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ON THE INCOMPLETENESS OF THE WEIL-PETERSSON METRIC ALONG DEGENERATIONS OF CALABI-YAU MANIFOLDS

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Introduction

The classical Weil-Petersson metric on the Teichmüller space of compact Riemann surfaces is a Kähler metric which is complete only in the case of elliptic curves. It has a natural generalization to higher dimansional polarized Kähler-Einstein manifolds (still Kähler). In the case of abelian varieties and K3 surfaces the Weil-Petersson metric is complete and in fact equals the Bergman metric of the (Hermitian symmetric) period domain. The completeness of the metric is important if one tries to study the moduli space via differential geometry. Naively one would guess that the completeness property will hold true for general compact Kähler manifolds with trivial canonical bundle (Calabi-Yau manifolds). However Candelas et. al. [Ca] found examples of nodal degenerations of Calabi-Yau 3-folds with finite Weil-Petersson distance!

The aim of this note is to characterize such finite distance degenerations, both Hodge theoretically and complex analytically, and to describe the possible picture of the completion. However, the results obtained here are too weak to answer the full question (although we do get a strong feeling that it is related to the minimal model program). This note thus should be regarded as merely a preliminary study of this completion program.

There are two parts. The first part, §1-§2, starts with Tian's description of the Weil-Petersson metric as the Chern form of the Hodge bundle F^n. This fits naturally into the framework of variation of Hodge structures (VHS) and we can extend the definition of the Weil-Petersson metric to this setting. By applying Schmid's theory of limiting mixed Hodge structure (MHS), we find in (1.1) that the center of a degeneration of polarized Hodge structures of weight n with $F^n \cong \mathbb{C}$ has finite Weil-Petersson distance if and only if $NF^n_\infty = 0$, where N is the nilpotent monodromy and F^n the limiting filtration. From this we see in (1.2) that the degeneration has finite distance if and only if the n-th flag period map extends continuously over the puncture.

In §2 we return to the geometric situation, namely the semi-stable degeneration of polarized Calabi-Yau manifolds. As a simple application of the "geometric genus formula", which is deduced from the Clemens-Schmid exact sequence, we

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show in (2.1) that the central fiber is at finite Weil-Petersson distance if and only if exactly one component of it has $H^{n,0} \neq 0$, (we should remark here that Theorem (2.1) is also claimed in a recent preprint of Hayakawa [H], however the proof given there seems to be incomplete and too complicated). As a corollary we deduce in (2.3) that smoothable Calabi-Yau varieties with canonical singularities are at finite distance.

In the second part, §3-§4, we deal with a more refined question: Is the finite distance degeneration birationally trivial (up to a base change)? This is always the case for K3 surfaces by (1.1) and Kulikov's classification theorem [Ku]! The completion program will be much easier if this is the only case we have. Unfortunately there are nontrivial examples. The simplest examples are given by degenerations with nontrivial nilpotent monodromy. This is indeed the case for nodal degenerations of Calabi-Yau 3-folds (see §3).

However, it is possible for a birationally nontrivial degeneration to have trivial monodromy. In §4, using several results in 3-fold birational geometry, we show in (4.1), (4.5) that smoothable Calabi-Yau 3-folds with nontrivial terminal singularities all provide nontrivial finite distance examples. This result also gives a negative answer to the so-called filling-in problem in dimension 3.

§0. Preliminaries

Here we collect briefly some basic definitions and well-known properties about Hodge theory (0.1)-(0.5), see [C, G, GS, S] for more details.

0.1 Polarized VHS. The m-th complex primitive cohomology $V = P^m_{\mathbb{R}} \otimes \mathbb{C} \subset H^m_{\mathbb{C}}$ of a compact Kähler manifold (X^n, ω) admits a Hodge structure of weight m: $V = \bigoplus_{p+q=m} P^{p,q}$ with $P^{p,q} = \overline{P^{q,p}}$, (equivalently in terms of Hodge filtrations $F : V = F^0 \supset F^1 \supset \cdots \supset F^m$ with $F^p = \bigoplus_{i \geq p} P^{i,m-i}$). For $m \leq n$, the Hodge-Riemann bilinear form

$$Q(u,v) := (-1)^{\frac{m(m-1)}{2}} \int_X u \wedge v \wedge \omega^{n-m},$$

polarizes V in the sense that (1) $Q(F^p, F^{m+1-p}) = 0$ and (2) $Q(Cv, \bar{v}) > 0$, where C, the *Weil operator*, acts on $P^{p,q}$ by multiplying $\sqrt{-1}^{p-q}$. Varying the above data, a family of polarized ($[\omega]$ fixed) Kähler manifolds $\mathcal{X}/S$ then gives rise to a polarized VHS over S. It satisfies the Griffiths transversality relation $\partial F^p_s/\partial s \in F^{p-1}_s$ in any holomorphic direction. It follows that $\partial F^p_s/\partial \bar{s} \in F^p_s$ and F^p_s $(s \in S)$ form a holomorphic subbundle of the flat bundle $\mathbb{V}$ which is polarized by the flat bilinear form Q.

0.2 The period map. The *period domain* D is the classifying space of all Hodge filtrations F of V polarized by Q. $D = G/K$ where $G = Aut(P_{\mathbb{R}}, Q)$ and K the stabilizer of a point. It comes naturally with the tautological homogeneous vector bundle $\mathbb{F}^p \subset \mathbb{V} := V \times D$. The compact dual $\check{D}$ is the set of all the F's which satisfy only (1). It contains D as an open subvariety. The family $\mathcal{X}/S$ gives the *period map* $\phi : S \to \Gamma\backslash D$ with Γ the representation of $\pi_1(S)$ in G and $\phi^*\mathbb{F}^p$ gives the holomorphic vector bundle mentioned before. The Griffiths transversality translates into the *horizontality* of ϕ: $d\phi(T_S) \subset \bigoplus_{p+q=m} \operatorname{Hom}(P^{p,q}, P^{p-1,q+1}) := T_h(D)$, the *horizontal tangent (sub)bundle* of T_D. Finally, we can formalize the above situation and define the polarized VHS as a locally liftable horizontal holomorphic map $\phi : S \to \Gamma\backslash D$ with Γ a representation of $\pi_1(S)$ in $G \cap Aut(H_{\mathbb{Z}})$, $H_{\mathbb{Z}}$ an integral lattice and $P_{\mathbb{R}} \subset H_{\mathbb{R}}$. In the case $F^n \cong \mathbb{C}$, we will also consider the "n-th flag period map" $\phi^n : S \to \Gamma\backslash\mathbb{P}(V)$ which in fact contains almost all the information we need.

0.3 Semi-stable degenerations, smoothings. We are interested in the case of a degeneration $\mathcal{X}/\Delta$ of polarized Kähler n-folds. By this we mean $\mathcal{X}$ is a Kähler $(n+1)$-fold and $\mathcal{X} \to \Delta$ is a proper flat holomorphic map with the general fiber X_t $(t \neq 0)$, a smooth Kähler n-fold. Notice that the resulting family over the punctured disk has a polarization induced from the Kähler form of $\mathcal{X}$. $\mathcal{X}/\Delta$ is called semi-stable if X_0 is a reduced divisor with normal crossings in $\mathcal{X}$. By a theorem of Mumford (in [K]), every degeneration has a semi-stable reduction by a sequence of blow-ups and base-changes ($t \to t^d$ on Δ).

A word about the terminology used in this note: $\mathcal{X}/\Delta$ is called a ? degeneration if it is a degeneration in the above sense and X_0 has only singularities of type ?. Also by "$\mathcal{X}/\Delta$ is a smoothing of X_0" we will mean $\mathcal{X} \to \Delta$ is a proper flat family with smooth X_t but without assuming the complex space $\mathcal{X}$ to be smooth.

0.4 Monodromy. Now $\pi_1(\Delta^\times) = \mathbb{Z}$ induces the Picard-Lefschetz transformation (monodromy) T on $H^m_{\mathbb{Z}}$, which is known to be quasi-unipotent. Under the semi-stable asssumption, T will be unipotent and we may consider $N := \log T$, the associated nilpotent operator acting on $H^m_{\mathbb{Q}}$ (and therefore on $V \subset H^m_{\mathbb{C}}$ since the polarization class is invariant under T). The corresponding quasi-unipotent statement is still true for a polarized VHS. We will always assume that T is unipotent by doing a base change implicitly.

0.5 Schmid's limiting MHS. For a polarized VHS $\phi : \Delta^\times \to \langle T\rangle\backslash D$; the map ϕ lifts to the upper half plane $\Phi : \mathbb{H} \to D$; the coordinates $t \in \Delta^\times$ and $z \in \mathbb{H}$ are related by $t = e^{2\pi\sqrt{-1}z}$. Set $A(z) = e^{-zN}\Phi(z) : \mathbb{H} \to \check{D}$ (instead of D) so that $A(z+1) = A(z)$, then A descends to a function $\alpha(t)$ on $\Delta^\times$. The very first part of Schmid's "nilpotent orbit theorem" says that $\alpha(t)$ extends holomorphically over $t = 0$. The special value $F_\infty := \alpha(0)$ is called the limiting filtration and is in general outside D. However, the nilpotent operator N uniquely defines a *monodromy weight filtration* on V: $0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m-1} \subset W_{2m} = V$ such that $N(W_k) \subset W_{k-2}$ and induces an isomorphism $N^\ell : Gr^W_{m+\ell} \cong Gr^W_{m-\ell}$ where $Gr^W_k := W_k/W_{k-1}$. These two filtrations F^p_∞ and W_k together define a *polarized mixed Hodge structure* on V in the following sense: $F^p_\infty Gr^W_k := F^p_\infty \cap W_k / F^p_\infty \cap W_{k-1}$ for $p = 0, \ldots, m$ defines a (pure) Hodge structure of weight k on Gr^W_k. The operator N acts on them as a morphism of MHS's of type $(-1,-1)$, that is, $N(F^p_\infty Gr^W_k) \subset F^{p-1}_\infty Gr^W_{k-2}$. Moreover for $\ell \geq 0$, the primitive part $P^W_{m+\ell} := \ker N^{\ell+1} \subset Gr^W_{m+\ell}$ is polarized by $Q(\cdot, N^\ell \bar{\cdot})$.

When ϕ comes from geometric situations, by also putting together the non-primitive part, the total cohomology $H^m(X_t, \mathbb{C})$ still admits (non-polarized) MHS.

Now we turn to the definition of the Weil-Petersson metric (0.6)-(0.7):

0.6 The Weil-Petersson metric. For a given family of polarized Kähler manifolds $\mathcal{X}/S$ with Kähler metrics $g(s)$ on X_s, one can define a hermitian metric G on S (possibly degenerate) as follows: at $s \in S$ with fiber $X = X_s$, Kodaira-Spencer theory gives a map $\rho : T_s(S) \to H^1(X, T_X) \cong \mathbb{H}^{0,1}_{\bar\partial}(T_X)$ (harmonic representatives). The Kähler metric $g(s)$ induces a metric on $\Lambda^{0,1}(T_X)$; so for v, $w \in T_s(S)$, we define

$$G(v, w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)}.$$

When $\mathcal{X}/S$ is a polarized Kähler-Einstein family ($g(s)$ is K-E) and ρ injective, G is called the Weil-Petersson metric on S and is denoted by G_{WP}.

0.7 The Calabi-Yau case. When X is Calabi-Yau there are two important structure theorems: (1) Yau's solution to Calabi's conjecture [Y]: X has an unique Ricci flat metric in each Kähler class and (2) the Bogomolov-Tian-Todorov theorem: the Kuranishi space of X is unobstructed [T1, To].

Let $\mathcal{X}/S$ be a maximal subfamily of the Kuranishi family with a fixed polarization class $[\omega]$, then ρ is clearly injective. Let $g(s)$ be the unique Ricci flat metric in the given polarization. Using the fact that the global holomorphic n-form $\Omega(s)$ is flat with respect to $g(s)$, it is not hard to see ([T1, To]) that

$$G_{WP}(v, w) = \frac{Q(C(i(v)\Omega), \overline{i(w)\Omega})}{Q(C\Omega, \bar\Omega)}, \tag{0.7.1}$$

where $H^1(X, T_X) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \cong H^{n-1,1}$ via the interior product $v \mapsto i(v)\Omega$ is the well-known isomorphism. The tangent space T_S is mapped to $P^{n-1,1}$ isomorphically (this leads to a proof that the n-th flag period map is an local embedding), so the Weil-Petersson metric is induced from the Hodge metric on the n-th piece of the horizontal tangent bundle. For convienence, let's write $\tilde Q = \sqrt{-1}^n Q$ $(= Q(C\cdot, \bar\cdot)$ on $H^{n,0} = P^{n,0})$. Tian goes further to show that $\tilde Q$ is a Kähler potential of G_{WP}, that is,

$$\omega_{WP} = \frac{\sqrt{-1}}{2}\,\mathrm{Ric}_{\tilde Q}(H^{n,0}) = -\frac{\sqrt{-1}}{2}\partial\bar\partial \log \tilde Q, \tag{0.7.2}$$

where ω_{WP} denotes the fundamental real 2-form of G_{WP} (this formula shows in particular that ω_{WP} is independent of the polarization). The proof of this observation is a straightforward differentiation and purely Hodge theoretic in nature, so we can extend the definition of G_{WP} to polarized VHS over S with $h^{n,0} = 1$ by (0.7.2). Notice that by Griffiths' calculation [G] we know it is semi-positive (possibly degenerate). It still makes sense to talk about geodesics and distances, so we will still call it the Weil-Petersson "metric".

§1. Hodge theoretic criterion for finite distance points

We now give the basic criterion for finite Weil-Petersson distance in the case of one parameter degenerations of polarized Hodge structures $\phi : \Delta^\times \to \langle T \rangle \backslash D$ with $h^{n,0} = 1$:

Theorem 1.1. *The center of a degeneration of polarized Hodge structures of weight n with $F^n \cong \mathbb{C}$ has finite Weil-Petersson distance if and only if $NF^n_\infty = 0$.*

Proof. Using the notation from §0 with $\Phi : \mathbb{H} \to D$ being the lifting. The only crucial point of the computation is a good choice of a holomorphic section Ω of $H^{n,0}$. By taking the projection to the F^n part $p^n : D \to \mathbb{P}(V)$ we get $\Phi^n(z) = (e^{zN}\alpha(t))^n = e^{zN}\alpha^n(t)$. Here $*^n := p^n(*) \in \mathbb{P}(V)$ means the n-th flag. Near $t = 0$, we can consider a vector (local homogeneous coordinates) representation $\mathbf{a}$ of α^n in V. Then $\mathbf{a}(t) = a_0 + a_1 t + \cdots$ is holomorphic in t. So correspondingly, $\mathbb{A}(z) = a_0 + a_1 e^{2\pi\sqrt{-1}z} + a_2 e^{4\pi\sqrt{-1}z} + \cdots$. Now the point is, $e^{2\pi\sqrt{-1}z} = e^{2\pi\sqrt{-1}x}e^{-2\pi y}$ has the property that all the partial derivatives in x and y exponentially decay to 0 as $y \to \infty$ with rate of decay independent of x. For ease of notation, let h be the function class satisfying the above property and $\mathbf{h}$ the corresponding function class with values in V. Now let $\Omega(z) = e^{zN}\mathbb{A}(z)$. This is the desired section because vector representations correspond to sections of the tautological line budle of $\mathbb{P}^n$ which pull back to $H^{n,0}$ by Φ. So the Kähler form ω_{WP} of the induced Weil-Petersson metric G_{WP} on $\mathbb{H}$ is given by

$$\omega_{WP} = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log \tilde{Q}(e^{zN}\mathbb{A}(z), e^{\bar{z}N}\overline{\mathbb{A}(z)}).$$

Since we are in one complex variable, write $G_{WP} = G|dz|^2$, then $G = -(1/4)\triangle \log \tilde{Q}$. We have $Q(Tu, Tv) = Q(u, v)$, it follows easily that $Q(Nu, v) = -Q(u, Nv)$ and $Q(e^{zN}u, v) = Q(u, e^{-zN}v)$. Since $\mathbb{A} = a_0 + \mathbf{h}$, we have $\tilde{Q}(e^{zN}\mathbb{A}, e^{\bar{z}N}\bar{\mathbb{A}}) = \tilde{Q}(e^{zN}a_0, e^{\bar{z}N}\bar{a_0}) + h = \tilde{Q}(e^{2\sqrt{-1}yN}a_0, \bar{a_0}) + h = p(y) + h$, where $p(y)$ is a polynomial in y with $d = \deg p(y) = \max\{\ell | N^\ell \alpha_0 \neq 0\}$. This a consequence of the polarization condition for the mixed Hodge structure and the fact that $a_0 \in Gr_{n+d}$. So

$$\begin{aligned} 4G &= \frac{(p' + h)^2 - (p + h)(p'' + h)}{(p + h)^2} = \frac{({p'}^2 - pp'') + h}{p^2 + h} \\ &\sim \frac{{p'}^2 - pp''}{p^2} + h \sim \frac{d^2 - d(d-1)}{y^2} + h = \frac{d}{y^2} + h. \end{aligned}$$

Here we have used the fact that $p^{-2}h \in h$. Obviously, if $NF^n_\infty = 0$ then $d = 0$ and $G = h$, so $\int_t^\infty \sqrt{G}|dz| < \infty$ for some curve (e.g. $x = c$). When $NF^n_\infty \neq 0$ we have $d \geq 1$ and for y large enough we can make $h < 1/y^3$ uniformly in x, then clearly $\int_t^\infty \sqrt{G}|dz| \sim 2\log y|_t^\infty = \infty$ for any path with $y \to \infty$. Q.E.D.

In terms of the period map ϕ^n, (1.1) can be rephrased as

Corollary 1.2. *The center is at finite distance if and only if the period map ϕ^n extends continuously over it.*

Proof. Using the notation from (1.1), we have $\Omega(z) = e^{zN}\mathbb{A}(z) = e^{zN}(a_0 + a_1 e^{2\pi\sqrt{-1}z} + \cdots) = e^{zN}a_0 + e^{zN}a_1e^{2\pi\sqrt{-1}z} + \cdots$. All terms except $e^{zN}a_0$ go to 0 exponentially when $y \to \infty$, so $\Omega(z)$ has a limit if and only if $Na_0 = 0$; this limit value is $a_0 \in F^n_\infty$. Since Ω is nonzero when y is large, it determines $F^n_z = \Phi^n(z)$. So $\lim_{t\to 0} F^n_t = \lim_{t\to 0}\phi^n(t) = \lim_{z\to i\infty}\Phi^n(z) \pmod T = F^n_\infty$. □

Remark 1.3. From the proof of (1.1) we know that in the case of infinite distance the Weil-Petersson metric is exponentially asymptotic to a scaling of the Poincaré metric. This is exactly the situation for the moduli space of elliptic curves.

§2. Geometric criterion for finite distance points

For a semi-stable degeneration, there is a well-known procedure to relate the limiting MHS and Deligne's *canonical MHS* on the singular cohomology of the central fiber, namely the *Clemens-Schmid exact sequence*. Let's briefly recall the constructions (see [C, GS] for more details). Let $X = \cup_i X_i$ be a simple normal crossing variety, for $I = \{i_0, \cdots, i_p\}$, $X_I := X_{i_0} \cap \cdots \cap X_{i_p}$. Also let $X^{[p]}$ be the disjoint union of all X_I with $|I| = p+1$. There is a spectral sequence with $\mathrm{E}_0^{p,q} = \Omega^q(X^{[p]})$ having differentials $d_0 := d$, the exterior differentiation of forms, and $d_1 := \delta$, the restriction operator of forms defined by

$$(\delta\phi)(X_{i_0\cdots i_{p+1}}) := \sum\nolimits_{j=0}^{p+1} (-1)^j \phi(X_{i_0\cdots\hat{i_j}\cdots i_{p+1}})\big|_{X_{i_0\cdots i_{p+1}}}.$$

This spectral sequence has $\mathrm{E}_1^{p,q} = H^q(X^{[p]})$, the $\mathrm{E}_2^{p,q}$ terms is computed from

$$H^q(X^{[p-1]}) \xrightarrow{\delta} H^q(X^{[p]}) \xrightarrow{\delta} H^q(X^{[p+1]}),$$

where the δ's are morphisms of Hodge structures. Moreover it degenerates at E_2. The weight filtration for the resulting MHS on $H^m(X_0)$ is $W_\ell := \bigoplus_{s\leq\ell}\mathrm{E}_2^{m-s,s}$, and the Hodge filtration is the usual Hodge filtration for each factor induced from E_1.

The Clemens-Schmid exact sequence for a semi-stable degeneration $\mathcal{X} \supset X_0$ is an exact sequence of MHS's:

$$\cdots \to H_{2n+2-m}(X_0) \xrightarrow{j} H^m(X_0) \xrightarrow{i} H^m \xrightarrow{N} H^m \xrightarrow{k} H_{2n-m}(X_0) \to \cdots$$

Notice that the inclusion $X_0 \subset \mathcal{X}$ is a homotopy equivalence, j is induced by inclusion and duality, i is induced by inclusion $X_t \subset \mathcal{X} \sim X_0$, H^m denotes the cohomology for the general fiber X_t, and N is the nilpotent monodromy oprator. Moreover, this exact sequence is compatible with the MHS's with types of morphisms $(n+1, n+1)$, $(0,0)$, $(-1,-1)$ and $(-n,-n)$ respectively, where type (p,q) means $F^* Gr_* \to F^{*+p} Gr_{*+2q}$. In addition, MHS for homology is defined by duality: $Gr_{-\ell}(H_q) = Gr_\ell(H^q)^*$ and $F^{-p} Gr_{-\ell}(H_q) = \mathrm{Ann}(F^{p+1} Gr_\ell(H^q))$.

When the degeneration of Hodge structures in (1.1) comes from a semi-stable degeneration of Calabi-Yau manifolds, we have:

Theorem 2.1. *The central fiber X has finite Weil-Petersson distance if and only if some irreducible component X_i of X has $H^{n,0}(X_i) \neq 0$ (equivalently, has exactly one component with $h^{n,0} = 1$.)*

Proof. By Schmid, F_∞ and N defines a MHS on V, and it follows that $NF^n_\infty = 0$ if and only if $F^n_\infty = Gr^W_n F^n_\infty$ (use $N^\ell : Gr^W_{n+\ell} \cong Gr^W_{n-\ell}$). The geometric genus formula ([C]) says $p_g(X_t) \geq \sum_i p_g(X_i)$ where $p_g(M) = h^0(\Omega^n_M) = h^{n,0}(M)$ and the right hand side corresponds exactly to all the invariant cycles (i.e. $\ker N$) in F^n_∞, so the theorem follows. Since $p_g(X_t) = 1$, there exists at most one component with $h^{n,0} \neq 0$ (and in fact must be 1). □

For the reader's convienence, we sketch the argument for the geometric genus formula. Applying the Clemens-Schmid exact sequences to $F^n Gr_n(H^n)$,

$$\cdots \to F^{-1} Gr_{-n-2} H_{n+2}(X_0) \to F^n Gr_n H^n(X_0) \to F^n_\infty Gr^W_n H^n \xrightarrow{N} 0.$$

Since $Gr_{n+2}(H^{n+2}) = \mathrm{E}_2^{0,n+2} = \mathrm{Ker}(\delta)$ with $\delta : H^{n+2}(X^{[0]}) \to H^{n+2}(X^{[1]})$ and $F^2 H^{n+2} = H^{n+2}$, we have $F^2 \mathrm{Ker}(\delta) \equiv \mathrm{Ker}(\delta)$. So $F^{-1} Gr_{-n-2}(H_{n+2}) = \mathrm{Ann}(F^2 Gr_{n+2}(H^{n+2})) = 0$ hence $F^n Gr_n(H^n(X_0)) \cong F^n_\infty Gr^W_n(H^n)$.

Now $F^n Gr_n(H^n(X_0)) = F^n \mathrm{E}_2^{0,n} = F^n \mathrm{E}_1^{0,n} = F^n H^n(X^{[0]}) = H^{n,0}(X^{[0]})$ because $\mathrm{E}_2^{0,n}$ is computed from $0 \to \mathrm{E}_1^{0,n} \to \mathrm{E}_1^{1,n} = H^n(X^{[1]})$, whose F^n part is zero. Writing out the meaning of both sides gives the result.

Remark 2.2. Both (1.1) and (2.1) are stated in the one parameter case, but the Weil-Petersson metric distance should be evaluated in the corresponding smoothing component of the central fiber, which is in general of many dimensions. However, finite distance in a special direction implies finite distance in the whole component, so (2.1) indeed provides a suifficient condition for the existence of finite distance points. The converse is not obvious, it needs more work to get the criterion for all finite distance points. This will be discussed in a subsequent paper.

Now we apply (2.1) to smoothable singular Calabi-Yau varieties. A Calabi-Yau variety is by definition a normal projective variety with trivial canonical (cartier) divisor. A normal variety V is said to have *canonical* (resp. *terminal*) singularities if K_V is $\mathbb{Q}$-Cartier and there is a (equivalently for any) resolution $f : W \to V$ such that $K_W =_{\mathbb{Q}} f^* K_V + \sum e_i E_i$ with $e_i \geq 0$ (resp. $e_i > 0$), where E_i's are the exceptional divisors. Canonical singularities in dimension 2 are exactly RDP's. Terminal singularities must be of codimension 3 and it is easy to see that they include all Kleinian singularities in any dimension ≥ 3. In dimension 3 they are completely classified (See Reid [R] for an introduction). Canonical singularities play an important role in birational geometry. In the case of Calabi-Yau 3-folds, birational *log extremal contractions* (also called *primitive contractions* by Wilson [W]) will create at most canonical singularities. It is conjectured that the moduli spaces of Calabi-Yau 3-folds with $h^1(\mathcal{O}) = 0$ and of different topological types can be connected by performing primitive contractions and smoothings. Our next result implies that this can happen only within finite Weil-Petersson distance.

Proposition 2.3. *Let X be a Calabi-Yau varieties which admits a smoothing to Calabi-Yau manifolds. If X has only canonical singularities then X has finite Weil-Petersson distance along any such smoothing.*

Proof. For any resolution $f : \tilde{X} \to X$ we have as in the above, $H^{n,0}(\tilde{X}, \mathbb{C}) = \Gamma(\tilde{X}, K_{\tilde{X}}) = \Gamma(\tilde{X}, \sum e_i E_i)$ (notice e_i's are integers). Since E_i's are exceptional it follows easily that $H^{n,0}(\tilde{X}, \mathbb{C}) \neq 0$ precisely when V has at most canonical singularities. Now let $\mathcal{X}/\Delta$ be a smoothing of X. Take a semi-stable reduction of it, then there is a component in the central fiber of the semi-stable reduction which corresponds to the proper transform of X. Then it has $h^{n,0} = 1$. Now apply Proposition (1.2) and notice that finite distance in a special smoothing implies finite distance in the whole smoothing component. □

Question 2.4. Is the converse of (2.3) true in the sense that if a degeneration of Calabi-Yau's has finite Weil-Petersson distance, is that true this degeneration is birational to another degeneration such that the central fiber is an irreducible Calabi-Yau with only canonical singularities? This would be an important step toward the completion program.

Remark 2.5. The problem of whether a singular Calabi-Yau X with canonical singularities has a flat deformation into nonsingular Calabi-Yau's X_t has been studied extensively in dimension 3. The first step was taken by Friedman [F1] in the case of ODP's (see also Tian [T2] and [F2]). Recent preprints of Namikawa-Steenbrink and Gross have provided quite satisfactory results in this direction.

Remark 2.6. In fact, all the statements in §1 and §2 are true in the following much more general setting. Given a smooth polarized family of varieties $\mathcal{X} \to S$ parametrized by a smooth base S and with $h^0(X_s, K_{X_s}) \geq 2$, we may consider the semi-definite "metric" ω on S given by the Chern form of the determinant line bundle of the Hodge bundle F^n, that is, $\det f_* K_{\mathcal{X}/S}$ ($K_{\mathcal{X}/S}$ is the relative canonical bundle). Using this metric, the main results (1.1), (2.1) and (2.3) generalize immediately. However even in the Kähler-Einstein case with K_X ample, this metric is not the Weil-Petersson metric defined in (0.6). There is a complicated relation between the two in terms of certain "Quillen metric". This issue will also be discussed in a subsequent paper.

§3. Incompleteness I: nontrivial monodromy

In §3 and §4, we work on the projective category and for Calabi-Yau varieties it is always assumed that $h^1(\mathcal{O}) = 0$ (so we have excluded the case of abelian varieties).

There exists smoothable Calabi-Yau 3-folds with canonical singularities such that the smoothing comes from a birational contraction of a smooth family over the disk which induces isomorphisms outside the puncture. The examples are due to Wilson [W] in his deep study of the jumping phenomenon of Kähler cone (the type III primitive contraction with the exceptional divisor a quasi-ruled surface over an elliptic curve provides such an example. See his proposition (4.1)).

In the surface case, these correspond to smoothings of K3 surfaces with RDP's (canonical singularities), which by Kulikov's classification theorem [Ku] and the fact that "$NF_\infty^2 = 0$ implies $N = 0$", are birational to smooth families (up to a base change).

For our purpose, the above examples should not be considered as incomplete points in the following sense: one can include these points by hand–just replace the degeneration by the smooth family by allowing the polarization line bundle to be only big and nef. In the case of K3 surfaces, an equivalent way is to add these points by allowing Ricci-flat orbifold metrics (this leads to the completeness of K3 moduli!). However we will see there are "nontrivial" examples. By this we mean a degeneration such that the complement of the central fiber can not be completed into a smooth family.

If the monodromy T is not of finite order ($N \neq 0$) then the degeneration is clearly nontrivial in the above sense. In this direction we have the following classical result due to Lefschetz and Poincaré:

Theorem 3.1. *The monodromy T of a nodal degeneration of smooth n-folds is trivial except possibly in the middle dimensional cohomology. In the middle dimension, $N^2 = 0$ if n is odd, and $T^2 = I$ (so $N = 0$) if n is even.*

The standard proof of (3.1) is to write down the explicit formula of T in terms of the "vanishing cycle". However, in order to see whether $N \neq 0$ in the odd case one needs to know whether the vanishing cycle is a nontrivial homology class, and this is usually not an easy problem. (The fact that the vanishing cycle can be homologically trivial was kindly pointed out to me by J. de Jong.)

Here is a modern proof of (3.1) as a corollary of the Clemens-Schmid exact sequence:

First of all, a semi-stable reduction can be obtained by first doing a degree 2 base change and then blowing up the ODP's of the total space. So $X^{[0]}$ is the union of n-quadrics and the proper transform X' of the original central fiber, $X^{[1]}$ is the union of $(n-1)$-quadrics and $X^{[2]} = \emptyset$. For $m < n$, we claim that for $\ell \leq m - 1$, $Gr_\ell H^m = 0$. Suppose it has been proved up to $\ell - 1$, then

$$Gr_{\ell-2n-2}H_{2n+2-m}(X_0) \to Gr_\ell H^m(X_0) \to Gr_\ell^W H^m \xrightarrow{N} 0.$$

Now $G_\ell H^m(X_0) = \mathrm{E}_2^{m-\ell,\ell} = 0$ if $m - \ell \geq 2$ since $X^{[2]} = \emptyset$. If $m - 1 = \ell$, $\mathrm{E}_2^{1,m-1} = \mathrm{Coker}(\delta)$ with $\delta : H^{m-1}(X^{[0]}) \to H^{m-1}(X^{[1]})$ which is surjective by explicit cohomologies of quadrics (or use hyperplane section theorem) since $m < n$. This proves $G_\ell H^m(X_0) = 0$ and so $G_\ell^W H^m = 0$ up to $\ell = m - 1$. This means the MHS is pure and so $N = 0$.

For H^n, the same argument shows that $G_\ell^W H^n = 0$ up to $\ell = n - 2$, so $N^2 = 0$. Since $\ell < n$, $Gr_{\ell-2n-2} H_{2n+2-n}(X_0) = (Gr_{2(n+1)-\ell} H^{2(n+1)-n}(X_0))^* = 0$. We have $Gr_{n-1} H^n(X_0) \cong Gr_{n-1}^W H^n = \mathrm{Coker}(\delta)$ with $\delta : H^{n-1}(X^{[0]}) \to H^{n-1}(X^{[1]})$. Now the middle cohomology of an $(n-1)$-quadric is zero if n is even, so $N = 0$ in this case. Q.E.D.

For the case n is odd, $N \neq 0$ if and only if δ is not surjective. The middle cohomology has rank 2 for an even dimensional quadric and the image of these n-quadrics under δ consist of suitable powers of the hyperplane class, which is also in the image of $H^{n-1}(X')$ if $n \geq 5$ (because for any of the $(n-1)$-quadrics E, $E|_E$ generates $H^2(E)$ which is only 1 dimensional). Therefore δ is surjective if and only if the induced map $\delta' : H^{n-1}(X') \to H^{n-1}(X^{[1]})$ is surjective. There are examples where δ' is surjective and also examples where it is not surjective, so we can not tell much about the monodromy without specifying the varieties under consideration.

We do have the following:

Proposition 3.2. *For a semi-stable degeneration of Calabi-Yau 3-folds with finite Weil-Petersson distance, or more generally, 3-folds with $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$ such that $NF_\infty^3 = 0$, $N \neq 0$ if $b^2(X^{[1]}) + b^2(X_t) - b^2(X^{[0]}) + (k - \ell - 1) > 0$, where k (resp. ℓ) is the number of components of $X^{[0]}$ (resp. $X^{[2]}$).*

Lemma 3.3. *For H^2, both Deligne's canonical MHS and Schmid's limiting MHS are pure with $H^{1,1}Gr_1$ the only nonzero term. They are related by $b^2(X_0) = b^2(X_t) + (k-1)$ and $H^2(X_0) = \mathrm{Ker}(\delta_0)$ with $\delta_0 : H^2(X^{[0]}) \to H^2(X^{[1]})$.*

Proof. For H^2, $h^1(\mathcal{O}) = 0$ implies $h^2(\mathcal{O}) = 0$ in the Calabi-Yau case, so we know the only nontrivial term for the limiting MHS is $H^{1,1}Gr_1^W H^2$. To see $F^2 Gr_2(H^2(X_0)) = 0$, apply the Clemens-Schmid exact sequence to $F^2 Gr_2$, we get

$$F^{-2} Gr_{-6} H_6(X_0) \to F^2 Gr_2 H^2(X_0) \to 0.$$

The first term is equal to $\mathrm{Ann}(F^3 Gr_6 H^6(X_0))$, which is zero because $Gr_6 H^6(X_0) = E_2^{0,6} = H^6(X^{[0]})$ and the F^3 piece equals the whole space. The cases $F^2 Gr_3(H^2(X_0)) = 0$ and $F^2 Gr_4(H^2(X_0)) = 0$ are similiar and in fact more trivial because the total space of the first term is actually 0. By Hodge symmetry we only need to look at $H^{1,1}(H^2(X_0))$.

Applying the Clemens-Schmid exact sequence to $F^1 Gr_2$, we obtain

$$0 \to F^0 Gr_0^W H^0 \to F^{-3} Gr_{-6} H_6(X_0) \to F^1 Gr_2 H^2(X_0) \to F_\infty^1 Gr_2^W H^2 \xrightarrow{N} 0.$$

It is clear that $F^0_\infty Gr^W_0 H^0 \cong \mathbb{C}$ and $F^1_\infty Gr^W_2 H^2 = H^2(X_t)$. Now $F^{-3}Gr_{-6}H_6(X_0)$ equals $\mathrm{Ann}(F^4 Gr_6 H^6(X_0))$, which is $\mathbb{C}^k$ because $Gr_6 H^6(X_0) = H^6(X^{[0]}) \cong \mathbb{C}^k$ whose F^4 is zero and $\mathrm{Ann}(0)$ is the whole space. A dimension counting gives the desired result. □

Lemma 3.4. *For H^3, Deligne's canonical MHS and Schmid's limiting MHS are isomorphic for all Gr_ℓ with $\ell \leq 2$. In particular the only nonzero term is $F^1_\infty Gr^W_2 = \mathrm{Ker}(\delta_1)/\mathrm{Im}(\delta_0)$ in*

$$H^2(X^{[0]}) \xrightarrow{\delta_0} H^2(X^{[1]}) \xrightarrow{\delta_1} H^2(X^{[2]}).$$

Proof. Since $NF^3_\infty = 0$ the limiting MHS has $F^3_\infty \neq 0$ only in Gr^W_3. By the Hodge symmetry of the MHS, we know that the only nontrivial terms for Gr^W_ℓ, $\ell \leq 3$ are Gr^W_3 and $H^{1,1}Gr^W_2$. The remainder of the the proof is now similiar to that of lemma (3.3), so we omit it. □

Proof of 3.2. Since $N \neq 0$ if and only if $Gr^W_2 H^3 \neq 0$, (3.2) then follows directly from lemma (3.3) and (3.4). □

Corollary 3.5. *$N \neq 0$ in the case of nodal degenerations.*

Proof. If there are r ODP's, using the semi-stable reduction discussed above, we find $\ell = 0$, $k = r + 1$, $b^2(X^{[1]}) = 2r$ and $b^2(X^{[0]}) = (b^2(X_t) + r) + r$, so $b^2(X^{[1]}) + b^2(X_t) - b^2(X^{[0]}) + (k - \ell - 1) = r > 0$. □

In principal one should be able to apply (3.3) to more general singularities. However the semi-stable reduction is usually much more complicated than that of ODP's.

§4. Incompleteness II: birational geometry

Now we go to the most technical part of this paper. Even in dimension 3, it is possible for a nontrivial degeneration to have $N = 0$! In [F2], Friedman remarks that a family of quintic hypersurfaces in $\mathbb{P}^4$ aquiring an A_2 singularity has $N = 0$ (due to Clemens), he also asks whether this family can be filled in smoothly (up to a base change).

In this section, by using several results of Reid, Kawamata and Kollar in the theory of 3-fold birational geometry along with Friedman's result on the simultaneous resolution of 3-fold double points, a negative answer to this question is given for any projective smoothing of a terminal Gorenstein 3-fold with numerical effective (nef) canonical bundle. As a consequence, any smoothable terminal Calabi-Yau 3-fold provides nontrivial incomplete points of the Weil-Petersson metric. Similiar statement is true for the general setting in remark (2.6). Here is the main theorem:

Theorem 4.1. *Let $\mathcal{X}/\Delta$ be a projective smoothing of a terminal Gorenstein 3-fold X_0 with K_{X_0} nef. Then $\mathcal{X}/\Delta$ is not birational to a projective smooth family $\mathcal{X}'/\Delta$ with $X_t = X'_t$ for $t \neq 0$.*

We start with the following important fact (true in any dimension):

Proposition 4.2. *Let $\mathcal{X}/\Delta$ and $\mathcal{X}'/\Delta$ be two projective families with smooth general fiber $X_t = X'_t$ for $t \neq 0$. Assume that*

(1) *$\mathcal{X}$ and $\mathcal{X}'$ have at most terminal singularities and*
(2) *$K_{\mathcal{X}}$ (resp. $K_{\mathcal{X}'}$) is nef on the central fiber,*

then the bimeromorphic map which identifies all fibers outside $t = 0$ extends to a map which is an isomorphism in codimension one. In particular, X_0 and X'_0 are birational to each other.

Proof. This is essentially the same as in [K1, lemma (4.3)] except they deal with the case where $\mathcal{X}$ and $\mathcal{X}'$ are both compact projective and Δ is not involved. The same proof applies to our relative situation basically because our families are assumed to be projective, so we will just sketch the proof here:

Let ϕ be the given bimeromorphic map and $\mathcal{Z}$ be a desingularization of the closure of the graph of ϕ with projection maps $p : \mathcal{Z} \to \mathcal{X}$ and $p' : \mathcal{Z} \to \mathcal{X}'$ over Δ. (Notice that $Z_t = X_t = X'_t$ for $t \neq 0$.)

If p and p' have the same exceptional divisors then the p-exceptional set and p'-exceptional set differ only in codimension two or higher, let E be the union of both set. Then we have the following isomorphisms

$$\mathcal{X} - p(E) \cong \mathcal{Z} - E \cong \mathcal{X}' - p'(E),$$

which is the extension of ϕ we want.

To see p and p' have the same exceptional divisors, consider the relation between canonical divisors:

$$K_{\mathcal{Z}} = p^* K_{\mathcal{X}} + E_1 + F = {p'}^* K_{\mathcal{X}'} + E_2 + G,$$

where E_i (resp. F, resp. G) denotes the part which are p and p' (resp. p but not p', resp. p' but not p) exceptional. We can then write

$$p^* K_{\mathcal{X}} = {p'}^* K_{\mathcal{X}'} + G + (E_2 - E_1 - F).$$

Because of the existence of relative hyperplane sections over Δ, the key reduction lemma in [K3, (5.2.5.3)] can be adapted for our purpose. And it implies that $E_2 - E_1 - F \geq 0$, hence $F = 0$ and $E_2 \geq E_1$. (The reduction lemma says that we only need to prove the above statement for the surface case; the nef condition is used here) Reversing the role of p and p' gives $G = 0$ and $E_1 \geq E_2$, so we have in fact $E_1 = E_2$. Since both $\mathcal{X}$ and $\mathcal{X}'$ have terminal singularities, all exceptional divisors must appear in E_i. So the proposition is proved. □

Proof of 4.1. Assume such a smooth family $\mathcal{X}'/\Delta$ exists. Now we check the conditions needed in (4.2). (2) is clearly satisfied since $K_{\mathcal{X}}|_{X_0} = K_{X_0}$ which is nef. To see (1), first notice that by a simple fact in commutative algebra, a small smoothing of Gorenstein singularities has Gorenstein total sapce. We then need the following nontrivial theorem. (Although the statement is not explicitly appeared in [K2], the proof is actually contained in [K2, (17.4), (17.6)], and so will not be given here.) □

Theorem 4.3. *The total space of a small smoothing of terminal singularities has at most terminal singularities.*

Since both conditions in (4.2) are satisfied, we know X_0 is birational to X_0'. We will show this is impossible.

If X_0 is $\mathbb{Q}$-factorial then X_0 and X_0' are birationally equivalent minimal models (minimal model is a variety which is $\mathbb{Q}$-factorial, terminal and has nef canonical class). By Kollar's theorem on flops [K1], they are related by a sequence of flops. But a flop does not change the singularities in the terminal case, so we get a contradiction.

If X_0 is not $\mathbb{Q}$-factorial, a theorem of Reid-Kawamata (see e.g. [K2, (6.7.4)]) says that we still have a projective small morphism $X \to X_0$ from a ($\mathbb{Q}$-factorial) minimal model X. X is birational to X_0 and so is birational to X_0'. As before this implies X is smooth and is related to X_0' by a sequence of flops. By [K1], X and X_0' have the same integral homologies and hence have the same homologies as the general fiber X_t in $\mathcal{X}$. Now we have the following "small contraction-smoothing" diagram

$$\begin{array}{c} X \\ \downarrow \\ X_0 \subset \mathcal{X}. \end{array}$$

If X_0 has ODP singularities, there is a well known explicit formula which relates the homologies of X and X_t and shows they can not be the same. We will state this formula in a slightly more general form which will be needed later, but the proof is exactly the same as that given in [F2].

Lemma 4.4. *Given a diagram as above in the C^∞ category such that near each singular point of X_0 it is a "small contraction-smoothing" diagram of a germ of ODP. Let C_i be the rational curves contracted to those ODP's and $e : \bigoplus_i \mathbb{Z}[C_i] \to H_2(X, \mathbb{Z})$ be the map which associates to each C_i its class in X, then $H_2(X_t) = \mathrm{Coker}(e)$.*

So $H_2(X_t) \cong H_2(X)$ means the image of e is zero which is impossible because X is projective. This is the desired contradiction in the case when X_0 has only ODP's.

In the general case, since the singularities are of index one, by Reid's classification they are exactly isolated cDV singular points (one parameter deformation of surface RDP's). By Friedman's result [F1], if $p \in V$ is a germ of an isolated cDV point and the curve $C \subset U$ is the corresponding germ of the exceptional set contracted to p, then $Def(p, V)$ and $Def(C, U)$ are both smooth and there is an inclusion map of complex spaces $Def(C, U) \to Def(p, V)$. Moreover one can deform the complex structure of a small neighborhood of C so that in this new complex structure, C decomposes into several $\mathbb{P}^1$'s where the contraction map becomes a (nontrivial) contraction of these $\mathbb{P}^1$'s down to ODP's, while a neighborhood of these ODP's remains in the versal deformations of the germ p. We can do this for all C's and p's simultaneously in the corresponding small neighborhoods analytically, then patch them together smoothly. As a result we obtain a deformed diagram with conditions as in lemma (4.4):

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ \tilde{X}_0 \subset \tilde{\mathcal{X}}. \end{array}$$

By construction, $\tilde{X}$ is diffeomorphic to X and $\tilde{X}_t$ is diffeomorphic to X_t for $t \neq 0$. The later is true because $Def(p, V)$ is smooth and the constructiuon is local. Now we have again,

$$H_2(\tilde{X}_t) \cong H_2(X_t) \cong H_2(X) \cong H_2(\tilde{X}),$$

hence the image of e is zero. Since the original exceptional curve has nontrivial homology class, at least one deformed rational curve C has nontrivial homology class. This leads to the desired contradiction again and we are done. □

In the case of Calabi-Yau 3-folds with canonical singularities, $h^1(\mathcal{O}) = 0$ implies $h^2(\mathcal{O}) = 0$, hence any smoothing $\mathcal{X}/\Delta$ (if it exists) is projective by semi-continuity and X_t must still be Calabi-Yau. So we obtain the following:

Theorem 4.5. *Let $\mathcal{X}/\Delta$ be a smoothing of a terminal Calabi-Yau 3-fold. Then $\mathcal{X}/\Delta$ is not birational to a smooth family.*

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