A FUNCTION THEORETIC VIEW OF THE MEAN FIELD EQUATIONS ON TORI

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ABSTRACT. The mean field equations on flat tori is analyzed purely in terms of the theory of elliptic functions. Some results which were known previously through PDE techniques are reproved here using more elementary methods. Also a general method to construct explicit solutions is developed when the topological degree is known to be non-zero. Hopefully our method will lead to all solutions. This article is a supplementary reading to our recent work [7].

1. INTRODUCTION

1.1. **A brief historical review.** The mean field equation on a flat torus $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ takes the form

$$\triangle u + \rho e^u = \rho \delta_0, \qquad \rho \in \mathbb{R}_+. \tag{1.1}$$

It is originated from the prescribed curvature problem in geometry. It also comes from statistical physics as the mean field limits of the Euler flow. Recently it was shown to be related to the self dual condensation of Chern-Simons-Higgs model. We refer to [2], [3], [4], [5], [8] and [9] for the recent development of this subject.

When $\rho \neq 8m\pi$ for $m \in \mathbb{Z}$, it has been recently proved in [3], [4], [5] that the Leray-Schauder degree is non-zero, so the equation always has solutions, regardless on the actual shape of T.

For the first *dangerous value* $\rho = 8\pi$ where the degree theory fails, the following results were recently obtained by the authors.

Theorem 1.1 ([7]). *For $\rho = 8\pi$, the mean field equation on a flat torus has solutions if and only if the Green function $G(z)$ has critical points other than the three half period points. Moreover, each extra pair $\pm z_0$ of critical points corresponds to an one parameter scaling family of solutions.*

It is known that for rectangular tori $G(z)$ has precisely the three obvious critical points, namely the half periods $\frac{1}{2}\omega_i$, hence for $\rho = 8\pi$ equation (1.1) has no solutions. However for $\omega_1 = 1$ and $\tau = \omega_2 = e^{\pi i/3}$, it is shown in [7] that there are five critical points and solutions of (1.1) exist. (The two others are $\pm\frac{1}{3}(\omega_1 + \omega_2)$.)

Theorem 1.2 ([7]). *The Green function has at most five critical points. Equivalently, for $\rho = 8\pi$, the mean field equation on a flat torus has at most one solution up to scaling.*

In fact no direct proof of the first statement is known. Instead, the uniqueness theorem was proved using PDE methods first, and then Theorem 1.1 implies that there are at most 5 critical points.

The critical point equation for $z_0 = t\omega_1 + s\omega_2$ is a very simple equation in elliptic functions (c.f. Lemma 2.3):

$$\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2. \tag{1.2}$$

However, it remains a challenge to analyze it within function theory.

The proof of Theorem 1.2 in [7] is based on the continuity method applied to the one parameter family of mean field equations

$$\triangle u + \rho e^u = \rho\delta_0, \quad \rho \in [4\pi, 8\pi]$$

on T within *even solutions*, where for the starting point $\rho = 4\pi$ the solution is uniquely constructed through integration of Weierstrass elliptic functions.

1.2. **The content of the paper.** As the title suggests, this paper concerns the function theoretic aspect of the mean field equations and Green functions on tori without appealing to the techniques of partial differential equations.

Specifically, we give a proof of Theorem 1.1 following the same line in [7] with one technical difference: For the proof in [7] we used the fact:

Theorem 1.3. *For $\rho = 8\pi$, the blow-up points of solutions to the mean field equation $\triangle u + \rho e^u = \rho\delta_0$ on a torus T are precisely those non half-period critical points of the Green function G.*

This now follows as a consequence (c.f. Corollary 5.1) of our proof.

Secondly we explore the construction of solutions for $\rho = 4\pi$ and $\rho = 8\pi$ in a more systematic manner aiming at investigating the general situations for $\rho = 4\pi l$, $l \in \mathbb{N}$. As in [7], we use the classical Liouville theory to find local developing maps f's of a solution u:

$$u = c_1 + \log \frac{|f'|^2}{(1 + |f|^2)^2}, \tag{1.3}$$

and glue them to a global meromorphic function on $\mathbb{C}$ (c.f. Lemma 3.1).

The developing maps are locally unique up to a $\mathrm{PSU}(1)$ action. By comparing $f(z + \omega_i)$ with $f(z)$ for $i = 1, 2$, we achieve two types of solutions. To describe them, it is convenient to consider the logarithmic derivative

$$g = (\log f)' = \frac{f'}{f} \tag{1.4}$$

in our discussions. Then we have (see §3)

Type I: Topological solutions, where g is elliptic on $T' = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$:

$$
\begin{aligned}
f(z+\omega_1) &= -f(z),\\
f(z+\omega_2) &= -\frac{\bar{q}^2}{f(z)}.
\end{aligned}
\tag{1.5}
$$

Type II: Blow-up solutions, where g is elliptic on T:

$$
\begin{aligned}
f(z+\omega_1) &= e^{2i\theta_1} f(z),\\
f(z+\omega_2) &= e^{2i\theta_2} f(z).
\end{aligned}
\tag{1.6}
$$

The type I solutions are called topological since a Leray-Schauder degree counting formula is known when $\rho \neq 8m\pi$ [3], [4], [5]. The answer is 1 for $\rho = 4\pi$ and 2 for $\rho = 12\pi$.

To warm up, we repeat in §5 the proof in [7] showing that there is a unique solution for $\rho = 4\pi$ and no solutions for $\rho = 8\pi$ of Type I.

In §6 we proceed to construct two explicit solutions for $\rho = 12\pi$ by investigating in details the properties satisfied by g and f. It is reasonable to believe that solutions constructed from the holomorphic data will contribute a positive degree in the degree counting. Thus the two solutions constructed should then be all the solutions.

The type II solutions are called blow-up solutions since if f is a developing map of u then $e^{\lambda} f$, $\lambda \in \mathbb{R}$ will also be a developing map for another solution

$$
u_\lambda(z) := c_1 + \log \frac{e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}.
\tag{1.7}
$$

Moreover any zero of f will be a blow-up point for the sequence u_λ.

The following result is well-known. Here we give a purely elementary proof of it through elliptic functions (c.f. Theorem 4.4).

Theorem 1.4. *For $\rho = 4\pi l$ with l being odd, there are no type II, i.e. blow-up, solutions to the mean field equation*

$$
\triangle u + \rho e^u = \rho \delta_0 \quad \text{in } T.
$$

We hope that the strategy developed here from §4 to §6 will be useful for later studies on the mean field equations on tori with $\rho = 4\pi l$, $l \geq 3$, especially on the explicit constructions of solutions.

2. Green functions via elliptic functions

2.1. **Green functions via theta functions.** Let $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a flat torus. As usual we let $\omega_3 = \omega_1 + \omega_2$. The Green function $G(z, w)$ is the unique function on T which satisfies

$$
-\triangle_z G(z,w) = \delta_w(z) - \frac{1}{|T|}
\tag{2.1}
$$

and $\int_T G(z,w)\,dA = 0$. It has the property that $G(z,w) = G(w,z)$ and it is smooth in (z,w) except along the diagonal $z = w$, where

$$G(z,w) = -\frac{1}{2\pi}\log|z-w| + O(|z-w|^2) + C \tag{2.2}$$

for a constant C which is independent of z and w. Moreover, due to the translation invariance of T we have that $G(z,w) = G(z-w,0)$. Hence it is also customary to call $G(z) := G(z,0)$ the Green function. It is an even function with the only singularity at 0.

The Green function can be explicitly solved in terms of elliptic functions. It takes a simple form using theta functions.

Consider a torus $T = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, a lattice with $\omega_1 = 1$ and $\omega_2 = \tau = a + bi$, $b > 0$. Let $q = e^{\pi i\tau}$ with $|q| = e^{-\pi b} < 1$. The theta function $\vartheta_1(z;\tau)$ is the exponentially convergent series

$$\begin{aligned}\vartheta_1(z;\tau) &= -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi iz}\\ &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z.\end{aligned} \tag{2.3}$$

For simplicity we also write it as $\vartheta_1(z)$. It is entire with

$$\begin{aligned}\vartheta_1(z+1) &= -\vartheta_1(z),\\ \vartheta_1(z+\tau) &= -q^{-1}e^{-2\pi iz}\vartheta_1(z),\end{aligned} \tag{2.4}$$

which has simple zeros at the lattice points (and no others).

As usual we denote $z = x + iy$.

Lemma 2.1. *Up to a constant $C(\tau)$, the Green function $G(z,w)$ for the Laplace operator $\triangle$ on T is given by*

$$G(z,w) = -\frac{1}{2\pi}\log|\vartheta_1(z-w)| + \frac{1}{2b}(\operatorname{Im}(z-w))^2 + C(\tau). \tag{2.5}$$

Proof. Let $R(z,w)$ be the right hand side. Clearly for $z \neq w$ we have $\triangle_z R(z,w) = 1/b$ which integrated over T gives 1. Near $z = w$, $R(z,w)$ has the correct behavior. So it remains to show that $R(z,w)$ is indeed a function on T. From the quasi-periodicity, $R(z+1,w) = R(z,w)$ is obvious. Also

$$R(z+\tau,w) - R(z,w) = -\frac{1}{2\pi}\log e^{\pi b + 2\pi y} + \frac{1}{2b}((y+b)^2 - y^2) = 0.$$

These properties characterize the Green's function up to a constant. $\square$

By the translation invariance of G, it is enough to consider $w = 0$. Let

$$G(z) = G(z,0) = -\frac{1}{2\pi}\log|\vartheta_1(z)| + \frac{1}{2b}y^2 + C(\tau).$$

If we represent the torus T as centered at 0, then the symmetry $z \mapsto -z$ shows that $G(z) = G(-z)$. By differentiation, we get $\nabla G(z) = -\nabla G(-z)$. If $-z_0 = z_0$ in T, that is $2z_0 = 0 \pmod{\Lambda}$, then we get $\nabla G(z_0) = 0$. Hence we obtain the half periods $\frac{1}{2}$, $\frac{1}{2}\tau$ and $\frac{1}{2}(1+\tau)$ as three obvious critical points of $G(z)$ for any T.

By computing $\partial G/\partial z = \frac{1}{2}(G_x - iG_y)$ we find

Corollary 2.2. *The equation of critical points $z = x + iy$ of $G(z)$ is given by*

$$\frac{\partial G}{\partial z} \equiv \frac{-1}{4\pi}\left((\log \vartheta_1)_z + 2\pi i \frac{y}{b}\right) = 0. \tag{2.6}$$

2.2. **Weierstrass elliptic functions and periods integrals.** Now we translate these to Weierstrass' elliptic functions. Recall that $\zeta = -\int \wp = z^{-1} + \cdots$ and $\sigma = \exp \int \zeta = z + \cdots$. From

$$(\log \sigma(z))' = \zeta(z),$$

it is known that $\sigma(z)$ is entire, odd with a simple zero on lattice points. Moreover,

$$\sigma(z \pm \omega_i) = -e^{\pm \eta_i (z \pm \frac{1}{2}\omega_i)} \sigma(z). \tag{2.7}$$

This is similar to the theta function transformation law, indeed

$$\sigma(z) = e^{\eta_1 z^2/2} \frac{\vartheta_1(z)}{\vartheta_1'(0)}. \tag{2.8}$$

Hence

$$\zeta(z) - \eta_1 z = \left[\log \frac{\vartheta_1(z)}{\vartheta_1'(0)}\right]_z = (\log \vartheta_1(z))_z. \tag{2.9}$$

It is extremely important to understand the geometric meaning of

$$F_i(z) := \omega_i \zeta(z) - \eta_i z, \qquad i = 1, 2,$$

due to the following

Proposition 2.3. *Write $z = t\omega_1 + s\omega_2$, then*

$$G_z = -\frac{1}{4\pi}(\zeta(z) - \eta_1 t - \eta_2 s). \tag{2.10}$$

In particular, z is a critical point of G if and only if

$$\zeta(t\omega_1 + s\omega_2) = t\eta_1 + s\eta_2. \tag{2.11}$$

Moreover, it is also equivalent to that

$$F_i(z) \in i\mathbb{R}, \qquad i = 1, 2. \tag{2.12}$$

Proof. Since $z = x + yi = t\omega_1 + s\omega_2 = t + sa + sbi$, we have

$$(\log \vartheta_1)_z + 2\pi i \frac{y}{b} = \zeta - \eta_1(t + s\omega_2) + 2\pi i s = \zeta - t\eta_1 - s\eta_2,$$

where the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ is used.

The calculation indeed shows that

$$\omega_1 \zeta - \eta_1 z = (\zeta - t\eta_1 - s\eta_2)\omega_1 - 2\pi i s.$$

And similarly

$$\omega_2\zeta - \eta_2 z = (\zeta - t\eta_1 - s\eta_2)\omega_2 + 2\pi i t.$$

If z is a critical point then it is clear that both $F_i(z)$'s are purely imaginary. Conversely if both $F_i(z)$'s are imaginary, then the above equalities implies that $(\zeta - t\eta_1 - s\eta_2)\omega_i$'s are linearly dependent. This is possible only if $\zeta - t\omega_1 - s\omega_2 = 0$. The lemma is proved. ☐

Finally, $F_i(z)$ can be interpreted as certain period integrals of the meromorphic one form

$$\Omega(z,\xi)\,d\xi := \frac{\wp'(z)}{\wp(\xi) - \wp(z)}\,d\xi.$$

Lemma 2.4. *Up to $2\pi i\mathbb{N}$, $2F_i(z)$ is the period integral*

$$\int_{L_i} \Omega\, d\xi$$

along the i-th fundamental cycle L_i (the i-th complete period).

Proof. Recall the addition law (c.f. [1])

$$\wp(z) - \wp(y) = -\frac{\sigma(z+y)\sigma(z-y)}{\sigma^2(z)\sigma^2(y)}. \tag{2.13}$$

Then

$$\Omega = \frac{d}{dz}\log(\wp(z) - \wp(\xi)) = 2\zeta(z) - \zeta(z+\xi) - \zeta(z-\xi).$$

Hence

$$\int_{L_i} \Omega\, d\xi = 2\omega_i\zeta(z) + \log\frac{\sigma(\xi - z)}{\sigma(\xi+z)}\bigg|_a^{a+\omega_i} = 2\omega_i\zeta(z) - 2\eta_i z + 2\pi i m$$

for some $m \in \{-1,0,1\}$, where the last equality uses (2.7). ☐

3. Liouville theory with singular data

3.1. **Theorem of Liouville.** We start with a quick review of the Liouville theory. A classical theorem of Liouville says that any solution u of

$$\triangle u + \rho e^u = 0$$

in a simply connected domain $D \subset \mathbb{C}$ can be represented by

$$u = c_1 + \log\frac{|f'|^2}{(1+|f|^2)^2}, \tag{3.1}$$

where f is a meromorphic function in D and $c_1 = \log 4/\rho$. Such an f is called a developing map of u and can be selected to be holomorphic.

Developing maps are not unique, different developing maps are related by Möbius transformations. Indeed it is easily checked that u and f satisfy

$$u_{zz} - \frac{1}{2}u_z^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{3.2}$$

The right hand side of (3.2) is the Schwartz derivative of f. Thus for any two developing maps f and $\tilde{f}$ of u, there exists $S = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix} \in \mathrm{PSU}(1)$ (i.e. $p, q \in \mathbb{C}$ and $|p|^2 + |q|^2 = 1$) such that

$$\tilde{f} = Sf := \frac{pf - \bar{q}}{qf + \bar{p}}. \tag{3.3}$$

Here is a useful observation: By conjugating a matrix $U \in \mathrm{PSU}(1)$ to $\tilde{f}$ and f with $USU^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$, we may achieve that

$$U\tilde{f} = e^{2i\theta} Uf. \tag{3.4}$$

3.2. **Liouville theory on tori with isolated singular data.** It is a challenging problem to extend the Liouville theory to Riemann surfaces and with singular source. Here we consider the simplest genus one case with isolated singularities, namely the mean field equation

$$\triangle u + \rho e^u = \rho \delta_0, \quad \rho \in \mathbb{R}^+ \tag{3.5}$$

in a flat torus T, with δ_0 being the Dirac measure at 0 and the volume of T is normalized to be 1.

Lemma 3.1. *For $T = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\rho = 4\pi l$, $l \in \mathbb{N}$, by gluing the f's among simply connected domains, (3.1) holds on the whole $\mathbb{C}$ with f being a meromorphic function.*

Proof. This is stated and proved in [5] for rectangular tori with $l = 2$, but the proof works for the general case which is included below.

It is sufficient to prove the lemma for any chosen f. Let $z = e^{2\pi i w}$ be the universal covering map $\mathbb{H} \to \Delta^\times$ from the upper half plane to the punctured disk (of some small radius which is assumed to be 1 to save notations) and let $F(w) = f(z) = f(e^{2\pi i w})$. Then

$$F(w+1) = SF(w)$$

for some $S \in \mathrm{PSU}(1)$. By the above observation, up to a conjugation, we may start with another f so that $F(w+1) = e^{-2i\theta} F(w)$ for some $\theta \in [0, \pi)$.

Now let $\Psi(w) = e^{-2i\theta w} F(w)$. Then

$$\Psi(w+1) = e^{-2i\theta(w+1)} F(w+1) = e^{-2i\theta w} F(w) = \Psi(w).$$

Hence $\Psi(w)$ comes from a meromorphic function $\psi(z)$ on $\Delta^\times$.

By [6], §2, Lemma 4, if ψ has essential singularity at $z = 0$ then $f = z^{\theta/\pi} \psi$ takes any value in $\mathbb{C}$ infinitely many times except at most one value.

In our case

$$\infty > \int_{\Delta^\times} e^u \, dA = \int_{\Delta^\times} \frac{|f'|^2}{(1+|f|^2)^2} \, dA$$

with the RHS being the spherical area, under the inverse stereographic projections, covered by $f(\Delta^\times)$. This implies that ψ extends to a meromorphic function on the whole disk Δ.

Let $n = \operatorname{ord}_{z=0} \psi \in \mathbb{Z}$ and $\psi = z^n g$. Then $f = z^a g$ with $a = n + \theta/\pi$ and

$$u = c_1 + 2 \log \frac{|z|^{a-1} |ag + zg'|}{1 + |z^a g|^2}. \tag{3.6}$$

If $a = 0$ then $n = 0$ and $\theta = 0$ (since $0 \le \theta < \pi$). In this case $f = g = \psi$ is holomorphic at 0. So we may assume that $a \neq 0$.

For $\rho = 4\pi l$ with $l \in \mathbb{N}$, the asymptotic of u at $z = 0$ is given by

$$u(z) \sim 2l \log |z|.$$

On the other hand, by (3.6) the asymptotic is given by

$$u(z) \sim 2(|a| - 1) \log |z|. \tag{3.7}$$

In particular, $|a| = l + 1 \in \mathbb{N}$, which forces $\theta = 0$ because $0 \le \theta < \pi$. Moreover $f = z^{\pm(l+1)} g$ is meromorphic at $z = 0$. □

Now we look for the constraints. The first type of constraints are imposed by the double periodicity of the equation. By applying (3.3) to $f(z + \omega_1)$ and $f(z + \omega_2)$, we find S_1 and S_2 in $\mathrm{PSU}(1)$ with

$$\begin{aligned} f(z + \omega_1) &= S_1 f, \\ f(z + \omega_2) &= S_2 f. \end{aligned} \tag{3.8}$$

These relations also force that $S_1 S_2 = \pm S_2 S_1$ since $A \equiv -A$ in $\mathrm{PSU}(1)$.

After conjugating a matrix in $\mathrm{PSU}(1)$, we may and will assume that $S_1 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$. Let $S_2 = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix}$ then (3.8) becomes

$$\begin{aligned} f(z + \omega_1) &= e^{2i\theta} f(z), \\ f(z + \omega_2) &= S_2 f(z). \end{aligned} \tag{3.9}$$

Since $S_1 S_2 = \pm S_2 S_1$, a direct computation shows that there are three possibilities:

(1) $p = 0$ and $e^{i\theta} = \pm i$;
(2) $q = 0$; and
(3) $e^{i\theta} = \pm 1$.

In case (3) we get that S_1 is the identity. So by another conjugation we may assume that S_2 is diagonal and it is reduced to case (2).

Explicitly the two cases read as

Type I:

$$\begin{aligned} f(z + \omega_1) &= -f(z), \\ f(z + \omega_2) &= -\frac{\bar{q}^2}{f(z)}. \end{aligned} \tag{3.10}$$

Type II:

$$f(z+\omega_1) = e^{2i\theta_1} f(z),$$
$$f(z+\omega_2) = e^{2i\theta_2} f(z). \tag{3.11}$$

The second type of constraints are imposed by the Dirac singularity of (3.5) at 0. By the argument from (3.6) to (3.7), we have

Lemma 3.2. (1) *If $f(z)$ has a pole at $z_0 \equiv 0 \pmod{\omega_1, \omega_2}$, then the order $k = l + 1$.*

(2) *If $f(z) = a_0 + a_k z^k + \cdots$ is regular at $z_0 \equiv 0 \pmod{\omega_1, \omega_2}$ with $a_k \neq 0$ then $k = l + 1$.*

(3) *If $f(z)$ has a pole at $z_0 \not\equiv 0 \pmod{\omega_1, \omega_2}$, then the order is 1.*

(4) *If $f(z) = a_0 + a_k (z - z_0)^k + \cdots$ is regular at $z_0 \not\equiv 0 \pmod{\omega_1, \omega_2}$ with $a_k \neq 0$ then $k = 1$.*

In particular, modulo ω_1, ω_2, f has only simple zeros and poles outside 0.

4. The general structure of solutions

In the study of mean field equations with $\rho = 4\pi l$, $l \in \mathbb{N}$, the overall important object to be considered is the logarithmic derivative g of the developing map f:

$$g := (\log f)' = \frac{f'}{f}.$$

It is clear that any zero or pole of f contributes a simple pole of g and the order $k \in \mathbb{Z}$ is simply the residue.

Thus the zeros of g must occur at a regular point z_0 of f with $f(z_0) \neq 0$. By Lemma 3.2, if $z_0 \not\equiv 0$ then $f'(z_0) \neq 0$ and z_0 is not a zero of g. Hence we must have $z_0 \equiv 0$ and it contributes a zero of g of order l.

4.1. Type I: Topological solutions.

In Type I, we have

$$g(z+\omega_1) = g(z),$$
$$g(z+\omega_2) = -g(z). \tag{4.1}$$

Hence g is an elliptic function on $T' = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ of the form

$$g(z) = A \frac{\sigma^l(z)\sigma^l(z-\omega_2)}{\prod_{i=1}^{l} \sigma(z-p_i) \prod_{i=1}^{l} \sigma(z-q_i)} \tag{4.2}$$

for some constant A. The location of poles p_i's and q_i's are $2l$ distinct points in T' by Lemma 3.2. They are those (simple) zeros and poles of f which are constrained by

$$\sum p_i + \sum q_i = l\omega_2. \tag{4.3}$$

The function g is determined by $p_i, q_i \pmod{\omega_1, 2\omega_2}$ and (4.3).

Since $g(z+\omega_2) = -g(z)$, after some reordering we may assume that

$$q_i \equiv p_i + \omega_2 \pmod{\omega_1, 2\omega_2}. \tag{4.4}$$

(This gives a correspondences between zeros and poles of f.)
Combining with (4.3) we find

$$p_1 + \cdots + p_l \equiv \frac{m}{2}\omega_1 + n\omega_2 \pmod{\omega_1, 2\omega_2} \tag{4.5}$$

for $m, n \in \{0, 1\}$. This gives rise to the first equation of p_i's.

Lemma 4.1. *We may assume that* $p_1 + \cdots + p_l = \frac{1}{2}\omega_1$.

Proof. By modifying p_i's and q_i's by elements in $\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$, there are only four cases of (m, n) in (4.5) needs to be considered. Namely

$$(m, n) = \quad \text{(i) } (0, 0), \quad \text{(ii) } (1, 0), \quad \text{(iii) } (0, 1) \quad \text{and} \quad \text{(iv) } (1, 1).$$

During the proof we may assume that $q_i = p_i + \omega_2$ for $i = 1, \ldots, l-1$.
For (i), we may further set $q_l = p_l + \omega_2$. Then

$$g(z+\omega_2) = A\frac{\sigma^l(z+\omega_2)\sigma^l(z)}{\prod_{i=1}^l \sigma(z+\omega_2-p_i)\prod_{i=1}^l \sigma(z+\omega_2-q_i)} \tag{4.6}$$

By transforming $g(z+\omega_2)$ into $g(z)$ using the periodicity property (2.7) on T', we get an automorphic factor

$$\frac{(-1)^l \exp l\eta_2 z}{(-1)^l \exp \eta_2 \sum_{i=1}^l (z-p_i)} = 1.$$

That is $g(z+\omega_2) = g(z)$, which is the not the case we want.
For (ii), we may set

$$q_l = p_l + \omega_2 - \omega_1$$

to match the sum constraint (4.3). Again by transforming $g(z+\omega_2)$ as in (4.6) into $g(z)$ in (4.2) we get the automorphic factor

$$\frac{(-1)^l \exp l\eta_2 z}{(-1)^{l+1} \exp \eta_2 \sum_{i=1}^{l-1}(z-p_i) + (\eta_2-\eta_1)(z-p_l-\frac{1}{2}\omega_1) + \eta_1(z-p_l-\frac{1}{2}\omega_1)}$$

which is -1 since $\sum p_i = \frac{1}{2}\omega_1$. Hence $g(z+\omega_2) = -g(z)$ as desired.
For (iii), we may set

$$q_l = p_l - \omega_2.$$

By changing the role of p_l and q_l this is reduced to case (i).
Similarly for (iv) we may set

$$q_l = p_l - \omega_2 - \omega_1.$$

By changing the role of p_l and q_l this is reduced to case (iii). □

Remark 4.2. For convenience sometimes we would like to assume that p_i's are simple zeros of f while q_i's are simple poles of f. In doing so we may also need to consider the possibility that

$$p_1 + \cdots + p_l = \frac{\omega_1}{2} + \omega_2.$$

Alternatively we may switch the role of p_l and q_l and still use

$$p_1 + \cdots + p_l = \frac{\omega_1}{2}.$$

Other equations come from the residue consideration. Let

$$r_j = \frac{\sigma^l(p_j)\sigma^l(p_j - \omega_2)}{\prod_{i=1,\neq j}^{l} \sigma(p_j - p_i) \prod_{i=1}^{l} \sigma(p_j - q_i)}. \tag{4.7}$$

Then the residue of g at $z = p_j$ is Ar_j and we must have, under the arrangement that p_i's are (simple) zeros of f except perhaps p_l, that

$$r_1 = r_2 = \cdots = r_{l-1} = \pm r_l. \tag{4.8}$$

There are (at most) two sets of $l - 1$ independent equations from it. Together with (4.3) we obtain two systems of non-linear system of l equations in l variables.

Since this system is holomorphic in the variables $p_i \in T'$. The solution set actually defines an analytic subvariety $V \in (T')^n$ (which is in fact algebraic by Chow's theorem). It is expected that V consists of a finite number of points.

Naively the solution structure in this case is topological because the intersection number N of the l hypersurfaces defined by the l equations is independent of the shape of the torus T.

However more care needs to be taken since some of the solutions may lead to $p_i = p_j$ for some $i \neq j$ and need to be excluded. Futhermore we need to integrate

$$f = \exp \int g$$

to check whether (3.10) is satisfied. In fact $f(z + \omega_1) = -f(z)$ corresponds to the statement that

$$\int_{L_1} g(z)\, dz \equiv \pi i \pmod{2\pi i}. \tag{4.9}$$

Later we will check these technical issues through explicit determinations for small values of l up to $l = 3$. In all the solutions we found, there must be some $p_i = \frac{1}{2}\omega_1$ and the function g is odd. This will lead to (4.9) (c.f. §5.1) and so the solution is topological (independent of the shape of T). For general odd l we expect this to be true but we do not know how to prove it in the full generality.

4.2. **Type II: Scaling families and blow-up solutions.** In Type II, it follows from (3.11) that g is an elliptic function on T. And by the discussion at the beginning of this section g must take the form

$$g(z) = \frac{f'(z)}{f(z)} = A\,\frac{\sigma^l(z)}{\prod_{i=1}^l \sigma(z - p_i)} \tag{4.10}$$

with $\sum p_i = 0$.

As in Type I, there are other $l - 1$ equations arising from the residues. Let

$$r_j = \frac{\sigma^l(p_j)}{\prod_{i=1,\neq j}^l \sigma(p_j - p_i)} \tag{4.11}$$

so that the residue of g at $z = p_j$ is given by Ar_j. Since f has only simple zeros and poles outside 0, we have $Ar_j = \pm 1$ and this leads to equations

$$r_1 = \pm r_j, \qquad j = 2, \ldots, l. \tag{4.12}$$

At this point the solutions seem to be topological. But in fact they are not. Recall that

$$f(z) = f(0) \exp \int_0^z g(w)\,dw.$$

Lemma 4.3. *In order for f to verify (3.11), it is equivalent to require that the periods integrals are purely imaginary:*

$$\int_{L_i} g(z)\,dz \in i\,\mathbb{R}, \quad i = 1, 2. \tag{4.13}$$

Later we will see how this lemma for $\rho = 8\pi$ links to the critical points of Green functions via Lemma 2.4. It is known [7] that the structure (or even the number) of critical points depends heavily on the geometry of T.

Another characteristic feature for Type II is that any solution must exist in an one parameter scaling family of solutions.

To see this, notice that if f is a developing map of solution u then $e^\lambda f$ also satisfies (3.11) for any $\lambda \in \mathbb{R}$. In fact $e^\lambda f$ is a developing map of u_λ defined by

$$u_\lambda(z) = c_1 + \log \frac{e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2} \tag{4.14}$$

and it is clear that u_λ is a scaling family of solutions of (3.5).

To understand the quantitative behavior of u_λ as λ varies, we notice also that f must have zero in T. For otherwise (3.11) implies that $1/f$ is a bounded entire function on $\mathbb{C}$ which by Liouville's elementary theorem f reduces to a constant, which is absurd.

Let z_0 be a zero of f. We know that $z_0 \not\equiv 0$ and $f'(z_0) \neq 0$. Thus

$$u_\lambda(z_0) \sim 2\lambda \to \infty \quad \text{as} \quad \lambda \to \infty \tag{4.15}$$

while if $f(z) \neq 0$ then

$$u_\lambda(z) \sim -2\lambda \to -\infty \quad \text{as} \quad \lambda \to -\infty. \tag{4.16}$$

It is customary to call points like z_0 blow-up points.

We close this section by noticing an elementary consequence.

Theorem 4.4 (= Theorem 1.4). *For $\rho = 4\pi l$ with l being odd, there are no type II, i.e. blow-up, solutions to the mean field equation*

$$\triangle u + \rho e^u = \rho \delta_0 \quad \textit{in } T.$$

Proof. If there is a solution u with developing map f, then $g = f'/f$ is elliptic on T with residues at p_i, $i = 1, \ldots, l$, to be ± 1. Since l is odd, the sum of residues is non-zero which is a contradiction. $\square$

5. Results in [7] revisited: $\rho = 4\pi$ or 8π

To illustrate how the general principle outlined in the last section works in practice, we will work out in this section solutions of the mean field equations for $\rho = 4\pi$ and $\rho = 8\pi$. These results are contained in [7], but here we will treat the 8π case slightly differently without using the knowledge that blow-up points are critical points of Green functions. In fact this will follow from our current treatment.

5.1. **The case $\rho = 4\pi$.** In Type II, g is elliptic on T. Since $l = 1$, g has a simple zero at $z = 0$, hence also a simple pole at $z = p_1$. However there is no such elliptic functions (or using $p_1 = 0$ to get a contradiction).

In Type I, g is elliptic on T' and

$$g(z) = A \frac{\sigma(z)\sigma(z - \omega_2)}{\sigma(z - p_1)\sigma(z - q_1)} \tag{5.1}$$

with $p_1 + q_1 = \omega_2$. By Lemma 4.1, there is a unique solution of (p_1, q_1) up to order modulo $\omega_1, 2\omega_2$ given by

$$p_1 = \frac{\omega_1}{2}; \quad q_1 = -\frac{\omega_1}{2} + \omega_2. \tag{5.2}$$

It is easily checked, using (2.7), that $g(z + \omega_1) = g(z)$, $g(z + \omega_2) = -g(z)$ and

$$g(-z) = -g(z). \tag{5.3}$$

The residues of g at p_1 and q_1 are equal to Ar and $-Ar$ respectively, where

$$r = \frac{\sigma(\frac{1}{2}\omega_1)\sigma(\frac{1}{2}\omega_1 - \omega_2)}{\sigma(\omega_1 - \omega_2)}.$$

Let $A = 1/r$, then $f(z)$ is uniquely defined up to a factor $f(0)$. There is an unique choice of $f(0)$ up to a norm one factor such that $c := f(\omega_2)f(0)$ has $|c| = 1$. Thus by integrating $g(z + \omega_2) = -g(z)$ we get $f(z + \omega_2) = c/f(z)$.

By integrating $g(z+\omega_1) = g(z)$ we get $f(z+\omega_1) = c' f(z)$ where

$$c' = \frac{f(\omega_1)}{f(0)} = \exp \int_0^{\omega_1} g(z)\,dz.$$

To evaluate the period integral, notice that the following anti-symmetric property at $\omega_1/2$ holds:

$$g\Big(\frac{\omega_1}{2} + u\Big) = g\Big(-\frac{\omega_1}{2} - u\Big) = -g\Big(\frac{\omega_1}{2} - u\Big).$$

By using the Cauchy principal value integral and the fact that the residue of g at $\frac{1}{2}\omega_1$ is ± 1, we get

$$\int_0^{\omega_1} g(z)\,dz = 0 \pm \frac{1}{2} \times 2\pi i = \pm \pi i \tag{5.4}$$

and so $c' = -1$.

Thus f gives rise to a topological solution of (3.5) for $\rho = 4\pi$. The developing map for the other choice $A = -1/r$ is $1/f(z)$ which leads to the same solution. Since equation (3.5) is invariant under $z \mapsto -z$, the unique solution is necessarily even. This can also be seen directly from our construction.

5.2. **The case $\rho = 8\pi$.** In Type I, let $\wp(z)$, $\zeta(z)$ and $\sigma(z)$ be the Weierstrass elliptic functions on T'. Recall that for $\tilde{\omega}_1 = \omega_1$, $\tilde{\omega}_2 = 2\omega_2$ and $\tilde{\omega}_3 = \omega_1 + 2\omega_2$,

$$\sigma(z \pm \tilde{\omega}_i) = -e^{\pm \eta_i (z \pm \frac{1}{2}\tilde{\omega}_i)} \sigma(z). \tag{5.5}$$

Now $l = 2$ and (4.10) reads as

$$g(z) = A \frac{\sigma^2(z)\sigma^2(z-\omega_2)}{\sigma(z-a)\sigma(z-b)\sigma(z-c)\sigma(z-d)} \tag{5.6}$$

with four distinct simple poles such that $a+b+c+d = 2\omega_2$. We will show that such a function $g(z)$ does not exist.

By Lemma 4.1, we have $(a,b,c,d) = (a, -a + \frac{1}{2}\omega_1, a + \omega_2, -a - \frac{1}{2}\omega_1 + \omega_2)$.

The residues of g at a, b, c and d are equal to $-Ar$, Ar', Ar and $-Ar'$ respectively, where

$$r = \frac{\sigma^2(a+\omega_2)\sigma^2(a)}{\sigma(\omega_2)\sigma(2a - \frac{1}{2}\omega_1 + \omega_2)\sigma(2a + \frac{1}{2}\omega_1)}$$

and

$$r' = \frac{\sigma^2(a - \frac{1}{2}\omega_1)\sigma^2(a - \frac{1}{2}\omega_1 + \omega_2)}{\sigma(2a - \frac{1}{2}\omega_1)\sigma(2a - \frac{1}{2}\omega_1 + \omega_2)\sigma(\omega_1 - \omega_2)}.$$

We must have $r = \pm r'$. Using (5.5), this is equivalent to

$$\frac{\sigma^2(a+\omega_2)\sigma^2(a)}{\sigma^2(a + \frac{1}{2}\omega_1)\sigma^2(a - \frac{1}{2}\omega_1 + \omega_2)} = \mp e^{\eta_1(-\frac{1}{2}\omega_1 + \omega_2)}. \tag{5.7}$$

To solve a from this equation, by substituting $y = \frac{1}{2}\tilde{\omega}_i$ into the addition formula (2.13) and using (5.5), we get

$$\wp(z) - e_i = \frac{\sigma^2(z + \frac{1}{2}\tilde{\omega}_i)}{\sigma^2(z)\sigma^2(\frac{1}{2}\tilde{\omega}_i)} e^{-\eta_i z}. \tag{5.8}$$

With (5.8), the "+" case in equation (5.7) simplifies to

$$\wp\Big(a - \frac{\omega_1}{2} + \omega_2\Big) - e_1 = \wp(a) - e_1.$$

That is, $2a \equiv \frac{1}{2}\omega_1 - \omega_2$. But this implies that $b \equiv c$, a contradiction.

Similarly, the "$-$" case in (5.7) simplifies to (using the Legendre relation)

$$\wp\Big(a + \frac{\omega_1}{2}\Big) - e_3 = \wp(a) - e_3.$$

That is, $2a + \frac{1}{2}\omega_1 \equiv 0$. This leads to $c \equiv d$, which is again a contradiction.

In Type II, g is elliptic on T and we have

$$g(z) = A\frac{\sigma^2(z)}{\sigma(z - z_0)\sigma(z + z_0)} \tag{5.9}$$

where f has a zero at z_0 and a pole at $-z_0$. In particular $z_0 \neq -z_0$ in T and we conclude that $z_0 \neq \omega_k/2$ for any $k \in \{1, 2, 3\}$.

Now the residue of g at z_0 is 1, hence

$$A\frac{\sigma^2(z_0)}{\sigma(2z_0)} = 1 \Longrightarrow A = \frac{\sigma(2z_0)}{\sigma^2(z_0)}. \tag{5.10}$$

Then we may transform g into Weierstrass functions as

$$\begin{aligned} g(z) &= \frac{\sigma(2z_0)}{\sigma^2(z_0)}\frac{\sigma^2(z)}{\sigma(z - z_0)\sigma(z + z_0)} \\ &= \frac{\sigma(2z_0)}{\sigma^4(z_0)}\frac{\sigma^2(z)\sigma^2(z_0)}{\sigma(z - z_0)\sigma(z + z_0)} = \frac{\wp'(z_0)}{\wp(z) - \wp(z_0)} \end{aligned} \tag{5.11}$$

where the addition law (2.13) and the duplication formula

$$\wp'(z) = -\sigma(2z)/\sigma^4(z)$$

are used.

Then

$$f(z) = f(0)\int_0^z g(\xi)\, d\xi = f(0)\int_0^z \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi$$

is a developing map for a solution precisely when

$$\int_{L_i} \frac{\wp'(z_0)}{\wp(\xi) - \wp(z_0)}\, d\xi \in i\,\mathbb{R}, \quad i = 1, 2.$$

By Proposition 2.3 and Lemma 2.4, this is equivalent to that z_0 is a critical point of the Green function $G(z)$ on T.

Together with §4.2, we have recovered the following known result which was first proved in [2] using PDE methods.

Corollary 5.1. *For $\rho = 8\pi$, the blow-up points of solutions to the mean field equation $\triangle u + \rho e^u = \rho\delta_0$ on a torus T are precisely those non-half period critical points of the Green function G.*

6. THE CASE $\rho = 4\pi l$ WITH $l = 3$

By Theorem 4.4, there are no Type II solutions.

6.1. **Constructing g.** For Type I solutions, $g(z) = (\log f(z))' = f'(z)/f(z)$ is elliptic on $T' = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ and we may write

$$g(z) = A\frac{\sigma^3(z)\sigma^3(z-\omega_2)}{\sigma(z-a)\sigma(z-b)\sigma(z-c)\sigma(z-d)\sigma(z-e)\sigma(z-f)}$$

where a, b, c are simple zeros and d, e, f are simple poles of $f(z)$.

From

$$g(z+\omega_2) = -g(z),$$

Lemma 4.1 gives rise to two possibilities: Let a, b be free variables. Then

$$d = a + \omega_2, \quad e = b + \omega_2$$

and either

$$c = -a - b + \frac{\omega_1}{2}, \quad f = -a - b - \frac{\omega_1}{2} + \omega_2 \tag{6.1}$$

or

$$c = -a - b + \frac{\omega_1}{2} + \omega_2, \quad f = -a - b - \frac{\omega_1}{2}. \tag{6.2}$$

The second case can be included into the first one if we change the role of c and f, namely to require that c is a pole and f is a zero of $f(z)$.

We fist consider the former case (6.1). Denote the residues of $g(z)$ at a, b, c by Ar_1, Ar_2 and Ar_3 respectively. Then $Ar_1 = Ar_2 = Ar_3 = 1$. Here

$$r_1 = -\frac{\sigma^3(a)\sigma^3(a-\omega_2)}{\sigma(a-b)\sigma(2a+b-\frac{\omega_1}{2})\sigma(\omega_2)\sigma(a-b-\omega_2)\sigma(2a+b+\frac{\omega_1}{2}-\omega_2)},$$

$$r_2 = -\frac{\sigma^3(b)\sigma^3(b-\omega_2)}{\sigma(b-a)\sigma(2b+a-\frac{\omega_1}{2})\sigma(\omega_2)\sigma(b-a-\omega_2)\sigma(2b+a+\frac{\omega_1}{2}-\omega_2)}$$

and $r_3 =$

$$\frac{\sigma^3(a+b-\frac{\omega_1}{2})\sigma^3(a+b-\frac{\omega_1}{2}+\omega_2)e^{-\eta_1(\frac{\omega_1}{2}-\omega_2)}}{\sigma(2a+b-\frac{\omega_1}{2})\sigma(2b+a-\frac{\omega_1}{2})\sigma(2a+b-\frac{\omega_1}{2}+\omega_2)\sigma(2a+b-\frac{\omega_1}{2}+\omega_2)\sigma(\omega_2)}.$$

We consider the equation $r_1 = r_2$, which simplifies to

$$\begin{aligned} &\frac{\sigma^3(a)\sigma^3(a-\omega_2)}{\sigma(2a+b-\frac{\omega_1}{2})\sigma(2a+b+\frac{\omega_1}{2}-\omega_2)} \\ &\qquad = -e^{\eta_2(a-b)}\frac{\sigma^3(b)\sigma^3(b-\omega_2)}{\sigma(2b+a-\frac{\omega_1}{2})\sigma(2b+a+\frac{\omega_1}{2}-\omega_2)}. \end{aligned} \tag{6.3}$$

The obvious candidate of solutions would be those coming from equating the triple powers of sigma functions, namely to force that $b = \pm a$ or

$b = \pm(a - \omega_2)$. By our assumption $b \neq a$ and also $b \neq a - \omega_2$ (otherwise r_1 is not defined). So the candidates are $b = -a$ and $b = -a + \omega_2$.

Indeed at most one choice will do since $g(z + \omega_2) = -g(z)$. It is easily checked, using (5.5), that for $b = -a$ the LHS = RHS in (6.3) while (hence) for $b = -a + \omega_2$ the LHS = −RHS.

Thus $b = -a$ and $c = \frac{1}{2}\omega_1$. Then $g(z) =$

$$A \frac{\sigma^3(z)\sigma^3(z - \omega_2)}{\sigma(z - a)\sigma(z + a)\sigma(z - \frac{\omega_1}{2})\sigma(z - a - \omega_2)\sigma(z + a - \omega_2)\sigma(z + \frac{\omega_1}{2} - \omega_2)}.$$

It is very important to notice, and easy to verify, that

$$g(-z) = -g(z). \tag{6.4}$$

Indeed the transformation factor from $g(-z)$ to $g(z)$ is

$$\frac{(-1)^3 \exp 3\eta_2 z}{\exp(\eta_1 z + \eta_2(z + a) + \eta_2(z - a) + (-\eta_1 + \eta_2)z)} = -1.$$

We proceed to solve $r_1 = r_3$ under $b = -a$, $c = \frac{1}{2}\omega_1$. (Another possibility that $r_1 = -r_3$ will be treated later.) The equation $r_1 = r_3$ reads as

$$\frac{\sigma^3(a)\sigma^3(a - \omega_2)}{\sigma(2a)\sigma(2a - \omega_2)} = -e^{-\eta_1(\frac{\omega_1}{2} - \omega_2) + (\eta_1 - \eta_2)a} \frac{\sigma^3(\frac{\omega_1}{2})\sigma^3(\frac{\omega_1}{2} - \omega_2)}{\sigma(a + \frac{\omega_1}{2})\sigma(a + \frac{\omega_1}{2} - \omega_2)}. \tag{6.5}$$

Recall the duplication formula

$$\frac{\sigma(2a)\sigma(w_1)\sigma(w_2)\sigma(w_3)}{2\sigma(a)\sigma(a + w_1)\sigma(a + w_2)\sigma(a + w_3)} = 1 \tag{6.6}$$

for any three half periods w_i with $w_1 + w_2 + w_3 = 0$ (c.f. [10]).

With this in mind, we may rewrite (6.5) as

$$\begin{aligned} \frac{\sigma^2(a)\sigma^2(a - \omega_2)}{\sigma(2a - \omega_2)\sigma(\omega_2)} &= -e^{-\eta_1(\frac{\omega_1}{2} - \omega_2) + (\eta_1 - \eta_2)a} \times \\ &\frac{\sigma(2a)\sigma(\frac{\omega_1}{2})\sigma(\frac{\omega_1}{2} - \omega_2)\sigma(\omega_2)}{\sigma(a)\sigma(a - \omega_2)\sigma(a + \frac{\omega_1}{2})\sigma(a + \frac{\omega_1}{2} - \omega_2)} \cdot \frac{\sigma^2(\frac{\omega_1}{2})\sigma^2(\frac{\omega_1}{2} - \omega_2)}{\sigma^2(\omega_2)}. \end{aligned} \tag{6.7}$$

Using the addition law (2.13) and its consequence (5.8)

$$\wp(z) - e_i = \frac{\sigma^2(z + \frac{\tilde{\omega}_i}{2})}{\sigma^2(z)\sigma^2(\frac{\tilde{\omega}_i}{2})} e^{-\eta_i z},$$

together with (6.6), we simplify (6.7) to

$$\frac{-1}{\wp(a) - \wp(a - \omega_2)} = -2 \frac{1}{\wp(\omega_2 - \frac{\omega_1}{2}) - \wp(\frac{\omega_1}{2})}.$$

That is

$$\wp(a) - \wp(a - \omega_2) = \frac{1}{2}(e_3 - e_1). \tag{6.8}$$

The LHS has 2 poles at $a = 0$ and $a = \omega_2$ with total order 4, hence the equation have 4 zeros $a = \pm p$ and $a = \pm p'$. Notice that a is not a half period. To see that $p \neq p'$, i.e. they are simple zeros, we use the half period formula (c.f. [10]):

$$\wp\left(a + \frac{\tilde{\omega}_2}{2}\right) = e_2 + \frac{(e_1 - e_2)(e_3 - e_2)}{\wp(a) - e_2}. \tag{6.9}$$

By substituting this into (6.8), a simple calculation gives

$$\wp(a) = e_2 + \frac{e_3 - e_1}{4} \pm \frac{1}{4}\sqrt{(e_3 - e_1)^2 + 16(e_1 - e_2)(e_3 - e_2)}. \tag{6.10}$$

It is then clear that there are 4 in-equivalent solutions. Since $b = -a$, these solutions give rise to 2 independent solutions of g.

To complete our discussion we still needs to consider the case $r_1 = -r_3$. In exactly the same calculation with sign changed, we arrive at the equation

$$\wp(a) - \wp(a - \omega_2) = -\frac{1}{2}(e_3 - e_1). \tag{6.11}$$

The important observation is that $a \mapsto a + \omega_2$ exchanges equations (6.8) and (6.11). Hence the solutions of (6.11) is given by

$$\pm p + \omega_2 \quad \text{and} \quad \pm p' + \omega_2.$$

By our general discussion of the structure of solutions in §4.1, this shows that in fact $r_1 = -r_3$ gives rise to the same set of solutions as $r_1 = r_3$.

To be concrete, if in $r_1 = r_3$ there is a solution of (a, b, c, d, e, f) to be

$$\left(p, -p, \frac{\omega_1}{2}, p + \omega_2, -p + \omega_2, \omega_2\right), \tag{6.12}$$

then a corresponding solution (a', b', c', d', e', f') in $r_1 = -r_3$ is

$$\left(p + \omega_2, -(p - \omega_2), \frac{\omega_1}{2}, p + 2\omega_2, -p, \omega_2\right). \tag{6.13}$$

These two sets are equivalent modulo $\omega_1, 2\omega_2$ hence they determine the same function g up to the choice of constants A and A' in both cases. By definition of the constant the choice is only up to a sign.

Indeed, in (6.12) a, b, c are viewed as simple zeros of f. They correspond to d', e', c' which according to Remark 4.2 are viewed as simple poles of, perhaps another, $\tilde{f}$. A closer look shows that $A' = -A$ and hence that

$$\tilde{f} = \frac{1}{f}.$$

Clearly $\tilde{f}$ gives rise to the same u as f.

6.2. **Constructing f and hence the solution u.** This step is similar to the case $\rho = 4\pi$ in §5.1. Let $A = 1/r_i$, then

$$f(z) := f(0) \exp \int_0^z g(w)\, dw$$

is uniquely defined up to a factor $f(0)$. There is an unique choice of $f(0)$ up to a norm one factor such that $c := f(\omega_2) f(0)$ has $|c| = 1$. Thus by integrating $g(z + \omega_2) = -g(z)$ we get $f(z + \omega_2) = c/f(z)$.

By integrating $g(z+\omega_1) = g(z)$ we get $f(z+\omega_1) = c' f(z)$ where

$$c' = \exp \int_{L_1} g(z)\, dz.$$

We would like to show that $c' = -1$. Notice that since g is odd (c.f. 6.4), the following anti-symmetric property at $\frac{1}{2}\omega_1$ still holds:

$$g\Big(\frac{\omega_1}{2} + u\Big) = g\Big(-\frac{\omega_1}{2} - u\Big) = -g\Big(\frac{\omega_1}{2} - u\Big).$$

By using the Cauchy principal value integral and the fact that the residue of g at $\frac{1}{2}\omega_1$ is ± 1, we get

$$\int_{L_1} g(z)\, dz = 0 \pm \frac{1}{2} \times 2\pi i = \pm \pi i \tag{6.14}$$

and so $c' = -1$. In fact it follows that the integral of g along any line $a + L_1$ is congruent to πi modulo $2\pi i$ since the residue at each pole is ± 1.

Thus f gives rise to a topological solution of the mean field equation (3.5) for $\rho = 12\pi$. The solutions as constructed, though not unique, are again even functions. Indeed $g(-z) = -g(z)$ implies that $f(-z) = f(z)$ and then $u(-z) = u(z)$.

It becomes clear that the key property is the oddness of g. Whether or not this is generally true for $\rho = 4\pi l$ with $l = 2k+1$ is a guiding problem for future studies.

We end this paper by raising the following

Conjecture 6.1. *For $\rho = 4\pi l$ with odd l, any type I solution constructed by solving the (holomorphic) residue equations contribute positively in the degree counting. Moreover all solutions are obtained in the way.*

Since the Leray-Schauder degree is 2 for $\rho = 12\pi$. The validity of this conjecture would imply that the two solutions constructed in this section are all the solutions.

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