction

In a former paper [15], the incompleteness phenomenon of the Weil-Petersson metric on Calabi-Yau moduli spaces was studied. In this note, I shall discuss some curvature properties of it. The first result is a simple explicit formula (Theorem 2.1) for the Riemann curvature tensor. While this problem was treated before in [6] and [10], the approach taken here is more elementary. Two simple proofs of Theorem 2.1 are offered in §2 and both are based on the Hodge-theoretic description of the Weil-Petersson metric [13]. The first one uses a trick to select suitable coordinate system and line bundle section to reduce the computation. The second proof uses Griffiths' curvature formula for Hodge bundles [2].

Direct consequences of Theorem 2.1 are relations between the Weil-Petersson metric and the Hodge metric for Calabi-Yau threefolds and various positivity results on the curvature tensor (see §3). §4 is devoted to the asymptotic analysis of the curvature near the boundary of moduli spaces. The method is modelled on the first proof and uses Schmid's theory on the degenerations of Hodge structures [7]. The final section §5 contains some remarks toward the completion and compactification problems of Calabi-Yau moduli spaces.

1. The Weil-Petersson metric

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. The local Kuranish family of polarized Calabi-Yau manifolds $\mathcal{X} \to S$ is smooth (unobstructed) by the Bogomolov-Tian-Todorov theorem [13]. One can assign the unique (Ricci-flat) Yau metric $g(s)$ on $\mathcal{X}_s$ in the polarization Kähler class [17]. Then, on a fiber $\mathcal{X}_s =: X$, the Kodaira-Spencer theory gives rise to an injective map $\rho : T_s(S) \to H^1(X, T_X) \cong \mathcal{H}^{0,1}_{\bar\partial}(T_X)$ (harmonic representatives). The metric $g(s)$ induces a metric on $\Lambda^{0,1}(T_X)$. For v, $w \in T_s(S)$, one then defines the *Weil-Petersson metric* on S by

$$g_{WP}(v, w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)}. \tag{1.1}$$

Let $\dim X = n$. Using the fact that the global holomorphic n-form $\Omega(s)$ is flat with respect to $g(s)$, it can be shown [13] that

$$g_{WP}(v, w) = -\frac{\tilde{Q}(i(v)\Omega, \overline{i(w)\Omega})}{\tilde{Q}(\Omega, \bar\Omega)}. \tag{1.2}$$

Here, for convenience, we write $\tilde{Q} = \sqrt{-1}^n Q(\cdot, \cdot)$, where Q is the intersection product. Therefore, $\tilde{Q}$ has alternating signs in the successive primitive cohomology groups $P^{p,q} \subset H^{p,q}$, $p + q = n$.
(1.2) implies that the natural map $H^1(X, T_X) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1})$ via the interior product $v \mapsto i(v)\Omega$ is an isometry from the tangent space $T_s(S)$ to $(H^{n,0})^* \otimes P^{n-1,1}$. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain. A simple calculation in formal Hodge theory shows that

$$\omega_{WP} = \mathrm{Ric}_{\tilde{Q}}(\mathcal{H}^{n,0}) = -\partial\bar\partial \log \tilde{Q}(\Omega, \bar\Omega), \tag{1.3}$$

where ω_{WP} is the 2-form associated to g_{WP}. In particular, g_{WP} is Kähler and is independent of the choice of Ω. In fact, g_{WP} is also independent of the choice of the polarization.
With this background, one can abstract the discussion by considering a polarized variations of Hodge structures $\mathcal{H} \to S$ of weight n with $h^{n,0} = 1$ and a smooth base S. In this note, I always assume that it is effectively parametrized in the sense that the infinitesimal period map (also called the second fundamental form [2])

$$\sigma : T_s(S) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \oplus \mathrm{Hom}(H^{n-1,1}, H^{n-2,2}) \oplus \cdots \tag{1.4}$$

is bijective in the first piece. Then the *Weil-Petersson metric g_{WP}* on S is defined by formula (1.2) (or equivalently, (1.3)).
One advantage to work with the abstract setting is that, instead of using $P^{p,q}$ in the geometric case, we may write $H^{p,q}$ directly in our presentation.

2. The Riemann curvature tensor formula

Here is the basic formula (compare with [6], [10] and [12]):

Theorem 2.1. *For a given effectively parametrized polarized variations of Hodge structures $\mathcal{H} \to S$ of weight n with $h^{n,0} = 1$, $h^{n-1,1} = d$ and smooth S, in terms of any holomorphic section Ω of $\mathcal{H}^{n,0}$ and the infinitesimal period map σ, the Riemann curvature tensor of the Weil-Petersson metric $g_{WP} = g_{i\bar{j}} dt_i \otimes d\bar{t_j}$ on S is given by*

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}) + \frac{\tilde{Q}(\sigma_i \sigma_k \Omega, \overline{\sigma_j \sigma_\ell \Omega})}{\tilde{Q}(\Omega, \bar{\Omega})}. \tag{2.1}$$

2.1. The first proof. The main trick in the proof is a nice choice of the holomorphic section Ω and special coordinate system on the base S. Since the problem is local, we may assume that S is a disk in $\mathbb{C}^d$ around $t = 0$. Specifically, we have

Lemma 2.2. *For any $k \in \mathbb{N}$, there is a local holomorphic section Ω of $\mathcal{H}^{n,0}$ such that in the power series expansion at $t = 0$*

$$\Omega(t) = a_0 + \sum\nolimits_i a_i t_i + \cdots + \sum\nolimits_{|I|=k} \frac{1}{I!} a_I t^I + \cdots \tag{2.2}$$

we have $a_0 \in H^{n,0}$, $\tilde{Q}(a_0, \bar{a_0}) = 1$ and $\tilde{Q}(a_0, \bar{a_I}) = 0$ for any multi-index $I \neq 0$ and $|I| \leq k$. (We always assume that $a_I = a_J$ if $I = J$ as unordered sets.)

Proof. Only the last statement needs a proof. Let

$$\begin{aligned} \tilde{\Omega} &= \sum\nolimits_I \tilde{a}_I t^I = \Big(1 + \sum\nolimits_i \lambda_i t_i + \cdots + \sum\nolimits_{|I|=k} \lambda_I t^I\Big)\Omega \\ &= a_0 + \sum\nolimits_i (\lambda_i a_0 + a_i) t_i + \cdots + \sum\nolimits_{|I|=k} (\lambda_I a_0 + a_I) t^I + \cdots \end{aligned}$$

Set $\lambda_I = -\tilde{Q}(a_0, \bar{a_I})$, then clearly $\tilde{Q}(\tilde{a}_0, \overline{\tilde{a}_I}) = 0$ for $I \neq 0$ and $|I| \leq k$. □

Lemma 2.3. *Pick Ω as in Lemma 2.2. For any $k' \in \mathbb{N}$ with $2 \leq k' \leq k$, there is a holomorphic coordinate system t such that a_i form an orthonormal basis of $H^{n-1,1}$, i.e. $\tilde{Q}(a_i, \bar{a_i}) = -\delta_{ij}$. Moreover, $\tilde{Q}(a_i, \bar{a_I}) = 0$ for all i and I with $2 \leq |I| \leq k'$.*

Proof. The Griffiths transversality says that

$$a_i = \frac{\partial}{\partial t_i} \Omega \Big|_{t=0} \in H^{n,0} \oplus H^{n-1,1}.$$

Lemma 2.2 then implies that $a_i \in H^{n-1,1}$. It is also clear that by a linear change of coordinates of t we can make a_i to form an orthonormal basis of $H^{n-1,1}$.

For the second statement, consider the following coordinates transformation:

$$t_i = s_i + \sum\nolimits_{1 \leq j,k \leq d} c_i^{jk} s_j s_k + \cdots + \sum\nolimits_{|I|=k'} c_i^I s^I, \quad 1 \leq i \leq d$$

with $c_i^I = c_i^J$ when $I = J$ as unordered sets.

It's easy to see that the number of coefficients to be determined is the same as the number of equations $\tilde{Q}(a_i, \bar{a_{jk}}) = 0$, hence the lemma. □

Proof. (of Theorem 2.1) Let Ω and t_i be as in the above lemmas. For multi-indices I and J, we set $q_{I,J} := \tilde{Q}(a_I, \bar{a_J})$. By Lemma 2.2 and 2.3,

$$\begin{aligned} q(t) &:= \tilde{Q}(\Omega(t), \overline{\Omega(t)}) \\ &= 1 - \sum\nolimits_i t_i \bar{t_i} + \cdots + \sum\nolimits_{i,j,k,\ell} \frac{1}{(ik)!(j\ell)!} q_{ik,j\ell}\, t_i t_k \bar{t_j} \bar{t_\ell} + O(t^5). \end{aligned}$$

To calculate $R_{i\bar{j}k\bar{\ell}}$, we only need to calculate $g_{k\bar{\ell}}$ up to degree 2 terms:

$$\begin{aligned} g_{k\bar{\ell}} &= -\partial_k \partial_{\bar{\ell}} \log q = q^{-2}(\partial_k q \partial_{\bar{\ell}} q - q \partial_k \partial_{\bar{\ell}} q) \\ &= \big(1 + 2\sum\nolimits_i t_i \bar{t_i} + \cdots\big) \times \\ &\qquad \Big[t_\ell \bar{t_k} - \big(1 - \sum\nolimits_i t_i \bar{t_i}\big)\big(- \delta_{k\ell} + \sum\nolimits_{i,j} q_{ik,j\ell} t_i \bar{t_j}\big) + \cdots\Big] \\ &= \delta_{k\ell} - \delta_{k\ell} \sum\nolimits_i t_i \bar{t_i} + t_\ell \bar{t_k} + 2\delta_{k\ell} \sum\nolimits_i t_i \bar{t_i} - \sum\nolimits_{i,j} q_{ik,j\ell} t_i \bar{t_j} + \cdots \\ &= \delta_{k\ell} + \delta_{k\ell} \sum\nolimits_i t_i \bar{t_i} + t_\ell \bar{t_k} - \sum\nolimits_{i,j} q_{ik,j\ell} t_i \bar{t_j} + \cdots . \end{aligned}$$

Here we have used the fact that degree 3 terms of mixed type (contain $t_k \bar{t_\ell}$) must be 0 by our choice of Ω.
As a result, we find that the Weil-Petersson metric g is already in its geodesic normal form, so the full curvature tensor at $t = 0$ is given by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial t_i \partial \bar{t_j}} = -\delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{kj} + q_{ik,j\ell}.$$

Rewriting this in its tensor form then gives the formula. $\square$

Remark 2.4. The proof does not require the full condition that $\mathcal{H} \to S$ is a variation of Hodge structures. The essential part used is the polarization structure on the indefinite metric $\tilde{Q}$ on $\mathcal{H}$. It has a fixed sign on $\mathcal{H}^{n-1,1}$ makes possible the definition of the Weil-Petersson metric. For such cases, in terms of the second fundamental form σ, Lemma 2.2 and 2.3 say that under this choice of Ω and t, ordinary differentiations approximate σ up to second order at $t = 0$. In particular, $a_0 = \Omega(0)$, $a_i = \sigma_i \Omega(0)$ and $a_{ij} = a_{ji} = \sigma_i \sigma_j \Omega(0) = \sigma_j \sigma_i \Omega(0)$.

2.2. THE SECOND PROOF. Now we give another proof of Theorem 2.1 via Griffiths' curvature formula for Hodge bundles.

Proof. Recall the isometry in §1:

$$T_s(S) \cong (H^{n,0})^* \otimes H^{n-1,1} \tag{2.3}$$

and Griffiths' curvature formula ([2], Ch.II Prop.4):

$$\langle R(e), e' \rangle = \langle \sigma e, \sigma e' \rangle + \langle \sigma^* e, \sigma^* e' \rangle. \tag{2.4}$$

Where R is the matrix valued curvature 2-form of $\mathcal{H}^{p,q}$, e and e' are any two elements of $H^{p,q}$ and $\langle , \rangle$ is the Hodge metric.

Let Ω be a holomorphic section of $\mathcal{H}^{n,0}$ and consider the basis of $H^{n-1,1}$ given by $\sigma_i\Omega$, then T has a basis $e_i = \Omega^* \otimes \sigma_i\Omega$ from (2.3). In this basis, the Weil-Petersson metric takes the form

$$g_{i\bar{j}} = \frac{\langle \sigma_i\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.5}$$

Let K, R_1 and R_2 be the curvature of T, $(\mathcal{H}^{n,0})^*$ and $\mathcal{H}^{n-1,1}$ respectively. Using the standard curvature formulae for tensor bundle and dual bundle, we find

$$\begin{aligned} K(e_i) &= (R_1 \otimes I_2 + I_1 \otimes R_2)(\Omega^* \otimes \sigma_i\Omega) \\ &= R_1(\Omega^*) \otimes \sigma_i\Omega + \Omega^* \otimes R_2(\sigma_i\Omega). \end{aligned}$$

By taking scalar product with $e_j = \Omega^* \otimes \sigma_j\Omega$ and using the definition of dual metric, we get

$$\begin{aligned} \langle K(e_i), e_j\rangle_{WP} &= \langle R_1(\Omega^*), \Omega^*\rangle\langle \sigma_i\Omega, \sigma_j\Omega\rangle + \langle \Omega^*, \Omega^*\rangle\langle R_2(\sigma_i\Omega), \sigma_j\Omega\rangle \\ &= -\langle \Omega, \Omega\rangle^{-2}\langle R(\Omega), \Omega\rangle\langle \sigma_i\Omega, \sigma_j\Omega\rangle + \langle \Omega, \Omega\rangle^{-1}\langle R(\sigma_i\Omega), \sigma_j\Omega\rangle. \end{aligned}$$

Now we evaluate this 2-form on $e_k \wedge \bar{e_\ell}$ and apply (2.4), we get (notice the order of ℓ, k and the sign)

$$-\frac{\langle \sigma_k\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\langle \sigma_i\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle} + \frac{\langle \sigma_k\sigma_i\Omega, \sigma_\ell\sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle} - \frac{\langle \sigma_\ell^*\sigma_i\Omega, \sigma_k^*\sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.6}$$

Since $h^{n,0} = 1$, $\sigma_p^*\sigma_q$ acts as a scalar operator on $H^{n,0}$:

$$\sigma_p^*\sigma_q = \frac{\langle \sigma_p^*\sigma_q\Omega, \Omega\rangle}{\langle \Omega, \Omega\rangle} = \frac{\langle \sigma_q\Omega, \sigma_p\Omega\rangle}{\langle \Omega, \Omega\rangle}.$$

Hence the last term in (2.6) becomes

$$-\frac{\langle \sigma_i\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\overline{\langle \sigma_j\Omega, \sigma_k\Omega\rangle}}{\langle \Omega, \Omega\rangle} = -\frac{\langle \sigma_i\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\langle \sigma_k\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.7}$$

Using (2.5) and (2.7), then (2.6) gives the formula (2.1). □

3. Some simple consequences of the curvature formula

3.1. Lower bounds of curvature. The immediate consequences of the general curvature formula are various positivity results of different types of curvature. We mention some of them here.

Theorem 3.1. *For the Weil-Petersson metric g_{WP}, we have*

(1) *The holomorphic sectional curvature $\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell \geq -2|\xi|^4$.*
(2) *The Ricci curvature $R_{i\bar{j}} \geq -(d+1)g_{i\bar{j}}$.*
(3) *The second term in (2.1) is "Nakano semi-positive".*

Proof. This is a pointwise question. For simplicity, let's use the normal coordinate system given by Lemma 2.2. (1) is obvious since $\sum_{i,j,k,\ell} \tilde{Q}(a_{ik}, \overline{a_{j\ell}})\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell = \tilde{Q}(A, \bar{A}) \geq 0$ for $A = \sum_{i,k} a_{ik}\xi^i\xi^k$.

For (2), we need to show that $\big(\sum_{k,\ell} g^{k\bar{\ell}}\tilde{Q}(a_{ik},\overline{a_{jl}})\big)_{i,j}$ is semi-positive. For any vector $\xi=(\xi_i)$, let A_k be the vector $\sum_i a_{ik}\xi_i$. Then

$$\sum\nolimits_{i,j,k,\ell} g^{k\bar{\ell}}\tilde{Q}(a_{ik},\bar{a_{j\ell}})\xi_i\bar{\xi}_j=\sum\nolimits_k \tilde{Q}(A_k,\bar{A}_k)\geq 0.$$

For (3), it simply means that for any vector $u=(u_{pq})$ with double indices,

$$\sum\nolimits_{i,j,k,\ell}\tilde{Q}(a_{ik},\overline{a_{j\ell}})u_{ik}\overline{u_{j\ell}}=\tilde{Q}(A,\bar{A})\geq 0$$

where $A:=\sum_{p,q} a_{pq}u_{pq}$. □

3.2. Relation to the Hodge metric. The period domain has a natural invariant metric induced from the Killing form. The horizontal tangent bundle also has a natural metric induced from the metrics on the Hodge bundles. These two metrics are in fact the same ([2], p.18) and we call it the Hodge metric. The *Hodge metric* g_H on S is defined to be the metric induced from the Hodge metric of the full horizontal tangent bundle.
In dimension three, e.g. the moduli spaces of Calabi-Yau threefolds, we can reconstruct the Hodge metric from the Weil-Petersson metric. This result was first deduced by Lu in 1996 through different method, see e.g. [4].

Theorem 3.2. *In the case $n=3$, we have*

$$g_H=(d+3)g_{WP}+\mathrm{Ric}(g_{WP}).$$

In particular, the Hodge metric g_H is Kähler.

Proof. The horizontal tangent bundle is

$$\mathrm{Hom}(H^{3,0},H^{2,1})\oplus\mathrm{Hom}(H^{2,1},H^{1,2})\oplus\mathrm{Hom}(H^{1,2},H^{0,3})$$

The first piece gives the Weil-Petersson metric on S. The third piece is dual to the first one, hence, as one can check easily, gives the same metric. Now

$$(d+3)g_{i\bar{j}}+R_{i\bar{j}}=2g_{i\bar{j}}+\sum\nolimits_{k,\ell} g^{k\bar{\ell}}\,\frac{\tilde{Q}(\sigma_i\sigma_k\Omega,\overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega,\bar{\Omega})}.$$

The last term gives the Hodge metric of the middle part of the horizontal tangent bundle since $\sigma_k\Omega$ form a basis of $H^{2,1}$ and the Hodge metric is defined to be the metric of linear mappings, which are exactly the infinitesimal period maps σ_i's. □

Remark 3.3. Moduli spaces of polarized complex tori (resp. hyperkähler manifolds) correspond to variations of polarized weight one (resp. weight two) Hodge structures. Their universal covering spaces are Hermitian bounded symmetric domains and the invariant (Bergman) metrics are Kähler-Einstein of negative Ricci curvature. In these cases, the weight n polarized VHS are completely determined by the weight one (resp. weight two) polarized VHS. Based on this observation, one can show that g_{WP} and g_H both coincide with the Bergman metric up to a positive constant (cf. [9] for the case of g_{WP}). However, as we will see in Theorem 4.4, the negativity of Ricci curvature fails for moduli spaces of general Calabi-Yau manifolds. In fact, the Hodge theory of Calabi-Yau threefolds with $h^1(\mathcal{O}) = 0$ may be regarded as the first nontrivial instance of Hodge theory of weight three.

3.3. Relation to the Bryant-Griffiths cubic form. In the case $n = 3$, Bryant and Griffiths [1] has defined a symmetric cubic form on the parameter space S:

$$F_{ijk} := \frac{\tilde{Q}(\sigma_i\sigma_j\sigma_k\Omega, \Omega)}{\tilde{Q}(\Omega, \bar{\Omega})}.$$

Strominger [12] has obtained a formula for the Riemann curvature tensor through this cubic form F_{ijk} (in physics literature it is called the Yukawa coupling), and it has played important role in the study of Mirror Symmetry. We may derive it from our formula (2.1):

Theorem 3.4. *For an effectively parametrized polarized variations of Hodge structures $\mathcal{H} \to S$ of weight 3, the curvature tensor of g_{WP} is given by*

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}}) + \sum\nolimits_{p,q} g^{p\bar{q}}\, F_{pik}\overline{F_{qj\ell}}.$$

Proof. Since $\tilde{Q}(\sigma_i\sigma_j\Omega, \Omega) = 0$ by the consideration of types, the metric compatibility implies that

$$0 = \partial_k\tilde{Q}(\sigma_i\sigma_j\Omega, \Omega) = \tilde{Q}(\sigma_k\sigma_i\sigma_j\Omega, \Omega) - \tilde{Q}(\sigma_i\sigma_j\Omega, \sigma_k\Omega).$$

Let us write $\overline{\sigma_j\sigma_\ell\Omega} = \sum_p a^p\,\sigma_p\Omega$, then

$$\sum\nolimits_p a^p\, g_{p\bar{q}} = -\sum\nolimits_p a^p\, \frac{\tilde{Q}(\sigma_p\Omega, \overline{\sigma_q\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = -\frac{\tilde{Q}(\overline{\sigma_j\sigma_\ell\Omega}, \overline{\sigma_q\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = \overline{F_{qj\ell}}.$$

So $a^p = \sum_q g^{p\bar{q}}\overline{F_{qj\ell}}$ and the second term in (2.1) becomes

$$\frac{\tilde{Q}(\sigma_i\sigma_k\Omega, \overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = \sum\nolimits_{p,q} g^{p\bar{q}}\,\overline{F_{qj\ell}}\,\tilde{Q}(\sigma_i\sigma_k\Omega, \sigma_p\Omega) = \sum\nolimits_{p,q} g^{p\bar{q}}\, F_{pik}\,\overline{F_{qj\ell}}.$$

□

Remark 3.5. In the geometric case, namely moduli of Calabi-Yau threefolds, the cubic form is usually written as

$$F_{ijk} = e^{-K}\int_X \partial_i\partial_j\partial_k\,\Omega \wedge \Omega,$$

where $K = \log\tilde{Q}$ and Ω is a relative holomorphic three-form over S.

4. Asymptotic behavior of the curvature along degenerations

To study the asymptotic behavior of the curvature, we may localize the problem and study degenerations of polarized Hodge structures. By taking a holomorphic curve transversal to the degenerating loci, or equivalently we study the limiting behavior of the holomorphic sectional curvature, we may consider the following situation (consult [2], [7] for more details): a period mapping

$$\phi : \Delta^\times \to \langle T\rangle\backslash D \to \langle T\rangle\backslash\mathbb{P}(V)$$

which corresponds to the degeneration. Here $V = H^n$ is a reference vector space with a quadratic form Q as in §1, and with $T \in \mathrm{Aut}(V, Q)$ the Picard-Lefschetz monodromy. Assume that T is unipotent and let $N = \log T$. There is an uniquely defined weight filtration $W : 0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2n} = V$ such that

$$NW_i \subset W_{i-2} \quad \text{and} \quad N^k : Gr^W_{n+k} \cong Gr^W_{n-k}$$

where $Gr^W_i := W_i/W_{i-1}$. This W, together with the limiting Hodge filtration $F_\infty := \lim_{t\to 0} e^{-zN} F_t$ ($z = \log t/2\pi\sqrt{-1}$ is the coordinates on the upper half plane, $t \in \Delta^\times$) constitute Schmid's polarized limiting mixed Hodge structures. This means that Gr^W_i admits a polarized Hodge structure $\bigoplus_{p+q=i} H^{p,q}_\infty$ of weight i induced from F_∞ and Q such that for $k \geq 0$, the primitive part $P^W_{n+k} := \mathrm{Ker} N^{k+1} \subset Gr^W_{n+k}$ is polarized by $Q(\cdot, N^k\bar{\cdot})$. Notice that N is a morphism of type $(-1,-1)$ in the sense that $N(H^{p,q}_\infty) \subset H^{p-1,q-1}_\infty$. This allows one to view the mixed Hodge structure in terms of a Hodge diamond and view N as the operator analogous to "contraction by the Kähler form".

By Schmid's nilpotent orbit theorem ([7], cf. [15], §0-§1), we can pick the (multi-valued) holomorphic section Ω of $\mathcal{F}^n$ over $\Delta^\times$ by

$$\Omega(t) = \Omega(z) := e^{zN}\mathbf{a}(t) = e^{\log t \tilde{N}}\mathbf{a}(t) \in F^n_t,$$

where $\mathbf{a}(t) = \sum a_i t^i$ is holomorphic over Δ with value in V and $\tilde{N} := N/2\pi\sqrt{-1}$. Also $0 \neq \mathbf{a}(0) = a_0 \in F^n_\infty$. (Notice that while $\Omega(z)$ is single-valued, $\Omega(t)$ is well-defined only locally or with its value mod T.)

Now we may summarize the computations done in [15], §1 in the following form:

Theorem 4.1. *The induced Weil-Petersson metric g_{WP} on $\Delta^\times$ is incomplete at $t = 0$ if and only if $F^n_\infty \subset \mathrm{Ker}\, N$.*

In the complete case, i.e. $Na_0 \neq 0$, let $k := \max\{\, i \mid N^i a_0 \neq 0\,\}$. Then $\tilde{Q}(\Omega, \bar{\Omega})$ blows up to $+\infty$ with order $c|\log|t|^2|^k$ and the metric g_{WP} blows up to $+\infty$ with order

$$\frac{k dt \otimes d\bar{t}}{|t|^2 \big|\log|t|^2\big|^2},$$

i.e. it is asymptotic to the Poincaré metric, where $c = (k!)^{-1}|\tilde{Q}(\tilde{N}^k a_0, \bar{a_0})| > 0$.

In the incomplete case, i.e. $Na_0 = 0$, the holomorphic section $\Omega(t)$ extends continuously over $t = 0$.

Idea of proof. We have the following well-known calculation: for any $k \in \mathbb{R}$,

$$-\partial\bar{\partial}\log|\log|t|^2|^k = \frac{k dt \wedge d\bar{t}}{|t|^2\big|\log|t|^2\big|^2}, \tag{4.1}$$

which is also true asymptotically if t is a holomorphic section of a Hermitian line bundle as the defining section of certain divisor, in a general smooth base S of arbitrary dimension. The main point is to prove that lower order terms are still of lower order whenever we take derivatives. This is done in [15] when $S = \Delta^\times$. This is also the main point of the remaining discussion in this section. □

To achieve the goal, we define operators $S_k = k + \tilde{N}$ for any $k \in \mathbb{Z}$. Then all S_k commute with each other and S_k is invertible if $k \neq 0$. By Theorem 4.1, we only need to study the incomplete case, i.e. $Na_0 = 0$. As in Lemma 2.2., we may assume that $\tilde{Q}(a_0, \bar{a_0}) = 1$ and $\tilde{Q}(a_0, \bar{a_i}) = 0$ for all $i \geq 1$.

Lemma 4.2. *If a_k is the first nonzero term other than a_0, then $S_k a_k \in F_\infty^{n-1}$. If moreover $Na_k = 0$ then $a_k \in H_\infty^{n-1,1}$, and for the next nonzero term $a_{k+\ell}$ we have $S_\ell S_{k+\ell} a_{k+\ell} \in F_\infty^{n-2}$.*

Proof. By the Griffiths transversality, we have

$$\Omega'(t) = e^{zN}\Big[\frac{1}{t}\tilde{N}\mathbf{a} + \mathbf{a}'\Big] \in F_t^{n-1}. \tag{4.2}$$

Since $Na_0 = 0$, this implies $\tilde{N}a_k t^{k-1} + k a_k t^{k-1} + \cdots \in e^{-zN}F_t^{n-1}$. Take out the factor t^{k-1} and let $t \to 0$, we get

$$S_k a_k \in F_\infty^{n-1}.$$

In fact, we have from (4.2), $e^{zN}(S_k a_k + S_{k+1}a_{k+1}t + \cdots) \in F_t^{n-1}$. Taking derivative and by transversality again, we get

$$e^{zN}\Big[\frac{1}{t}\tilde{N}S_k a_k + S_1 S_{k+1} a_{k+1} + S_2 S_{k+2} a_{k+2} t + \cdots\Big] \in F_t^{n-2}.$$

So, if $Na_k = 0$, then for the next nonzero term $a_{k+\ell}$, we have by the same way

$$S_\ell S_{k+\ell} a_{k+\ell} \in F_\infty^{n-2}.$$

We also know the following equivalence: $Na_k = 0$ iff $NS_k a_k = 0$ iff $S_k a_k \in H_\infty^{n-1,1}$ (because $S_k a_k \in F_\infty^{n-1}$). Since N is a morphism of type $(-1,-1)$ and the only nontrivial part of F_∞^n is in $H_\infty^{n,0}$, this is equivalent to $a_k \in H_\infty^{n-1,1}$. (This follows easily from the Hodge diamond.) $\square$

Define $q_{ij} = \tilde{Q}(e^{zN}a_i, \overline{e^{zN}a_j}) \equiv \tilde{Q}(e^{\log|t|^2\tilde{N}}a_i, \bar{a_j})$, which are functions of $\log|t|$. The following basic lemma is the key for all the computations, which explains "lower order terms are stable under differentiations".

Lemma 4.3. *Let $Q^{S_k}(a, \bar{b}) := Q(S_k a, \bar{b}) \equiv Q(a, \overline{S_k b})$, then for all $k, \ell \in \mathbb{Z}$,*

$$t\frac{\partial}{\partial t}(q_{ij}t^k\bar{t}^\ell) = q_{ij}^{S_k}t^k\bar{t}^\ell \quad \text{and} \quad \bar{t}\frac{\partial}{\partial \bar{t}}(q_{ij}t^k\bar{t}^\ell) = q_{ij}^{S_\ell}t^k\bar{t}^\ell.$$

Proof. Straightforward. $\square$

Now we state the main result of this section:

Theorem 4.4. *For any degeneration of polarized Hodge structures of weight n with $h^{n,0} = 1$, the induced Weil-Petersson metric g_{WP} has finite volume. For one parameter degenerations with finite Weil-Petersson distance, if $Na_1 \neq 0$ then g_{WP} blows up to $+\infty$ with order $c|\log|t|^2|^k$ and the curvature form $K_{WP} = K dt \wedge d\bar{t}$ blows up to $+\infty$ with order*

$$\frac{k dt \wedge d\bar{t}}{|t|^2|\log|t|^2|^2},$$

where $k = \max\{\, i \mid N^i S_1 a_1 \neq 0 \,\} \leq n-1$, $c = (k!)^{-1}|\tilde{Q}(\tilde{N}^k S_1 a_1, \overline{S_1 a_1})| > 0$.

Remark 4.5. The statement about finite volume is a standard fact in Hodge theory. By the nilpotent orbit theorem, the Hodge metrics on the Hodge bundles degenerate at most logarithmically. So g_H, and hence g_{WP}, has finite volume. In fact, in view of (4.1), the method of the following proof implies that g_H is asymptotic to the Poincaré metric of the punctured disk along transversal directions toward boundary divisors of the base space.

Proof. Pick $\mathbf{a}(t)$ and $\Omega(t)$ as before, (i.e. $Na_0 = 0$, $q_{00} = 1$, $q_{0i} = 0$ for $i \geq 1$ and all other q_{ij} are functions of $\log|t|$). We have

$$\begin{aligned} q(t) &= \tilde{Q}(\Omega(t), \bar{\Omega(t)}) \\ &= 1 + q_{11}t\bar{t} + (q_{12}t\bar{t}^2 + q_{21}t^2\bar{t}) + (q_{22}t^2\bar{t}^2 + q_{31}t^3\bar{t} + q_{13}t\bar{t}^3) + \cdots \end{aligned}$$

Applying Lemma 4.3, we can compute

$$\begin{aligned} \frac{\partial q}{\partial t}\frac{\partial q}{\partial \bar{t}} - q\frac{\partial^2 q}{\partial t \partial \bar{t}} = q_{11}^{S_1}\bar{t}q_{11}^{S_1}t \;-\; &(1 + q_{11}t\bar{t})(q_{11}^{S_1S_1} + q_{12}^{S_1S_2}\bar{t} + q_{21}^{S_2S_1}t \\ &+ q_{22}^{S_2S_2}t\bar{t} + q_{31}^{S_3S_1}t^2 + q_{13}^{S_1S_3}\bar{t}^2) + \cdots \end{aligned}$$

(Since we will assume that $Na_1 \neq 0$, q_{11} will be the only term needed. However we have calculated more terms in order for later use.) So the metric $ds^2 = g|dt|^2$ is given by

$$\begin{aligned} g &= -\partial_t\partial_{\bar{t}}\log q = q^{-2}(\partial_t q\partial_{\bar{t}}q - q\partial_t\partial_{\bar{t}}q) \\ &= -q_{11}^{S_1S_1} - (q_{12}^{S_1S_2}t + q_{21}^{S_2S_1}\bar{t}) \\ &\quad + (q_{11}q_{11}^{S_1S_1} + |q_{11}^{S_1}|^2 - q_{22}^{S_2S_2})t\bar{t} - q_{31}^{S_3S_1}t^2 - q_{13}^{S_1S_3}\bar{t}^2 + \cdots \end{aligned} \tag{4.3}$$

As $t \to 0$, the first term determines the behavior of g. If $Na_1 \neq 0$ (so $a_1 \neq 0$), let $\xi = S_1a_1 \in F_\infty^{n-1}$ (by Lemma 4.2) and let $k = \max\{\, i \mid N^i\xi \neq 0\,\}$. ($k \leq n-1$ simply because $Gr_{2n}^W = 0$.) Then the highest order term of $-q_{11}^{S_1S_1}$, with respect to $\log|t|^2$, is given by (for this purpose we can ignore all operators S_i with $i \neq 0$)

$$-\frac{1}{k!}\big(\log|t|^2\big)^k\tilde{Q}(\tilde{N}^k\xi, \bar{\xi}). \tag{4.4}$$

This term has nontrivial coefficient and is in fact positive. This follows from the fact that $\tilde{Q}$ polarizes the limiting mixed Hodge structures. Hence g blows up with the expected order, with $c = (k!)^{-1}|\tilde{Q}(\tilde{N}^kS_1a_1, \overline{S_1a_1})|$.
For the curvature form $K dt \wedge d\bar{t} = -\partial\bar{\partial}\log g$, the most singular term of K is given by

$$\begin{aligned} &\big(-q_{11}^{S_1S_1}\big)^{-2}\Big[\big(-q_{11}^{S_1S_1S_0}t^{-1}\big)\big(q_{11}^{S_1S_1S_0}\bar{t}^{-1}\big) - \big(-q_{11}^{S_1S_1}\big)\big(-q_{11}^{S_1S_1S_0S_0}t^{-1}\bar{t}^{-1}\big)\Big] \\ = &\big(-q_{11}^{S_1S_1}\big)^{-2}\Big[|q_{11}^{S_1S_1S_0}|^2 - q_{11}^{S_1S_1}q_{11}^{S_1S_1S_0S_0}\Big]|t|^{-2}. \end{aligned}$$

We need to show that this term is nontrivial. As before, we may ignore all S_i with $i \neq 0$. So the highest order terms of $\big(|q_{11}^{S_1S_1S_0}|^2 - q_{11}^{S_1S_1}q_{11}^{S_1S_1S_0S_0}\big)$ are

$$\frac{1}{(k-1)!}\big(\log|t|^2\big)^{2(k-1)}\big|\tilde{Q}(\tilde{N}^k\xi, \bar{\xi})\big|^2 - \frac{1}{k!(k-2)!}\big(\log|t|^2\big)^{k+(k-2)}\big|\tilde{Q}(\tilde{N}^k\xi, \bar{\xi})\big|^2.$$

It is clear that the coefficient $\frac{1}{(k-1)!} - \frac{1}{k!(k-2)!} = \frac{1}{k!(k-1)!} > 0$. Taking into account the order of $(-q_{11}^{S_1 S_1})^{-2}$ given by (4.4) shows that K blows up to $+\infty$ with the expected order $k|t|^{-2}\big|\log|t|^2\big|^{-2}$. □

Question 4.6. In the original smooth case in §2, a_1 corresponds to the Kodaira-Spencer class of the variation, so we know that $a_1 \neq 0$. For the degenerate case, what is the geometric meaning of a_1? Is it some kind of Kodaira-Spencer class for singular varieties?

Remark 4.7. Assume that $a_1 \neq 0$. If $Na_1 = 0$, then by Lemma 4.2, $a_1 \in H_\infty^{n-1,1}$ and

$$-q_{11}^{S_1 S_1} = -\tilde{Q}(e^{\log|t|^2 \tilde{N}} S_1 a_1, \overline{S_1 a_1}) \equiv -\tilde{Q}(a_1, \bar{a_1}) > 0.$$

From (4.3), g_{WP} has a non-degenerate continuous extension over $t = 0$.
In the case $n = 3$, if the moduli is one-dimensional ($h^{2,1} = 1$), then $a_1 \neq 0$ and $Na_1 = 0$ imply that $N \equiv 0$. That is, the variation of Hodge structures does not degenerate at all. More generally, if we have a positive answer to Question 4.6, then we also have a similar statement for multi-dimensional moduli. Namely, $N_i a_0 = 0$, $N_i a_{1,j} = 0$ for all i, j implies that $N_i \equiv 0$ for all i, where N_i's are the local monodromies, $a_{1,j}$'s are coefficients of the linear terms in $\mathbf{a}(t)$.

We conclude this section by a partial result:

Proposition 4.8. *Assume that $a_1 \neq 0$, $Na_1 = 0$ (so g_{WP} is continuous over $t = 0$). If $Na_2 = 0$ then the curvature tensor has a continuous extension over $t = 0$. The converse is true for $n = 3$. If the curvature tensor does not extend continuously over $t = 0$, it has a logarithmic blowing-up.*

Proof. Following Remark 4.7 and Lemma 2.3, we may assume that $-q_{11} = 1$, $q_{12} = 0$ and q_{1j}'s are constants for $j \geq 2$. So among the degree two terms of g, only the $t\bar{t}$ term contributes to the curvature. Namely from (4.3),

$$g = 1 + (2 - q_{22}^{S_2 S_2})t\bar{t} + \cdots.$$

Again we are in the "normal coordinates", so

$$K = -2 + q_{22}^{S_2 S_2 S_1 S_1} + \cdots.$$

It is clear that $Na_2 = 0$ implies that $q_{22}^{S_2 S_2 S_1 S_1}$ is a constant ($= 4\tilde{Q}(a_2, \bar{a_2})$). We will show that the converse is true if $n = 3$. We may assume that $a_2 \neq 0$. By Lemma 4.2 we have $a_1 \in H_\infty^{2,1}$ and $S_1 S_2 a_2 \in F_\infty^1$, so we may write (from the Hodge diamond) $S_1 S_2 a_2 = \lambda a_1 + \alpha + \beta$ with $\alpha \in H_\infty^{2,2}$ and $\beta \in H_\infty^{1,1}$. Now the constancy of $q_{22}^{S_2 S_2 S_1 S_1}$ implies that

$$\tilde{Q}(N(\lambda a_1 + \alpha + \beta), \overline{\lambda a_1 + \alpha + \beta}) = 0$$

as it is the highest order term ($N^2 \equiv 0$). So the fact $Na_1 = 0$ and $N\beta = 0$ implies that $\tilde{Q}(N\alpha, \bar{\alpha}) = 0$. This in turn forces $\alpha = 0$ by the polarization condition. So $S_1 S_2 a_2 = \lambda a_1 + \beta$, and $S_1 S_2 N a_2 = \lambda N a_1 + N\beta = 0$. That is, $Na_2 = 0$. The remaining statement about the logarithmic blowing-up is clear. □

5. Concluding Remarks

Based on Yau's solution to the Calabi Conjecture [17], the existence of the coarse moduli spaces of polarized Calabi-Yau manifolds in the category of separated analytic spaces was proved by Schumacher [8] in the 80's. Combined with the Bogomolov-Tian-Todorov theorem [13], these moduli spaces are smooth Kähler orbifolds equipped with the Weil-Petersson metrics.

In the algebraic category, the coarse moduli spaces can also be constructed in the category of Moishezon spaces. Moreover, for polarized (projective) Calabi-Yau manifolds, the quasi-projectivity of such moduli spaces has been proved by Viehweg [14] in late 80's.

From the analytic viewpoint, there is also a theory originated from Siu and Yau [11] dealing with the projective compactification problem for complete Kähler manifolds with finite volume. Results of Mok, Zhong [5] and Yeung [18] say that a sufficient condition is the negativity of Ricci curvature and the boundedness of sectional curvature. In general, the Weil-Petersson metric does not satisfy these conditions. This leads to some puzzles since the ample line bundle constructed by Viehweg ([14], Corollary 7.22) seems to indicate that ω_{WP} will play important role in the compactification problem. Since the curvature tends to be negative in the infinite distance boundaries, the puzzle occurs only at the finite distance part. This leads to two different aspects:

The first one is the geometrical metric completion problem. In [15], it is proposed that degenerations of Calabi-Yau manifolds with finite Weil-Petersson distance should correspond to degenerations with at most canonical singularities in a suitable birational model. Now this is known to follow from the minimal model conjecture in higher dimensions [16]. With this admitted, one may then go ahead to analyze the structure of these completed spaces. Are they quasi-affine varieties?

Another aspect is the usage of Hodge metric. Näively, since the Ricci curvature of the Weil-Petersson metric has a lower bound $-(d+1)g_{WP}$ and blows up to ∞ at some finite distance boundary points, for any $k > 0$ one may pull these points out to infinity by considering the following new Kähler metric

$$\tilde{g}_{i\bar{j}} = (d+1+k)g_{i\bar{j}} + R_{i\bar{j}} = kg_{i\bar{j}} + \sum\nolimits_{k,\ell} g^{k\bar{\ell}} \frac{\tilde{Q}(\sigma_i\sigma_k\Omega, \overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})}.$$

When the blowing-up is faster than logarithmic growth (e.g. $Na_1 \neq 0$), $\tilde{g}$ is then complete (at these boundaries). Otherwise one may need to repeat this process. This inductive structure is implicit in Theorem 4.1 and 4.4 and is explicitly expressed in Theorem 3.2 in the case $n = 3$, $k = 2$. It suggests that the resulting metric will be quasi-isometric to the Hodge metric.

If we start with the Hodge metric directly, the coarse moduli spaces being Moishezon allows us to assume that the boundary has a local model as normal crossing divisors. Then a similar asymptotic analysis as in §4 implies that the metric behaves like the Poincaré metric in the transversal direction toward the codimension one boundaries (i.e. points with $N \not\equiv 0$ where N is the local monodromy). In particular, it admits bounded sectional curvature. The main problem here is the higher codimensional boundaries. In this direction, we should mention that the negativity of the Ricci curvature for Hodge metrics has recently been proved by Lu [3]. We expect that the Hodge metric would eventually provide projective compactifications of the moduli spaces through the recipe of [5], [18].

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