QUANTUM INVARIANCE UNDER FLOP TRANSITIONS

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ABSTRACT. We survey some recent progress on the invariance of quantum cohomology and Gromov-Witten theory under flop transitions up to analytic continuations over the extended Kähler moduli. In particular we discuss in more details the case of ordinary flops of splitting type among Calabi-Yau manifolds.

This is partly based on my lecture delivered at Harvard University on August 28th, 2008 in celebration of Professor Shing-Tung Yau's 60th birthday. I would like to dedicate this article to him for his encouragement during the past fifteen years.

1. INTRODUCTION

The content of this article is based on joint works with Yuan-Pin Lee and Hui-Wen Lin [9, 10] in recent years. The origin of the problem, however, goes back to an old question on the finite Weil-Petersson distance boundary points in the Calabi-Yau moduli spaces. The necessity to study these boundary points was suggested by string theory in order to study the space-time topology change under extremal transitions. In fact this was suggested to me by Yau around 1994 as part of my thesis project.

It is generally hoped, also known as Reid's fantasy, that Calabi-Yau 3-folds (or n-folds) can be connected through extremal transitions along these finite distance boundaries. The exact study of the change of quantum effect under extremal transitions is still in a rather preliminary stage. An easier problem which compares the quantum effect among various different extremal transitions had caught a lot more attention in recent years. An extremal transition is a degeneration of Calabi-Yau manifolds into canonical singularities then followed by a crepant resolution. However, crepant resolutions are generally not unique. Two crepant resolutions are related by a flop, and flops are believed to form the building blocks of birational maps between Calabi-Yau manifolds (recently confirmed in [6]) or more general K equivalent manifolds. Thus it is a reasonable question to study the variation of quantum effect under a flop transition.

There are several reasons to study flops first. A major one is that flops preserve cohomology groups (the number of fields). Thus the quantum effects split perfectly into A model and B model respectively and we are led to claim the invariance of quantum cohomology for the A side and invariance of Kodaira-Spencer theory for the B side in a certain sense. This is in contrast to another famous transition in Calabi-Yau moduli, namely the mirror symmetry where the two theories are expected to switch roles after transition. Even in that context, historically the construction of mirror manifolds relies on orbifold or toric constructions where crepant resolutions are also needed as a final step. Thus the mirror family is only determined up to birational equivalence and the study of flops is needed to complete the story.

Let

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X' \\ & \searrow^{\psi} \quad {}^{\psi'}\swarrow & \\ & \bar{X} & \end{array}$$

be a smooth flop in the complex projective category. We had studied three types of such flops including the ordinary flops [9, 10, 7], Mukai flops [9] and stratified Mukai flops [3]. In all these cases the fiber product

$$[X \times_{\bar{X}} X'] \in A^*(X \times X')$$

induces canonical isomorphism-correspondence $\mathscr{F}$ of Chow motives and hence on the cohomology realization $H(X) \cong H(X')$ which we denote by H. Here the term canonical means that $\mathscr{F}$ preserves the Poincaré pairing.

The main problem is that $\mathscr{F}$ in general does not preserve the classical cup product. Thus *quantum corrections* are expected! The natural candidate comes from the *quantum product* in Gromov-Witten theory.

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps to X from genus g nodal curves with n marked points, and let $e_i : \overline{M}_{g,n}(X, \beta) \to X$ be the evaluation maps. The Gromov-Witten potential

$$\begin{aligned} F_g^X &= \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}^X \\ &= \sum_{n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t \end{aligned}$$

and the partition function (total potential)

$$Z^X = \exp \sum_{g=0}^\infty \hbar^{g-1} F_g^X(t),$$

are formal functions in $t \in H$ and Novikov variables q^β, with β in the Mori cone of effective classes of one cycles. Modulo convergence issue, they are functions on the complexified Kähler cone $\omega \in \mathcal{K}_X^{\mathbb{C}} := H_{\mathbb{R}}^{1,1} + i\mathcal{K}_X$ via

$$q^\beta = e^{2\pi i(\beta.\omega)}.$$

To compare Z^X and $Z^{X'}$, one notices that they share the same variable $t \in H$ but different variables in $NE(X)$ and $NE(X')$. In the formal level

$$\mathscr{F} q^\beta = q^{\mathscr{F}\beta}.$$

But for ℓ (resp. ℓ') being the ψ (resp. ψ') extremal ray, it is easy to check that

$$\mathscr{F}\ell = -\ell'$$

which is not effective. By duality this implies that

$$\mathcal{K}_X^{\mathbb{C}} \cap \mathcal{K}_{X'}^{\mathbb{C}} = \emptyset \quad \text{in } H_{\mathbb{C}}^2,$$

hence Z^X and $Z^{X'}$ have different domains of definition and the comparison can make sense only after analytic continuations over $\mathcal{K}_X^{\mathbb{C}} \cup \mathcal{K}_{X'}^{\mathbb{C}} \subset H_{\mathbb{C}}^2$. For this reason, the extended Kähler moduli $\mathcal{K}$ is defined to be the union of all $\mathcal{K}_{X'}^{\mathbb{C}}$'s with X' being smooth projective and K equivalent to X.

Let $\{T_i\}$ be a basis of H with $\{T^i\}$ being the dual basis with respect to the Poincaré pairing. Denote by $t = \sum t^i T_i$. The big quantum ring $(QH(X), *)$ uses only the genus zero potential with $n \geq 3$ marked points:

$$\begin{aligned} T_i *_t T_j &= \sum_k \frac{\partial^3 F_0^X}{\partial t^i \partial t^j \partial t^k}(t) T^k \\ &= \sum_{n \geq 0, \beta \in NE(X)} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle_{0,n,\beta}^X T^k. \end{aligned}$$

The condition $2g + n \geq 3$ is known as the stable range, so that the point case $X = \mathrm{pt}$ — the Deligne-Mumford space of stable curves $\overline{M}_{g,n}$ — is defined.

Conjecture 1.1. [16] *Among K equivalent manifolds, there exists canonical correspondences so that in the stable range the quantum invariance holds up to analytic continuations over the extended Kähler moduli.*

Quantum corrections attached to the extremal ray for Calabi-Yau 3-folds seems to first appear in Witten's article [17]. The first complete mathematical result in this direction is due to Li and Ruan on the invariance of big quantum ring for P^1 flop of three-folds [14], where they developed the degeneration/gluing formula of Gromov-Witten invariants to reduce the problem to local models. A version of Conjecture 1.1 had since then been raised by Ruan, known as the *quantum minimal model conjecture*.

Subsequently the study had been extended to higher dimensional cases as well as some higher genus case [9, 7, 10, 3]. The Gromov-Witten theory can be extended to allow *descendent* insertions in general, as well as *ancestor* insertions in the stable range. The quantum invariance is expected to hold for the ancestor potential [7].

Here is a summary of results we recently obtained:

Theorem 1.2. *For flops under fiber product correspondence $\mathscr{F}$, the quantum invariance up to analytic continuations holds in the following cases.*

(1) *For simple ordinary flops the quantum invariance holds for all genera in the stable range* [9, 7], *including the ancestor invariants.*
(2) *For ordinary flops over a general base, the big quantum ring restricted to the extremal ray is invariant. In general the big quantum ring is invariant if the flop is of splitting type* [10].
(3) *For Mukai flops the full Gromov-Witten theory is absolutely invariant without the need of analytic continuation*[9].
(4) *For stratified Mukai flops of type $A_{n,2}$, D_5 and $E_{6,I}$, the big quantum ring restricted to the extremal ray is invariant* [3, 10].

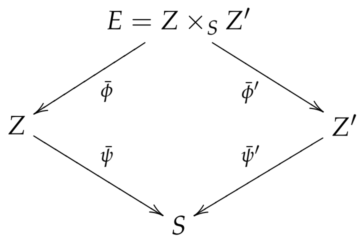
The purpose of this article is however modest. It is limited to the genus zero theory of ordinary flops. We start by reviewing the general framework initiated in [9] for the case of simple ordinary flops, and then discuss the main ideas used in the extension to the non-simple case [10].

In the second half we present in details a special case of (2), namely the quantum invariance under ordinary flops of *splitting type* among Calabi-Yau manifolds — as this is the most interesting case in connection with string theory.

2. Ordinary flops: Genus zero theory

2.1. **The canonical correspondence.**

The local geometry of ordinary P^r flops are based on two rank $r+1$ vector bundles F, F' over a smooth base S. Let $Z = P_S(F)$ and $Z' = P_S(F')$. The exceptional loci are identified with



where $\psi : X \to \bar X$ is the extremal contraction with $\bar\psi = \psi|_Z$ and

$$N := N_{Z/X} = \bar\psi^* F' \otimes \mathscr{O}_Z(-1).$$

The flop $f : X \dashrightarrow X'$ is obtained by blowing up $\phi : Y = \mathrm{Bl}_Z X \to X$ followed by the contraction $\phi' : Y \to X'$ of the exceptional divisor E in the $\bar\phi'$ fiber direction. The existence of ϕ' is deduced from the existence of ψ by the cone theorem.

Since $\dim Z \times_S Z' = \dim X - 1$, we have $Y = X \times_{\bar X} X' = \bar\Gamma_f \subset X \times X'$. As a graph correspondence, we have

$$\mathscr{F} := [\bar\Gamma_f]_* = \phi'_* \circ \phi^* : A(X) \to A(X').$$

One check directly [9] that for $a \in A(X)$, $\phi^* a = \phi'^* \mathscr{F} a + e$ where e is both ϕ and ϕ' exceptional, hence it induces the Chow groups as well as Chow motives isomorphism $h(X) \cong h(X')$ by considering $X \times T \dashrightarrow X' \times T$ and using the identity principle. Moreover, the Poincaré pairing

$$(a,b)^X = (\phi^* a, \phi^* b)^Y = (\phi'^* \mathscr{F} a, \phi^* b)^Y + (e, \phi^* b)^Y = (\mathscr{F} a, \mathscr{F} a)^{X'}$$

is preserved by the projection formula.

2.2. **The case of simple flops.** [9]

A flop f is simple if $S = \mathrm{pt}$. The study of analytic continuation of $QH(X)$ under simple ordinary flops is the starting point of the whole project. The general framework developed there involves four major steps:

(1) Determination of the defect of cup product under $\mathscr{F}$. Let $a_i \in A^{k_i}(X)$ with $1 \le k_i \le r$, $k_1 + k_2 + k_3 = \dim X = 2r+1$. Then

$$\begin{aligned} &(\mathscr{F} a_1 . \mathscr{F} a_2 . \mathscr{F} a_3)^{X'} - (a_1.a_2.a_3)^X \\ &\quad = (-1)^r (a_1.h^{r-k_1})^X (a_2.h^{r-k_2})^X (a_3.h^{r-k_3})^X. \end{aligned} \tag{2.1}$$

This was done by a brute force calculation using standard intersection theory [4]. Since the cup product is determined by the triple product via the Poincaré pairing, its defect under $\mathscr{F}$ is completely characterized by the above formula.

(2) Determination of $g = 0$, $n \ge 3$ points GW invariants attached to $\beta \in \mathbb{Z}_{\ge 0}\ell$ and showing that it corrects the topological defect up to analytic continuation.

The virtual dimension $D_{g,n,\beta} := \dim[\overline{M}_{g,n}(X,\beta)]^{vir}$ is given by

$$D_{g,n,\beta} = c_1(X).\beta + \dim X(1-g) + n + 3g - 3.$$

Extremal rays of flopping type are K-trivial. Let $\beta = d\ell$, $d \geq 1$ and $a_i \in A^{k_i}(X)$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} k_i = D_{0,n,d\ell} = 2r + 1 + (n-3)$, then

$$\begin{aligned}(2.2)\qquad &\langle a_1, \ldots, a_n \rangle^X_{0,n,d\ell} \\ &= (-1)^{(r+1)(d-1)} N_{k_1,\ldots,k_n} d^{n-3} (a_1.h^{r-k_1})^X \cdots (a_n.h^{r-k_n})^X,\end{aligned}$$

where the universal constants $N_{k_1,\ldots,k_n} = 1$ for $n = 2$ or 3. This is the generalization of the well-known multiple cover formula for $\mathscr{O}(-1) \oplus \mathscr{O}(-1) \to P^1$. The proof uses the reconstruction theorem (divisor relation) [8] to reduce to the $n = 1$ descendent invariants which can be calculated from [11, 2].

Consider the rational form of the geometric series

$$\mathbf{f}(q) := \frac{q}{1 - (-1)^{r+1} q} = \sum_{d \geq 1} (-1)^{(r+1)(d-1)} q^d.$$

It satisfies the functional equation $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$ which serves as the source of analytic continuations: For extremal functions $\langle A \rangle_0 := \sum_{d=0}^{\infty} \langle A \rangle_{d\ell} q^{d\ell}$,

$$\begin{aligned}(2.3)\qquad &\langle \mathscr{F}a_1, \mathscr{F}a_2, \mathscr{F}a_3 \rangle^{X'}_0 - \mathscr{F} \langle a_1, a_2, a_3 \rangle^X_0 \\ &= (a_1.h^{r-k_1})(a_2.h^{r-k_2})(a_3.h^{r-k_3})\big((-1)^r - \mathbf{f}(q^{\ell'}) - \mathbf{f}(q^{-\ell'})\big) = 0.\end{aligned}$$

And for $n \geq 4$ points extremal functions,

$$\begin{aligned}(2.4)\qquad &\langle \mathscr{F}a_1, \cdots, \mathscr{F}a_n \rangle^{X'}_0 - \mathscr{F} \langle a_1, \cdots, a_n \rangle^X_0 \\ &= (-1)^n N_{k_1,\ldots,k_n} \prod_{i=1}^{n} (a_i.h^{r-k_i}) \delta^{n-3} \big(\mathbf{f}(q^{\ell'}) + \mathbf{f}(q^{-\ell'})\big) = 0,\end{aligned}$$

where $\delta = q\partial/\partial q$ is the power operator.

(3) Using degeneration analysis [14, 13] and deformation to the normal cone to reduce the general case to projective local model

$$X_{loc} = P_Z(N \oplus \mathscr{O}).$$

(4) Determination of the $g = 0$ GW theory of local models using the toric mirror theorem [5, 12]. A quasi-linearity for one point "f-special" descendent invariant

$$\mathscr{F} \langle \tau_k \xi.a \rangle^{X_{loc}} = \langle \tau_k \xi'.\mathscr{F}a \rangle^{X'_{loc}}$$

is established where $\xi = E_\infty \cong P_Z(N)$ being the infinity divisor of X_{loc}. For n-point invariants an induction verifying the consistency between functional equations and the reconstruction is used.

A nice expository survey with explicit examples can be found in [15].

This scheme of proof is recently generalized to the non-simple case [10]. With the appearance of data (S, F, F'), called the type of the flop f, there are various new difficulties in each of the four steps which we now describe. Only the main ideas will be presented. The detailed proofs are referred to original paper.

2.3. **The topological defect.**

Explicit formula for e is determined to compare the triple products on X and X'. Let $\{t_i^k\}$ be a basis of $A^k(S)$ and $\{\hat{t}_i^k\} \subset A^{s-k}(S)$ be its dual basis where $s = \dim S$. Let $h = c_1(\mathscr{O}_Z(1))$, $c_i := c_i(F)$ and $H_k = c_k(Q_F) = h^k + c_1 h^{k-1} + \cdots + c_k$ where

$$0 \to \mathscr{O}_Z(-1) \to \bar{\psi}^* F \to Q_F \to 0.$$

Similarly we define h', c_i' and H_k' on the X' side. The elements H_k's are of fundamental importance since

$$\mathscr{F} H_k = (-1)^{r-k} H_k', \tag{2.5}$$

and the basis $\{t_i^{k-j} h^j\}_{j \le \min\{k,r\}}$ of $A^k(Z)$ has its dual in $A^{r+s-k}(Z)$ to be

$$\{\hat{t}_i^{k-j} H_{r-j}\}_{j \le \min\{k,r\}}.$$

With this, then

$$\phi'^* \mathscr{F} a = \phi^* a + j_* \sum_i \sum_{1 \le j \le \min\{k,r\}} (a.\hat{t}_i^{k-j} H_{r-j}) t_i^{k-j} \frac{x^j - (-y)^j}{x+y} \tag{2.6}$$

where $x = \bar{\phi}^* h$, $y = \bar{\phi}'^* h'$ and $j : E \hookrightarrow Y$. This leads to

Theorem 2.1. [10] *Let $a_i \in A^{k_i}(X)$ with $k_1 + k_2 + k_3 = \dim X = s + 2r + 1$. Then*

$$\begin{aligned}(\mathscr{F} a_1.\mathscr{F} a_2.\mathscr{F} a_3)^{X'} &- (a_1.a_2.a_3)^X \\ &= (-1)^r \times \sum (a_1.\hat{t}_{i_1}^{k_1-j_1} H_{r-j_1})^X (a_2.\hat{t}_{i_2}^{k_2-j_2} H_{r-j_2})^X (a_3.\hat{t}_{i_3}^{k_3-j_3} H_{r-j_3})^X \\ &\qquad \times (\tilde{s}_{j_1+j_2+j_3-2r-1} t_{i_1}^{k_1-j_1} t_{i_2}^{k_2-j_2} t_{i_3}^{k_3-j_3})^S,\end{aligned}$$

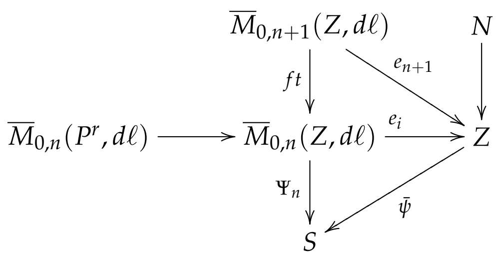
where the sum is over all i_1, i_2, i_3 and j_1, j_2, j_3 subject to $1 \le j_p \le \min\{r, k_p\}$ for $p = 1, 2, 3$ and $j_1 + j_2 + j_3 \ge 2r + 1$. Here

$$\tilde{s}_i := s_i(F + F'^*)$$

is the i-th Segre class of $F + F'^*$.

2.4. The extremal functions.

It is a bit surprising how the Segre classes $s_i(F + F'^*)$ may enter into the calculation of Gromov-Witten invariants. This is possible only if the stable map moduli has related bundle structures over S, and it is indeed the case for extremal invariants via



Let $a_i \in A^{k_i}(X)$, $i = 1, \ldots, n$, with $\sum_{i=1}^n k_i = 2r + 1 + s + (n-3)$. Since

$$a_i|_Z = \sum_{s_i} \sum_{j_i \le \min\{k_i, r\}} (a_i.\hat{t}_{s_i}^{k_i-j_i} H_{r-j_i}) t_{s_i}^{k_i-j_i} h^{j_i},$$

we compute

$$\begin{aligned}
&\langle a_1, \dots, a_n\rangle^X_{0,n,d\ell} \\
&= \sum_{\vec{s},\vec{j}} \int_{\overline{M}_{0,n}(Z,d\ell)} \prod_{i=1}^n \Big((a_i.\hat{t}_{s_i}^{k_i-j_i} H_{r-j_i})\; e_i^*(\bar\psi^* t_{s_i}^{k_i-j_i}.h^{j_i})\Big).e(R^1 f t_* e_{n+1}^* N) \\
&= \sum_{\vec{s},\vec{j}} \prod_{i=1}^n (a_i.\hat{t}_{s_i}^{k_i-j_i} H_{r-j_i}) \left[\prod_{i=1}^n t_{s_i}^{k_i-j_i}.\Psi_{n*}\Big(\prod_{i=1}^n e_i^* h^{j_i}.e(R^1 f t_* e_{n+1}^* N)\Big)\right]_S,
\end{aligned}$$

with the sum over all $\vec{s} = (s_1 \dots, s_n)$ and admissible $\vec{j} = (j_1, \dots, j_n)$. Here we make use of

$$[\overline{M}_{0,n}(X, d\ell)]^{virt} = [\overline{M}_{0,n}(Z, d\ell)] \cap e(R^1 f t_* e_{n+1}^* N)$$

and the fact that classes in S are constants among bundle morphisms.

We must have $\sum(k_i - j_i) \le s$ to get nontrivial invariants. That is, $\sum_{i=1}^n j_i \ge 2r + 1 + n - 3$. If the equality holds, then $\prod_{i=1}^n t_{s_i}^{k_i-j_i}$ is a zero dimensional cycle in S and the invariant reduces to the simple case:

$$(t_{s_1}^{k_1-j_1} \cdots t_{s_n}^{k_n-j_n})^S \langle h^{j_1}, \dots, h^{j_n}\rangle^{\text{simple}}_{0,n,d\ell} = (\textstyle\prod t_{s_i})^S N_{\vec{j}}\, d^{n-3}. \tag{2.7}$$

On the contrary, if the strict inequality holds, the fiber integral is represented by a cycle $S_{\vec{j}} \subset S$ of codimension

$$\mu := \sum j_i - (2r + 1 + n - 3).$$

The structure of $S_{\vec{j}}$ necessarily depends on the bundles F and F'.

Notice that the new phenomenon $\mu > 0$ does not occur for $n = 2$. In that case, $k_1 + k_2 = 2r + s$, $j_1 = j_2 = r$ and we may assume that t_{s_2} is running through the dual basis of t_{s_1}. Then

$$\begin{aligned}
\langle a_1, a_2\rangle^X_{0,2,d\ell} &= \sum_s (a_1.t_s)(a_2.\hat{t}_s)\langle h^r, h^r\rangle^{\text{simple}}_d \\
&= (-1)^{(d-1)(r+1)} \frac{1}{d} \sum_s (a_1.t_s)(a_2.\hat{t}_s).
\end{aligned} \tag{2.8}$$

To deal with general μ, define the fiber integral

$$\Big\langle \prod_{i=1}^n h^{j_i} \Big\rangle^{/S}_d := \Psi_{n*}\Big(\prod_{i=1}^n e_i^* h^{j_i}\Big) \in A^*(S)$$

as a *$\bar\psi$-relative invariant over S*. The absolute invariant is obtained by

$$\langle h^{j_1}, \cdots, t h^{j_n}\rangle^X_d = (\langle h^{j_1}, \cdots, h^{j_n}\rangle^{/S}_d.t)_S$$

For 3-point extremal functions, let $W_\mu := \langle h^{j_1}, h^{j_2}, h^{j_3}\rangle^{/S}_+ \in A^\mu(S)$ with $1 \le j_i \le r$ and $\mu \le r - 1$. (Here $+$ means sum over $\mathbb{N}\ell$.) Using reconstruction (divisor relation), this can be shown to be independent of the choices of j_i's. Moreover, the reconstruction and exercises in Chern classes lead to a recursion

$$W_\mu = s_\mu \mathbf{f} + \sum_{j=1}^{\mu} W_{\mu-j}\big((-1)^r c_j \mathbf{f} - (-1)^{r+j} c'_j \mathbf{f} - c_j\big) \tag{2.9}$$

starting with $W_0 = \mathbf{f}$. Together with the basic relation

$$\delta\mathbf{f} = \mathbf{f} + (-1)^{r+1}\mathbf{f}^2, \tag{2.10}$$

we may convert polynomials in $\mathbf{f}$ into polynomials in $\delta\mathbf{f}$. This allows to show

Theorem 2.2. [10] *The function $W(F, F') = \sum_{\mu=0}^{r-1} W_\mu$ is the action on $\mathbf{f}$ by a Chern classes valued polynomial in the operator δ. It satisfies the functional equation*

$$W_\mu - (-1)^{\mu+1} W'_\mu = (-1)^r \tilde{s}_\mu$$

for $0 \le \mu \le r-1$. In particular the topological defect is corrected by the extremal functions up to analytic continuation.

Once this is done, the case with $n \ge 4$ points follows in much the same spirit as the simple flop case by reconstruction.

2.5. **Degeneration Analysis.**

In order to compare GW invariants of non-extremal rays, the application of degeneration formula in [9] based on [13] and deformation to the normal cone is well suited for ordinary flops with base S. It reduces the problem to local models $X_{loc} = \tilde{E} = P_Z(N \oplus \mathscr{O})$, $X'_{loc} = \tilde{E}' = P_{Z'}(N' \oplus \mathscr{O})$ with induced flop

$$f : \tilde{E} \dashrightarrow \tilde{E}'.$$

The reduction has two steps, with relative GW invariants as medium.

Proposition 2.3. *To prove $\mathscr{F}\langle\alpha\rangle^X \cong \langle\mathscr{F}\alpha\rangle^{X'}$ for all α, it is enough to show that*

$$\mathscr{F}\langle A \mid \varepsilon, \mu\rangle^{(\tilde{E}, E)} \cong \langle \mathscr{F} A \mid \varepsilon, \mu\rangle^{(\tilde{E}', E)}$$

for all A and contact data (ε, μ) (an $H(E)$-valued weighted partition).

The local model $\bar{p} := \bar{\psi} \circ p : \tilde{E} \to Z \to S$ and the flop f are over S, with fibers being isomorphic to the simple case. Thus

$$\bar{p}_* : N_1(\tilde{E}) \to N_1(S)$$

has kernel spanned by the p-fiber line class γ and $\bar{\psi}$-fiber line class ℓ.

The difficulties with S are that $NE(Z)$ could be complicate and $NE(\tilde{E})$ is in general larger than $i_* NE(Z) \oplus \mathbb{Z}^+\gamma$. For $\beta = \beta_Z + d_2(\beta)\gamma \in NE(\tilde{E})$, while $\beta_Z = p_*\beta$ is effective, $d_2(\beta)$ could possibly be negative if $\beta_Z \neq 0$. Nevertheless, the correspondence $\mathscr{F}$ is compatible with $N_1(S)$. Namely

$$\begin{array}{ccc} N_1(\tilde{E}) & \xrightarrow{\quad\mathscr{F}\quad} & N_1(\tilde{E}') \\ & {\scriptstyle \bar{p}_* \oplus d_2}\searrow \qquad \swarrow{\scriptstyle \bar{p}'_* \oplus d'_2} & \\ & N_1(S) \oplus \mathbb{Z} & \end{array}$$

is commutative. This leads to the following observation:

Proposition 2.4. *Functional equation of a generating series $\langle A\rangle$ over Mori cone on local models $f : \tilde{E} \dashrightarrow \tilde{E}'$ is equivalent to functional equations of its various subseries (fiber series) $\langle A\rangle_{\beta_S, d_2}$ labelled by $NE(S) \oplus \mathbb{Z}$.*

The fiber series is a sum over the affine ray $\beta \in (d_2\gamma + \bar{\psi}^*\beta_S.H_r + \mathbb{Z}\ell) \cap NE(\tilde{E})$. Here $\bar{\psi}^*\beta_S.H_r$ is called the *canonical lift* characterized by "$\beta_S.h = 0$". For relative invariants, d_2 is the total contact order which is fixed for a given (ε, μ).

Proposition 2.5. *For the ordinary flop $\tilde{E} \dashrightarrow \tilde{E}'$, to prove*

$$\mathscr{F}\langle A \mid \varepsilon, \mu\rangle_{\beta_S} \cong \langle \mathscr{F} A \mid \varepsilon, \mu\rangle_{\beta_S}$$

for any A, $\beta_S \in NE(S)$ and (ε, μ), it is enough to show that

$$\mathscr{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}}_{\beta_S, d_2} \cong \langle \mathscr{F} A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho\rangle^{\tilde{E}'}_{\beta_S, d_2}$$

for any $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$, $\varepsilon_j \in H^*(E)$ and $\beta_S \in NE(S)$, $d_2 \geq 0$.

2.6. **The local models.**

For a flop $f : X \to X'$, GW invariants such that all the descendent insertions are coupled with cycles in the isomorphism loci are called of f-special type. It remains to prove analytic continuations for such invariants on local models. For notational convenience we assume now that $X = X_{loc}$.

Since $X \to S$ is a double projective bundle, $H(X)$ is generated by $H(S)$ and the relative hyperplane classes h for $Z \to S$ and ξ for $X \to Z$. Further applications of the reconstruction as in [9] by moving all the divisor powers of h, ξ as well as ψ classes into the last insertion reduces the problem to

$$\langle t_1, \ldots, t_{n-1}, \tau_k t_n h^j \xi^i\rangle^X_{\beta_S, d_2} \tag{2.11}$$

where $t_i \in H(S)$, $d_2 \in \mathbb{Z}$.

The one point descendent invariants are encoded by its generating function, the so called J function or the Euler data Q_β's. The precise definition will be given in the next section. The actual determination of $J = \sum J_\beta q^\beta$ can be carried out mainly in cases when X admits torus group actions. Certain localization data I_β or P_β (equivariant cohomology classes) coming from the stable map moduli (indeed the graph space) are of hypergeometric type. In good cases, say X is a semi-Fano toric manifold, these hypergeometric data are already enough to determine J through the so called "mirror map" [5, 11, 12]. This is well known as the *Mirror Theorem*.

For simple flop, X is indeed semi-Fano toric and the classical Mirror Theorem is sufficient for us to proceed, as is done in [9]. For general base S, even for projective bundles the determination of GW theory is still an unsolved question. The first main property we need can nevertheless be phrased as a conjecture:

Conjecture 2.6 (Quasi-linearity)**.** [10]

(1) *If $d_2 < 0$ then*

$$\mathscr{F} J^X_\beta = J^{X'}_{\mathscr{F}\beta}$$

term-wise. And for any $\alpha \in H^*(X)$, $t_i \in H^*(S)$,

$$\langle t_1, \ldots, t_{n-1}, \tau_k \alpha\rangle^X_\beta = \langle t_1, \ldots, t_{n-1}, \tau_k \mathscr{F}\alpha\rangle^{X'}_{\mathscr{F}\beta}.$$

(2) *If there is no restriction on d_2 then*

$$\mathscr{F}(J^X_\beta.\xi) = J^{X'}_{\mathscr{F}\beta}.\xi'$$

term-wise. And thus for any $\alpha \in H^*(X)$, $t_i \in H^*(S)$,

$$\langle t_1, \ldots, t_{n-1}, \tau_k \alpha.\xi\rangle^X_\beta = \langle t_1, \ldots, t_{n-1}, \tau_k \mathscr{F}\alpha.\xi'\rangle^{X'}_{\mathscr{F}\beta}.$$

For general S, the torus action on X can possibly exist only in the fiber direction. This is the case if the flop is of splitting type, namely $F = \bigoplus L_i$ and $F' = \bigoplus L_i'$ being direct sum of line bundles. In the splitting case the quasi-linearity was proved in [10] using a recent result on toric fibration in [1].

The only fiber series in (2.11) which are not covered by Conjecture 2.6, (2) are the cases when $i = 0$ and thus $k = 0$ by our assumption on f-special type. If $d_2 \neq 0$, then by the divisor axiom we have

$$\langle t_1, \ldots, t_n h^j \rangle^X_{\beta_S, d_2} = \frac{1}{d_2} \langle t_1, \ldots, t_n h^j \xi \rangle^X_{\beta_S, d_2}$$

which then reduces to the known cases.

The final remaining cases are *twisted extremal functions* of the form

$$\langle t_1, \ldots, t_n h^j \rangle^X_{\beta_S, d_2 = 0} \tag{2.12}$$

with $\beta_S \neq 0$. Notice that the case $\beta_S = 0$ is the extremal function which is solved in Theorem 2.2 where the analytic continuation holds for (and only for) $n \geq 3$.

The analytic continuations of them from X to X' are solved in [10] by induction on Mori cone $\beta_S \in NE(S)$, using *Birkhoff factorizations* and *generalized mirror maps*. As this step is rather technical, it would be convenient to give explicit proofs in some typical examples.

In the remaining of this article, a special case when X is a Calabi-Yau manifold will be presented, where only the classical mirror map is needed. Nevertheless some of the characteristic feature — the process of renormalization — already appears in the Calabi-Yau case.

3. Calabi-Yau flops

3.1. **The basic setup.**

A local P^r flop $f : X \dashrightarrow X'$ with data (S, F, F') is called of Calabi-Yau type if

$$c_1(X)|_Z = c_1 + c_1' + c_1(S) = 0.$$

Projective local models of P^r flop among Calabi-Yau manifolds lead to such flops.

We only need to consider genus zero n-point fiber functions of the form

$$\langle t_1, \ldots, t_n h^j \rangle^X_{\beta_S, d_2 = 0} \tag{3.1}$$

where $t_i \in A^*(S)$, $j \leq r$ and $\beta = \beta_S + d\ell + d_2\gamma$.

By the virtual dimension count,

$$d^v = c_1(X).\beta + \dim X + n - 3 = \sum |t_i| + j. \tag{3.2}$$

Lemma 3.1. *For P^r flop of CY type, if $r \geq 2$ then there are no one-point invariants of the form as in* (3.1) *with $n = 1$.*

For $r = 1$, the only such one-point invariants are of the form $\langle hp \rangle_{\beta_S, 0}$ where $p \in A^{\dim S}(S)$ is the point class.

Proof. Since $\dim X = \dim S + 2r + 1$ and $c_1(X).\beta = 0$, we get $\dim S + (2r - 1) = \deg t + j$. The only possibility is that $\deg t = \dim S$ and $j = r = 1$. $\square$

We will see in this section that for flops of CY type, the study of functional equations for one-point invariants will involve at most classical mirror maps. We will also carry out one explicit example to demonstrate how the mirror map gives renormalization which leads to analytic continuations.

To handle n-point invariants as in (3.1), this process needs to be extended to (a very special type of) generalized mirror maps. We will not discuss this issue here. Also it can be avoided if $\dim S = 1$ by the divisor axiom.

3.2. I, P, J **and their degrees.**

We first recall the J function for n-point invariants

$$J_\beta(u_1, \ldots, u_{n-1}) = e_{n*}\frac{\prod_{i=1}^{n-1} e_i^* u_i}{z(z-\psi_n)} \in H^*(X) \tag{3.3}$$

and

$$J^X = \sum_\beta J_\beta\, q^\beta = 1 + \frac{J_2}{z^2} + o(z^{-2}).$$

Defnition 3.2. We set $\deg q^\beta = c_1(X).\beta$ and $\deg z = 1$. All pure dimensional cohomology classes are given with their Chow degrees.

Lemma 3.3. *We have* $\deg J = \sum_{i=1}^{n-1} \deg u_i - (n-1)$. *In particular* $\deg J = 0$ *for* $n = 1$ *(one-point invariants).*

Proof. The virtual relative dimension of $e_n : M_{0,n}(X, \beta) \to X$ is given by $c_1(X).\beta + n - 3$, hence $J_\beta\, q^\beta$ has degree

$$\sum_{i=1}^{n-1} \deg u_i - 2 - (c_1(X).\beta + n - 3) + c_1(X).\beta = \sum_{i=1}^{n-1} \deg u_i - (n-1).$$

□

To compute J, we assume that (S, F, F') is of splitting type with $F = \bigoplus_{i=0}^r L_i$, $F' = \bigoplus_{i=0^r} L'_i$ and $c(F) = \prod_{i=0}^r (1+\lambda_i)$, $c(F') = \prod_{i=1}^r (1+\lambda'_i)$ be the Chern roots decompositions. Then we have the hypergeometric modification

$$P_\beta = I_\beta \bar{\psi}^* J_{\beta_S}, \tag{3.4}$$

where the relative factor $I_\beta = I_\beta^{X/S} = \prod A_i \prod B_i C$ is given by

$$\prod_{i=1}^r \frac{1}{\prod_0^{\beta.(h+\lambda_i)} (h+\lambda_i+mz)} \prod_{i=1}^r \frac{1}{\prod_0^{\beta.(\xi-h+\lambda'_i)} (\xi - h + \lambda'_i + mz)} \frac{1}{\prod_0^{\beta.\xi} (\xi+mz)}. \tag{3.5}$$

Also $P = \sum_\beta P_\beta\, q^\beta$.

The product is in $m \in \mathbb{Z}$, which is *directed* in the sense that

$$\prod_0^s \equiv \prod_{m=0+}^{s+} := \prod_{m=-\infty}^s \Big/ \prod_{m=-\infty}^0.$$

Thus for each i with $\beta.(h+\lambda_i) \le -1$, the corresponding subfactor is understood as in the numerator. The subfactor is 1 if $\beta.(h+\lambda_i) = 0$ since there is no such m.

Lemma 3.4. *We have* $\deg P = \deg J$ *for any given* $u_1, \ldots, u_{n-1}$.

Proof. It is clear that

$$\begin{aligned}\deg I_\beta &= -([(r+1)h + c_1] + [(r+1)\xi - (r+1)h + c'_1] + \xi).\beta \\ &= -c_1(X).\beta + c_1(S).\beta.\end{aligned}$$

The lemma follows from the observation that q^{β_S} has degree $c_1(S).\beta_S$ in S while it has degree $c_1(X).\beta_S$ in X. That is, the hypergeometric modification also gives rise to modification of degree from S to X. □

Under $d_2 = \beta.\xi = 0$ and set $s_i = \beta_S.\lambda_i$, $s'_i = \beta_S.\lambda'_i$, then

$$P_\beta = \prod_{i=1}^{r} \frac{\prod_{-d+s'_i}^{0} (\xi - h + \lambda'_i + mz)}{\prod_{0}^{d+s_i} (h + \lambda_i + mz)} \bar{\psi}^* J_{\beta_S}. \tag{3.6}$$

The positions of the A_i, B_i factors are correct (polynomial) only for d large. The subtle point is to study the starting range when going-up/down phenomenon occurs as d varies.

3.3. **The CY condition and the mirror map.**

The CY condition $c_1(X)|_Z = 0$ has the consequence that $c_1(X).\beta = 0$ for any $\beta \in NE(Z)$. That is $\deg q^\beta = 0$. In the case of one-point invariants with $d_2 = 0$, this allows us to write

$$P = P_0 + \frac{P_1}{z} + \frac{P_2}{z^2} + o(z^{-2})$$

where P_k is a cohomology valued series in q^β, $\beta \in NE(Z)$. Since $\deg P = 0$, we have (Chow degree)

$$\deg P_k = k \quad \text{for all} \quad k \geq 0.$$

By Lemma 3.1, we may assume that $r = 1$ with cohomology insertion hp.

The (generalized) mirror theorem says that J and P are related by a change of variables. We need a generalized theorem since S is allowed to be arbitrary. In good cases when $P_0 = 1$, which will be the case for our later calculations, this change of variables is particularly easy to describe.

Let D_i's be a cohomology basis of divisor classes with dual curve class basis β_i's. Let also $t = \sum_i t_i D_i$ be a general divisor with coordinates t_i's. We use the following formal identification

$$q^{\beta_i} = e^{t_i}.$$

Then the mirror theorem says that

$$\text{“}e^{t/z} J\text{”} = e^{t/z} P$$

after the change of variables (mirror map)

$$M : t_i \mapsto t_i + (\beta_i.P_1) \tag{3.7}$$

on the J side. First of all this makes sense as P_1 is a divisor valued power series in $q^{\beta_i} = e^{t_i}$. More importantly this equates the z^{-1} term on both sides. Indeed the mirror map is equivalent to

$$t = \sum_i t_i D_i \mapsto \sum_i t_i D_i + (\beta_i.P_1) D_i = t + P_1.$$

After the mirror map on $e^{t/z} J$ and by removing the common $e^{t/z}$ we get

$$e^{P_1/z} J^M = P. \tag{3.8}$$

And then

$$
\begin{aligned}
&1+\frac{J^M_2}{z^2}+\frac{J^M_3}{z^3}+\cdots \\
&=\Big(1+\frac{P_1}{z}+\frac{P_2}{z^2}+\cdots\Big)\Big(1-\frac{P_1}{z}+\frac{1}{2}\frac{P_1^2}{z^2}-\cdots\Big) \\
&=1+\frac{1}{z^2}\Big(P_2-\frac{1}{2}P_1^2\Big)+\cdots .
\end{aligned} \tag{3.9}
$$

We may write (3.7) as

$$M : q^{\beta_i} \mapsto q^{\beta_i} e^{(\beta_i . P_1)}.$$

Then the z^{-2} term in (3.9) takes the form

Lemma 3.5 (Recursive Relations)**.**

$$\sum_\beta J_{\beta;2}\, q^\beta e^{(\beta.P_1)} = \sum_\beta P_{\beta;2}\, q^\beta - \frac{1}{2}P_1^2.$$

We will use this to compare $J_2.hp$ and $J'_2.\mathscr{F}hp = J'_2.(\xi' - h')p$.

3.4. **Example: Flops of type** $(P^1, \mathscr{O}\oplus\mathscr{O}(-7)), \mathscr{O}(3)\oplus\mathscr{O}(2))$**.**

We emphasize that everything so far, as well as the remaining argument in this article, holds for general base S. But in order to keep things concrete and to avoid notational complexity we will assume that $S = P^1$ in the sequel. Moreover the choice of bundles $F = \mathscr{O}\oplus\mathscr{O}(-7)$ and $F' = \mathscr{O}(3)\oplus\mathscr{O}(2)$ is already general enough to demonstrate the general cases.

Indeed the CY condition says that $c_1 + c'_1 + 2 = 0$. Since the flop is defined up to a twist $(F\otimes L, F'\otimes L^*) \sim (F, F')$ [9], we may assume that $F = \mathscr{O}\oplus\mathscr{O}(-k)$ and then $F' = \mathscr{O}(a)\oplus\mathscr{O}(b)$ with $a + b = k - 2$. The choice made here is indeed the most complicate case among all the possibilities with $k = 7$. For the complete treatment, as well as the two lemmas below, we refer to [10].

We start by describing the Mori cone of projective bundles.

Let $V = \bigoplus_{i=0}^r \mathscr{O}(a_i) \to S = P^1$ with $a_0 \ge a_1 \ge \cdots \ge a_r$, $\bar\psi : P(V) \to P^1$ with $h = c_1(\mathscr{O}_{P(V)}(1))$ be the relative hyperplane class, $b = \bar\psi^*[S].H_r$ be the canonical lift of the base curve and ℓ be the fiber curve class. Then

Lemma 3.6. *$NE(P(V))$ is generated by ℓ and $b - a_0\ell$.*

By suitable applications to our flop with $F = \bigoplus_{i=0}^r \mathscr{O}(a_i)$, $F' = \bigoplus_{i=0}^r \mathscr{O}(a'_i)$:

Lemma 3.7. *A class $\beta = sb + d\ell + d_2\gamma$ is $\mathscr{F}$-effective, that is $\beta \in NE(X)$ and $\mathscr{F}\beta \in NE(X')$, if and only if*

$$d + a_0 s \ge 0 \quad \textit{and} \quad d_2 - d + a'_0 s \ge 0.$$

Let $\beta = \beta_S + d\ell$ with $\beta_S = sb$. In the unstable (that is, $\mathscr{F}$-effective) range $0 \le d \le 3s$, we have for the initial part $0 \le d \le 2s$:

(3.10)

$$P_{s,d} = \frac{1}{\prod_0^s (p+mz)^2}\,\frac{\prod_{d-7s}^{0}(h-7p+mz)}{\prod_0^d (h+mz)\prod_0^{-d+3s}(\xi-h+3p+mz)\prod_0^{-d+2s}(\xi-h+2p+mz)},$$

(3.11)

$$P'_{s,-d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{d-7s}^{0} (\xi' - h' - 7p + mz)}{\prod_0^d (\xi' - h' + mz) \prod_0^{-d+3s} (h' + 3p + mz) \prod_0^{-d+2s} (h' + 2p + mz)},$$

while for the remaining part $2s + 1 \le d \le 3s$:

$$P_{s,d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{d-7s}^{0} (h - 7p + mz) \prod_{-d+2s}^{0} (\xi - h + 2p + mz)}{\prod_0^d (h + mz) \prod_0^{-d+3s} (\xi - h + 3p + mz)}, \tag{3.12}$$

$$P'_{s,-d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{d-7s}^{0} (\xi' - h' - 7p + mz) \prod_{-d+2s}^{0} (h' + 2p + mz)}{\prod_0^d (\xi' - h' + mz) \prod_0^{-d+3s} (h' + 3p + mz)}. \tag{3.13}$$

Lemma 3.8. *Let $s \ge 1$. In the initial unstable range $0 \le d \le 2s$,*

$$\begin{aligned} P_{s,d} = (-1)^{s+d+1} &\frac{(7s-d-1)!}{(s!)^2 d!(3s-d)!(2s-d)!} \frac{h-7p}{z} \\ &\times \Big(1 - \frac{1}{z}\big(2pH_s + hH_d + (h-7p)H_{7s-d-1} \\ &\qquad + (\xi - h + 3p)H_{3s-d} + (\xi - h - 2p)H_{2s-d}\big)\Big) + o(z^{-2}), \\ P'_{s,-d} = (-1)^{s+d+1} &\frac{(7s-d-1)!}{(s!)^2 d!(3s-d)!(2s-d)!} \frac{\xi' - h' - 7p}{z} \\ &\times \Big(1 - \frac{1}{z}\big(2pH_s + (\xi' - h')H_d + (\xi' - h' - 7p)H_{7s-d-1} \\ &\qquad + (h' + 3p)H_{3s-d} + (h' - 2p)H_{2s-d}\big)\Big) + o(z^{-2}). \end{aligned}$$

Here $H_n = \sum_{j=1}^n 1/j$ denotes the harmonic series with $H_0 := 0$.

Lemma 3.9. *Let $s \ge 1$. In the remaining unstable range $2s + 1 \le d \le 3s$,*

$$\begin{aligned} P_{s,d} &= (-1)^s \frac{(7s-d-1)!(d-2s-1)!}{(s!)^2 d!(3s-d)!} \frac{(h-7p)(\xi - h + 2p)}{z^2} + o(z^{-2}), \\ P'_{s,-d} &= (-1)^s \frac{(7s-d-1)!(d-2s-1)!}{(s!)^2 d!(3s-d)!} \frac{(\xi' - h' - 7p)(h' + 2p)}{z^2} + o(z^{-2}). \end{aligned}$$

In the stable range, $\beta = sb + d\ell \in NE(X)$ with $d \ge 3s + 1$ and

$$P_{s,d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d+3s}^{0} (\xi - h + 3p + mz) \prod_{-d+2s}^{0} (\xi - h + 2p + mz)}{\prod_0^d (h + mz) \prod_0^{d-7s} (h - 7p + mz)}. \tag{3.14}$$

For $\beta' = sb + d\ell' \in NE(X')$ in the stable range, that is $s \geq 0$, $d \geq 0$, then

$$P'_{s,d} = \frac{1}{\prod_0^s (p+mz)^2} \frac{\prod_{-d}^{0}(\xi' - h' + mz) \prod_{-d-7s}^{0} (\xi' - h' - 7p + mz)}{\prod_0^{d+3s} (h' + 3p + mz) \prod_0^{d+2s} (h' + 2p + mz)}. \tag{3.15}$$

Lemma 3.10. *Let $s \geq 1$. In the stable range,*

$$P_{s,d} = (-1)^s \frac{(d-3s-1)!(d-2s-1)!}{(s!)^2 d!(d-7s)!} \frac{(\xi - h + 3p)(\xi - h + 2p)}{z^2} + o(z^{-2})$$
$$P'_{s,d} = (-1)^s \frac{(d-1)!(d+7s-1)!}{(s!)^2(d+3s)!(d+2s)!} \frac{(\xi' - h')(\xi' - h' - 7p)}{z^2} + o(z^{-2}).$$

Notice that in the first formula it is understood that $P_{s,d;2} = 0$ when $3s + 1 \leq d \leq 7s - 1$. Moreover The two rational functions of degree $2(s-1)$ in d:

$$\frac{(d-3s-1)!(d-2s-1)!}{(s!)^2 d!(d-7s)!} = \frac{(d-(3s+1))(d-(3s+2))\cdots(d-(7s-1))}{(s!)^2 d(d-1)\cdots(d-2s)},$$
$$\frac{(d-1)!(d+7s-1)!}{(s!)^2(d+3s)!(d+2s)!} = \frac{(d+(3s+1))(d+(3s+2))\cdots(d+(7s-1))}{(s!)^2 d(d+1)\cdots(d+2s)}$$

coincide after a sign change $d \mapsto -d$.

Defnition 3.11. Let $W_0(x) = 1/d^2$ and for $s \geq 1$,

$$W_s(x) = (-1)^s \frac{(x-(3s+1))(x-(3s+2))\cdots(x-(7s-1))}{(s!)^2 x(x-1)\cdots(x-2s)}$$
$$\equiv \frac{(-1)^s}{(s!)^2} \prod_{j=3s+1}^{7s-1} (x-j) \Big/ \prod_{j=0}^{2s} (x-j)$$

be the fundamental rational function associated to P and $W'_s(x)$ be the corresponding one for P'. Thus $W'_s(x) = W_s(-x)$.

The most important observation is the following proposition. It is used to show that the mirror map regularize (re-normalize) the singularity caused by the hypergeometric data into the the regular J function:

Proposition 3.12. *Let $s \geq 1$.*

(1) *In the initial unstable range $0 \leq d \leq 2s$, the coefficient of z^{-1} is given by the residue at d,*

$$P_{s,d} = \operatorname{Res}_{x=d} W_s(x) \frac{h-7p}{z} + (h-7p)o(z^{-1}).$$

(2) *In the remaining unstable range $2s+1 \leq d \leq 3s$,*

$$P_{s,d} = -W_s(d) \frac{(h-7p)(\xi - h + 2p)}{z^2} + o(z^{-2}).$$

(3) *And by its very definition, in the stable range $d \geq 3s+1$,*

$$P_{s,d} = W_s(d) \frac{(\xi - h + 3p)(\xi - h + 2p)}{z^2} + o(z^{-2}).$$

Proof. For (1), we calculate the residue at $x = d$ by rewriting the products split by d as

$$\begin{aligned}\operatorname{Res}_{x=d} W_s(x) &= \lim_{x\to d}(x-d)W_s(x)\\ &= \frac{(-1)^s}{(s!)^2}\frac{(d-(3s+1))\cdots(d-(7s-1))}{d\cdots 1\times(-1)\cdots(d-2s)}\\ &= \frac{(-1)^{s+d+1}}{(s!)^2}\frac{(7s-1-d)\cdots(3s+1-d)}{(d)!(2s-d)!}\\ &= \frac{(-1)^{s+d+1}}{(s!)^2}\frac{(7s-d-1)!}{(d)!(2s-d)(3s-d)!}.\end{aligned}$$

The proof of (2) is in exactly the same way except that there is no pole of $W_s(x)$ in this range. □

Similar statements hold for $P'_{s,d}$ by using $W'_s(x)$. Notice that $W'_s(x) = W_s(-x)$ implies $-\operatorname{Res} W'_s(x) = \operatorname{Res} W_s(-x)$.

Proposition 3.13. *Let $s \ge 1$.*

(1) *In the initial unstable range $0 \ge d \ge -2s$, the coefficient of z^{-1} is given by the residue at d,*

$$P'_{s,d} = -\operatorname{Res}_{x=d} W'_s(x)\frac{\xi'-h'-7p}{z} + (\xi'-h'-7p)o(z^{-1}).$$

(2) *In the remaining unstable range $-(2s+1) \ge d \ge -3s$,*

$$P'_{s,d} = -W'_s(d)\frac{(\xi'-h'-7p)(h'+2p)}{z^2} + o(z^{-2}).$$

(3) *And by its very definition, in the stable range $d \ge 1$,*

$$P'_{s,d} = W'_s(d)\frac{(\xi'-h')(\xi'-h'-7p)}{z^2} + o(z^{-2}).$$

In particular we have

$$\begin{aligned}P_1 &= (h-7p)\operatorname{Res} W\\ &= (h-7p)\sum_{s\ge 0}\sum_{d=0}^{2s}\operatorname{Res} W_s(d)\, q^{sb+d\ell},\\ P'_1 &= -(\xi'-h'-7p)\operatorname{Res} W'\\ &= -(\xi'-h'-7p)\sum_{s\ge 0}\sum_{d=-2s}^{0}\operatorname{Res} W'_s(d)\, q^{sb+d\ell'}.\end{aligned}$$

3.5. **Proof of the main result in the example.**

Theorem 3.14. *For $s \ne 0$, we have analytic continuations under $\mathscr{F}$:*

$$\langle hp\rangle^X_{sb} \cong \langle(\xi'-h')p\rangle^{X'}_{sb}.$$

Proof. We define the numerical versions of P and J by adding a ˆ on it:

$$\hat{P} = (P.hp)^X, \qquad \hat{J} = (J.hp)^X,$$
$$\hat{P}' = (P'.(\xi'-h')p)^{X'}, \qquad \hat{J}' = (J'.(\xi'-h')p)^{X'}.$$

Since $(h-7p)^2 p = 0$ and $(\xi' - h' - 7p)^2(\xi' - h')p = \xi'^3 p - 3\xi'^2 h' p = -1$, by Lemma 3.5, $\hat{J}$ and $\hat{P}$ are related by

$$(3.16)\qquad \begin{aligned} \sum_{s\geq 0}\sum_{d\geq 0} \hat{J}_{s,d;2}\, q^{sb+d\ell} e^{(d-7s)\operatorname{Res} W} &= \sum_{s\geq 0}\sum_{d\geq 0} \hat{P}_{s,d;2}\, q^{sb+d\ell}, \\ \sum_{s\geq 0}\sum_{d\geq -3s} \hat{J}'_{s,d;2}\, q^{sb+d\ell'} e^{(d+7s)\operatorname{Res} W'} &= \sum_{s\geq 0}\sum_{d\geq -3s} \hat{P}'_{s,d;2}\, q^{sb+d\ell'} + \frac{1}{2}(\operatorname{Res} W')^2. \end{aligned}$$

For $\beta = sb + d\ell$ in the unstable range ($0 \leq d \leq 3s$), by Lemma 3.8 and Lemma 3.9 we have $\hat{P}_{s,d;2} = 0$. Since $\operatorname{Res} W$ provides only non-negative powers in q^ℓ, the summation in d in the first equation may be written as in $d \geq 3s+1$. In this range we formally set

$$\hat{J}'_{s,-d;2} := -\hat{J}_{s,d;2}.$$

On the other hand if $\beta = sb + d\ell$ is in the stable range $d \geq 3s+1$ then by Lemma 3.10 and noticing that $W'_s(-d) = W_s(d)$ and

$$\begin{aligned} (\xi - h + 2p)(\xi - h + 3p)hp &= \xi^2 hp = 1, \\ (\xi' - h')(\xi' - h' - 7p)(\xi' - h')p &= \xi'^3 p - 3\xi'^2 h' p = -1, \end{aligned}$$

we may formally define

$$\hat{P}'_{s,-d;2} := -\hat{P}_{s,d;2} = -W_s(d) = -W'_s(-d),$$

Together with the identification $q^\ell \mapsto \mathscr{F} q^\ell = q^{-\ell'}$, the two recursive relations in (3.16) can be merged into a single relation on the X' side. Namely let

$$\hat{P}'_{s;2} - \mathscr{F}\hat{P}_{s,2} = \sum_{d\in\mathbb{Z}} \hat{P}'_{s,d;2}\, q^{d\ell'}$$

be the combined series. The actual value of $\hat{P}'_{s,d;2}$ is given by

Lemma 3.15.

$$\begin{aligned} \hat{P}'_{s,d;2} &= -W'_s(d), \quad d \leq -(2s+1) \text{ or } d \geq 1, \\ \hat{P}'_{s,d;2} &= -\operatorname{Reg} W'_s(d), \quad -2s \leq d \leq 0. \end{aligned}$$

Here $\operatorname{Reg} W'_s(d)$ *denotes the constant term in the Laurent expansion of* $W'_s(x)$ *at* $x = d$.

Proof. For the first statement, the only range not covered in the above discussion is when $-3s \leq d \leq -(2s+1)$, which again follows from Proposition 3.13, (2).

For the second statement, by Lemma 3.8 and straightforward intersection calculations we get for $0 \leq d \leq 2s$,

$$(3.17)\qquad \hat{P}'_{s,-d;2} = \frac{(-1)^{s+d+1}(7s-d-1)!}{(s!)^2 d!(3s-d)!(2s-d)!}(H_d + H_{7s-d-1} - H_{3s-d} - H_{2s-d}).$$

On the other hand, the Laurent expansion of $W_s(x)$ at $x = d$ is obtained from

$$(3.18)\qquad \frac{(-1)^s}{(s!)^2}\frac{1}{x-d}\prod_{j=3s+1}^{7s-1}((x-d)-(j-d))\prod_{j\neq d;\, j=0}^{2s}\frac{-1}{j-d}\Big(1+\frac{x-d}{j-d}+\cdots\Big).$$

The residue term clearly gives rise to

$$\frac{(-1)^{s+1+d}}{(s!)^2}\prod_{j=3s+1}^{7s-1}(j-d)\Big/ d!(2s-d)! = \frac{(-1)^{s+d+1}(7s-d-1)!}{(s!)^2 d!(3s-d)!(2s-d)!}$$

as we had seen in Proposition 3.12. And then it is also clear from (3.18) that the constant term is given by $-\hat{P}'_{s,-d;2}$. □

Since $\operatorname{Reg} W'_s(d) = W'_s(d)$ if d is not a pole of $W'_s(x)$, the above Lemma may be rephrased as

$$\hat{P}'_{s,d;2} = -\operatorname{Reg} W'_s(d)$$

for any s, d. Then (3.16) can be combined into

$$\sum_{s\geq 0}\sum_{d\in\mathbb{Z}} \hat{J}'_{s,d;2}\, q^{sb+d\ell'} e^{(d+7s)\operatorname{Res} W'} = -\sum_{s\geq 0} \operatorname{Reg} W'_s\, q^{sb} + \frac{1}{2}(\operatorname{Res} W')^2. \tag{3.19}$$

(Recall that $-\operatorname{Res} W_s(-d) = -\operatorname{Res} W'_s(d)$.) Here we formally set "$W'_0(0) = 0$" to include the case $(s, d) = (0, 0)$ since $\hat{J}_{0,0;2} = 0 = \hat{P}'_{0,0;2}$.

The remaining proof is based on the following idea: Consider the formal expression due to Euler:

$$E(q) := \sum_{d=-\infty}^{\infty} q^d = \cdots + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots .$$

If we denote $\mathbf{f}(q) = q/(1-q)$, then the analytic continuation

$$\mathbf{f}(q) + \mathbf{f}(q^{-1}) = -1$$

simply means the formal assignment $E(q) = 0$. For any polynomial $w(d)$, we have

$$\sum_{d=-\infty}^{\infty} w(d)\, q^d = w(\delta)E(q) \tag{3.20}$$

where $\delta = q\,d/dq$. Thus we may establish the analytic continuation result on $\hat{J}_{s;2}$ if we show that $\hat{J}'_{s,d;2}$ is polynomial in d for all $d \in \mathbb{Z}$.

It would be instructive to look at the case $s = 1$ first. We compare the terms with q^{sb} in (3.19). For $s = 0$, this simply reduces to the case of extremal rays:

$$\hat{J}'_{0,d;2} = -W'_0(d) = -\frac{1}{d^2} \qquad \text{for } d \neq 0. \tag{3.21}$$

For $s = 1$, since

$$e^{d\operatorname{Res} W'} = 1 + d\sum_{d_1=-2}^{0} \operatorname{Res} W'_1(d_1)\, q^{b+d_1\ell'} + o(q^b),$$

by comparing $q^{b+d\ell'}$ terms from both sides of (3.19) we have

$$\hat{J}'_{1,d;2}\, q^{b+d\ell'} + \sum_{d_1=-2}^{0} \hat{J}'_{0,d-d_1;2}(d-d_1)\operatorname{Res} W'_1(d_1)\, q^{b+d\ell'} = -\operatorname{Reg} W'_1(d) q^{b+d\ell'}.$$

In the sum it is assumed that $d_1 \neq d$. By (3.21), this leads to

$$\hat{J}'_{1,d;2} = -\operatorname{Reg} W'_1(d) + \sum_{d_1=-2}^{0} \frac{\operatorname{Res} W'_1(d_1)}{d-d_1}.$$

Again $d_1 = d$ is excluded if $d \in [-2, 0]$.

Lemma 3.16. *Let $F(x)$ be a rational function with simple poles at a_j's and with polynomial part $P(x)$. Then*

$$P(a) = \operatorname{Reg} F(a) - \sum_{a_j \neq a} \frac{\operatorname{Res} F(a_j)}{a - a_j}.$$

Proof. By division and taking partial fractions, we have

$$F(x) = P(x) + \frac{R(x)}{\prod_{a_j}(x - a_j)} = P(x) + \sum_{a_j} \frac{\operatorname{Res} F(a_j)}{x - a_j}.$$

If $a \notin \{a_j\}$, then $\operatorname{Reg} F(a) = F(a)$ and the lemma holds. If $a = a_i$ for some i, then

$$F(x) = \frac{\operatorname{Res} F(a)}{x - a} + \left(P(x) + \sum_{j \neq i} \frac{\operatorname{Res} F(a_j)}{x - a_j} \right)$$

and the lemma again holds. □

Thus $\hat{J}'_{1;2}$ is precisely the polynomial part of $-\hat{P}'_{1;2}$, hence is polynomial in d.

For $s = 2$, the relation becomes a bit more lengthy, though it inherits essentially the same structure. Thus we shall treat the recursive relation in a slightly clever way to make it works for all $s \geq 1$.

We rewrite (3.19) by separating the terms with $s = 0$:

$$\begin{aligned} &\sum_{d \neq 0} \hat{J}'_{s,0;2} q^{d\ell'} e^{d \operatorname{Res} W'} + \sum_{s \geq 1} \sum_{d \in \mathbb{Z}} \hat{J}'_{s,d;2}\, q^{sb + d\ell'} e^{(d+7s) \operatorname{Res} W'} \\ &\qquad = -\sum_{d \neq 0} \frac{1}{d^2} q^{d\ell'} - \sum_{s \geq 1} \operatorname{Reg} W'_s\, q^{sb} + \frac{1}{2} (\operatorname{Res} W')^2. \end{aligned} \tag{3.22}$$

The first sum gives

$$-\sum_{d \neq 0} \frac{1}{d^2} q^{d\ell'} \Big(1 + d \operatorname{Res} W' + \frac{d^2}{2} (\operatorname{Res} W')^2 + \sum_{k \geq 3} \frac{d^k}{k!} (\operatorname{Res} W')^k \Big).$$

The first summand cancels with the first sum in the right hand side. The third summand combines with the third term of the right hand side give rise to

$$\frac{1}{2} (\operatorname{Res} W')^2 E(q).$$

Also the fourth summand leads to

$$\sum_{k \geq 3} \frac{1}{k!} (\operatorname{Res} W')^k \delta^{k-2} E(q).$$

These terms are well-behaved since they contain the Euler series.

Now the second summand combines with the second term in the right hand side give rise to

$$\begin{aligned} &\sum_{d \neq 0} \sum_{s \geq 1} \sum_{d_1 = -2s}^{0} \frac{\operatorname{Res} W'_s(d_1)}{d} q^{sb + (d_1 + d)\ell'} - \sum_{s \geq 1} \operatorname{Reg} W'_s(d)\, q^{sb + d\ell'} \\ &= -\left(\sum_{s \geq 1} \operatorname{Reg} W'_s(d)\, q^{sb + d\ell'} - \sum_{s \geq 1} \sum_{d_1 = -2s}^{0} \sum_{d \neq d_1} \frac{\operatorname{Res} W'_s(d_1)}{d - d_1} q^{sb + d\ell'} \right). \end{aligned}$$

By Lemma 3.16, this is simply

$$-\sum_{s\geq 1} \operatorname{Poly} W'_s\, q^{sb}.$$

To summarize, the recursive relation (3.22) is reduced to

$$\begin{aligned}\sum_{s\geq 1}\sum_{d\in\mathbb{Z}} \hat{J}'_{s,d;2}\, q^{sb+d\ell'}\, e^{(d+7s)\operatorname{Res} W'} \\ = -\sum_{s\geq 1} \operatorname{Poly} W'_s\, q^{sb} + \sum_{k\geq 2} \frac{(\operatorname{Res} W')^k \delta^{k-2} E(q)}{k!}.\end{aligned}$$

This allows to apply induction on $s \geq 1$ to show that $\hat{J}'_{s,d;2}$ is polynomial in d. Indeed by looking at terms with q^{sb}, we get

$$\hat{J}'_{s,d;2}\, q^{sb+d\ell'} = \text{terms with lower } s + \text{polynomials in } d.$$

By induction the proof of the theorem is thus complete. □

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