ON THE MINIMALITY OF EXTRA CRITICAL POINTS OF GREEN FUNCTIONS ON FLAT TORI

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ABSTRACT. This is a sequel to [8, 2] to study the *geometry of flat tori*. In [8], we showed that the solvability of the mean field equation (MFE)

$$\triangle u + e^u = \rho\,\delta_0$$

on a flat torus E_τ with $\rho = 8\pi$ is equivalent to the existence of extra pair of critical points $\pm p$ of the Green function G. And such a pair, if exists, is unique. It was also announced there that G actually attains its minimum at $\pm p$. Here our first main result is to confirm this statement by way of the variational form of the MFE. It implies that the solution u is a minimizer of the corresponding non-linear functional $J_{8\pi}(u)$ (c.f. (1.1)), hence settles the existence problem of minimizers posed in [12].

We also prove the uniqueness of solution to the MFE when $0 < \rho < 8\pi$ and get the exact counting result of the number of solutions in terms of the number of critical points of G when ρ is close to 8π. This allows us to analyze the bifurcation structure of the MFE when ρ crosses 8π.

CONTENTS

0. INTRODUCTION

Consider the flat torus $E = E_\tau = \mathbb{C}/\Lambda_\tau$, $\tau = a + bi$, $b > 0$ and $\Lambda = \Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1 = 1$ and $\omega_2 = \tau$. Let G be the Green function on E:

$$\begin{cases} -\triangle G = \delta_0 - \dfrac{1}{|E|} & \text{on } E, \\ \int_E G = 0, \end{cases} \tag{0.1}$$

where δ_0 is the Dirac measure at the lattice point $0 \in E$. We continue our study, initiated in [8, 2, 6], on the critical points of G:

$$\nabla G(z) = 0. \tag{0.2}$$

Since G is an even function on E, all the half-periods $\frac{\omega_k}{2}$ are critical points of G. A critical point p is called a *non-trivial critical point of G if p is not one of the three half-periods*. Clearly, non-trivial critical points appear in pair $\pm p$. It is natural to ask: *how many pairs of non-trivial critical points might G have*? This has been answered completely in our previous paper [8]:

Theorem A. *For any $\tau \in \mathbb{H}$, the Green function $G(z;\tau)$ on the flat torus E_τ has at most one pair of non-trivial critical points.*

Thus G has either 3 or 5 critical points. Following [8] we denote by Ω_3 (resp. Ω_5) the subset of the moduli $\mathcal{M}_1 = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ where $G(z;\tau)$ on the flat torus E_τ has exactly 3 (resp. 5) critical points. See [8, 7] for the actual shape of the (simply connected) domain Ω_5.

What is the nature of those extra critical points? We answer it in the following theorem, which had been announced in [8, §1 Theorem A]:

Theorem 0.1. *Suppose that the pair of non-trivial critical points $\{\pm p\}$ of G exists, then $\pm p$ are the minimal points of G.*

We will present a proof of it in §1 based on the mean field equation

$$\triangle u + e^u = 8\pi\,\delta_0 \quad \text{on } E. \tag{0.3}$$

In fact our proof shows that any solution to (0.3) must be a minimizer of the non-linear functional

$$J_{8\pi}(u) = \frac{1}{2}\int_E |\nabla u|^2 - 8\pi \log \int_E e^{-8\pi G + u}$$

on $u \in H^1(E) \cap \{u \mid \int_E u = 0\}$. This completely solves the existence problem on minimizers raised in [12] when the two vortex points collapse into one.

One important application of Theorem 0.1 is the following result:

Corollary 0.2. *Suppose that the Green function G has non-trivial critical points, then all the three half periods are saddle points of G. That is, the Hessian of G is non-positive:* $\det D^2 G(\frac{\omega_k}{2}) \le 0$ *for $k = 1, 2, 3$.*

Remark 0.3. Based on Corollary 0.2, a stronger result is proved in [7]. Namely $\det D^2 G(\frac{\omega_k}{2}) < 0$ for $k = 1, 2, 3$ if G satisfies the hypothesis of Corollary 0.2.

From the Weierstrass elliptic curve model $y^2 = 4x^3 - g_2 x - g_3$ of E_τ, we know that the half periods $E_\tau[2]$ are precisely the branch points of the map $x = \wp(z) : E_\tau \to \mathbb{P}^1$. A quantity $D(q)$ defined at any branch point is strongly related to the geometry of E_τ at q. In [1, 6] it was proved that if u_k is a bubbling sequence of solutions to (0.3) with $\rho = \rho_k \to 8\pi$ (as $k \to \infty$), $\rho_k \ne 8\pi$ for large k, and with q the blow-up point, then q must be a half period point. In fact, asymptotically

$$\rho_k - 8\pi = (D(q) + o(1))e^{-\lambda_k} \tag{0.4}$$

where $\lambda_k = \max_{E_\tau} u_k$ and

$$D(q) := \int_{E_\tau} \frac{h(z)e^{8\pi(\tilde{G}(z,q)-\phi(q))} - h(q)}{|z-q|^4} - \int_{E_\tau^c} \frac{h(q)}{|z-q|^4}.$$

Here $h(z) = e^{-8\pi G(z)}$, $\tilde{G}(z,q)$ is the regular part of the Green function, and $\phi(q) = \tilde{G}(q,q)$. See §2 for more details. $D(q)$ plays an important role in the construction of bubbling solutions to (0.3), as well as in other non-linear PDEs, with $\rho_k \to 8\pi$. The sign of $D(q)$ determines the direction where the bubbling may take place, namely $\rho_k < 8\pi$ or $\rho_k > 8\pi$. If q is a not half-period critical point then $D(q)$ is still defined. But then $D(q) = 0$ since $\rho_k = 8\pi$ for all k (large).

In general it is difficult to compute $D(q)$ for a given torus. Nevertheless we will prove the following result in §2:

Theorem 0.4. *For any half period $q \in E_\tau$, $\tau = a + bi$, we have*

$$D(q) = -4\pi^2 b e^{-8\pi G(q)} \det D^2 G(q). \tag{0.5}$$

By Remark 0.3, we have $D(q) > 0$ if q is a saddle point. In particular if $\tau \in \Omega_5$ then $D(q) > 0$ for all half-periods. For any $\tau \in \mathbb{H}$, $D(q) \le 0$ if and only if q is the minimal point.

Combining with a recent technique in analyzing uniqueness of blow-up solutions [11], we will be able to classify all solutions to (0.3) for ρ in the range $(0, 8\pi + \epsilon_0)$ for some $\epsilon_0 > 0$:

Theorem 0.5. *For any torus E_τ, there is a small number $\epsilon_0 > 0$ such that*

(i) *If $\tau \in \Omega_3$ then (0.3) has only one solution for $\rho < 8\pi$, no solution for $\rho = 8\pi$, and two solutions for $8\pi < \rho < 8\pi + \epsilon_0$.*
(ii) *If $\tau \in \Omega_5$ then (0.3) has only one solution for $\rho < 8\pi$, infinitely many solutions for $\rho = 8\pi$, and four solutions for $8\pi < \rho < 8\pi + \epsilon_0$.*

In particular, the topological Leray–Schauder degree d_ρ, which is 2 for $\rho \in (8\pi, 16\pi)$ [3, 4, 5, 6], does not reflect the actual number of solutions. The proof is presented in §3, which relies also on the theory of Lamé equations in [2] accompanied with (0.3) as well as the blow-up analysis in [4, 11].

Remark 0.6. In [9] (see also [2]), we proved that (0.3) with $\rho = 12\pi$ has exactly two solutions on E_τ for $\tau \ne e^{\pi i/3}$. By Theorem 0.5, we see that when $\tau \in \Omega_5$ the bifurcation diagram of (0.3) is complicate for ρ ranging from 8π to 12π. It is a natural question to ask if (0.3) has exactly two solutions for $\rho \in (8\pi, 16\pi)$ when $\tau \in \Omega_3$. Theorem 0.5 also reflects the difficulty in the study the corresponding Lamé equations for the case $n \notin \frac{1}{2}\mathbb{N}$.

1. On the minimality of extra critical points

Theorem 1.1. *Let p be a critical point of G which is not a half period point, then p is a minimal point of G.*

Proof. Consider the even, normalized, L^1_2 Sobolev space

$$H^1_{ev}(E) = \{u \in H^1(E) \mid u(-z) = u(z), \int_E u = 0\}$$

and the non-linear functional

$$J_\rho(u) = \frac{1}{2}\int_E |\nabla u|^2 - \rho \log \int_E e^{-\rho G + u}, \qquad u \in H^1_{ev}(E). \tag{1.1}$$

It is well known that, as a consequence of the Moser–Trudinger inequality, J_ρ attains its minimum for $\rho < 8\pi$. Let v_ρ be a minimizer of J_ρ. Then v_ρ is an even solution of

$$\triangle v + \rho\Big(\frac{e^{-\rho G + v}}{\int_E e^{-\rho G + v}} - \frac{1}{|E|}\Big) = 0 \quad \text{in } E.$$

By the result of [8], when $\rho \to 8\pi$, v_ρ converges to a smooth function v which satisfies

$$\triangle v + 8\pi\Big(\frac{e^{-8\pi G + v}}{\int_E e^{-8\pi G + v}} - \frac{1}{|E|}\Big) = 0 \quad \text{in } E. \tag{1.2}$$

It is then obvious that

$$u(z) = -8\pi G(z) + v(z) - \log \int_E e^{-8\pi G + v}$$

is an even solution to the Liouville equation

$$\triangle u + 8\pi e^u = 8\pi \delta_0 \quad \text{in } E.$$

Since

$$J_\rho(v_\rho) = \inf_{\varphi \in H^1_{ev}} J_\rho(\varphi),$$

we have

$$J_{8\pi}(v) = \inf_{\varphi \in H^1_{ev}} J_{8\pi}(\varphi).$$

Let f be the developing map of u, that is,

$$u(z) = \log \frac{8\pi |f'(z)|^2}{(1 + |f(z)|^2)^2} \quad \text{for } z \in E.$$

As before, for $\lambda \in \mathbb{R}$ we define u^λ and v^λ by

$$u^\lambda(z) := \log \frac{8\pi e^{2\lambda} |f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2} =: 8\pi G(z) + v^\lambda(z) + c^\lambda, \tag{1.3}$$

where the constant c^λ is chosen so that $\int_E v^\lambda = 0$. Thus v^λ is also a solution to (1.2) and $v^\lambda(z)$ blows up at $z = p$ as $\lambda \to +\infty$ (i.e. p is a zero of f).

Next we would like to compute $J_{8\pi}(v^\lambda)$. By differentiation with respect to λ, we have by (1.2)

$$\begin{aligned}\frac{d}{d\lambda}J_{8\pi}(v^\lambda) &= \int_E \nabla v^\lambda \cdot \nabla\Big(\frac{\partial v^\lambda}{\partial\lambda}\Big) - 8\pi\frac{\int_E e^{-8\pi G+v^\lambda}\frac{\partial v^\lambda}{\partial\lambda}}{\int_E e^{-8\pi G+v^\lambda}} \\ &= -\int_E (\triangle v^\lambda)\frac{\partial v^\lambda}{\partial\lambda} - 8\pi\frac{\int_E e^{-8\pi G+v^\lambda}\frac{\partial v^\lambda}{\partial\lambda}}{\int_E e^{-8\pi G+v^\lambda}} \\ &= -\frac{8\pi}{|E|}\int_E \frac{\partial v^\lambda}{\partial\lambda} = 0.\end{aligned}$$

That is, $J_{8\pi}(v^\lambda)$ is independent of λ. In particular,

$$\lim_{\lambda\to+\infty} J_{8\pi}(v^\lambda) = \inf_{\varphi\in H^1_{ev}} J_{8\pi}(\varphi). \tag{1.4}$$

Using (1.4), we shall obtain an upper bound of $\lim J_{8\pi}(v^\lambda)$ by a choice of suitable test function φ_ϵ.

We fix a half period point $q \in E$ and small $\delta > 0$. For any $\epsilon > 0$ we define

$$\varphi_\epsilon(z) = \begin{cases} 2\log\dfrac{\epsilon^2/\delta^2+1}{\epsilon^2+|z-q|^2} + 8\pi\tilde{G}(z,q), & \text{if } z\in B_\delta(q), \\ 8\pi G(z,q), & \text{if } z \in E\backslash B_\delta(q), \end{cases}$$

where

$$\tilde{G}(z,q) = G(z-q) + \frac{1}{2\pi}\log|z-q|$$

is the regular part of $G(z,q)$ which is defined on $z \in T(q)$, the fundamental domain of E centered at q. Notice that the above two expressions for $\varphi_\epsilon(z)$ coincide when $|z-q| = \delta$. Since $\tilde{G}(z,q)$ depends only on $w = z-q$, we also denote $\tilde{G}(z,q) = \tilde{G}(z-q) = \tilde{G}(w)$, which is defined on the fundamental domain $T(0)$ centered at 0.

Obviously φ_ϵ is an even function. Since $\int_E G = 0$, direct integration gives

$$c_\epsilon := \frac{1}{|E|}\int_E \varphi_\epsilon = \frac{2}{|E|}\int_{B_\delta(q)} \log\frac{(\epsilon^2/\delta^2+1)|z-q|^2}{\epsilon^2+|z-q|^2} = O(\epsilon^2\log\epsilon), \tag{1.5}$$

where the notation O is with respect to the limit $\epsilon \to 0$. Thus $\varphi_\epsilon - c_\epsilon \in H^1_{ev}(E)$ and

$$J_{8\pi}(\varphi_\epsilon - c_\epsilon) = \frac{1}{2}\int_E |\nabla\varphi_\epsilon|^2 - 8\pi\log\int_E e^{-8\pi G+\varphi_\epsilon} + O(\epsilon^2\log\epsilon).$$

We will estimate the energy term and the non-linear term separately.

By Green's theorem, we have for $w = z - q$,

$$\begin{aligned}
&\int_E |\nabla \varphi_\epsilon|^2 = \int_{B_\delta(q)} |\nabla \varphi_\epsilon|^2 + (8\pi)^2 \int_{E\setminus B_\delta(q)} |\nabla G(z-q)|^2 \\
&= \int_{B_\delta(0)} \frac{16|w|^2}{(\epsilon^2+|w|^2)^2} \\
&\quad - 32\pi \int_{B_\delta(0)} \log\frac{1}{\epsilon^2+|w|^2} \triangle \tilde{G}(w) + 32\pi \int_{\partial B_\delta(0)} \log\frac{1}{\epsilon^2+|w|^2}\frac{\partial \tilde{G}(w)}{\partial \nu} \\
&\quad - (8\pi)^2 \int_{B_\delta(0)} \tilde{G}\triangle\tilde{G} + (8\pi)^2 \int_{\partial B_\delta(0)} \tilde{G}\frac{\partial \tilde{G}}{\partial \nu} \\
&\quad - (8\pi)^2 \int_{E\setminus B_\delta(0)} G\triangle G - (8\pi)^2 \int_{\partial B_\delta(0)} G\frac{\partial G}{\partial \nu}.
\end{aligned}$$

To estimate these terms, we first notice that (for $\delta > 0$ fixed)

$$\begin{aligned}
\int_{B_\delta(0)} \frac{16|w|^2}{(\epsilon^2+|w|^2)^2} &= 16\pi\log(1+\delta^2/\epsilon^2) - 16\pi\delta^2/(\epsilon^2+\delta^2) \\
&= 16\pi(\log(1+\delta^2/\epsilon^2) - 1) + O(\epsilon^2), \\
\int_{B_\delta(0)} \log\frac{1}{\epsilon^2+|w|^2} &= O(\epsilon^2\log\epsilon) + O(\delta).
\end{aligned} \tag{1.6}$$

Since $\triangle G = \delta_0 - 1/|E|$, $\triangle\tilde{G} = -1/|E|$, and $\int_E G = 0$, it is easy to see that each of three integrals involving G or $\tilde{G}$ is $O(\delta)$ and all boundary terms are $O(\delta)$ except

$$\frac{32\pi}{\delta}\int_{\partial B_\delta(0)} G = 32\pi\big(-\log\delta + 2\pi\gamma\big) + O(\delta), \tag{1.7}$$

where $\gamma = \tilde{G}(0) = \tilde{G}(q,q)$ is a constant independent of q.

Next we compute the non-linear term.

Since both $\nabla G(q) = 0$ and $\nabla\tilde{G}(z,q)|_{z=q} = \nabla\tilde{G}(0) = 0$, we have

$$\tilde{G}(z,q) - G(z) = \gamma - G(q) + O(|z-q|^2) \tag{1.8}$$

and

$$\begin{aligned}
\int_{B_\delta(q)} e^{-8\pi G(z)+\varphi_\epsilon(z)} &= e^{8\pi(\gamma - G(q))}\int_{B_\delta(0)} \frac{(\epsilon^2/\delta^2+1)^2}{(\epsilon^2+|w|^2)^2} + O(\epsilon^2\log\epsilon) \\
&= e^{8\pi(\gamma-G(q))}\Big(\frac{\pi}{\epsilon^2} - \frac{\pi}{\delta^2+\epsilon^2}\Big) + O(\epsilon^2\log\epsilon).
\end{aligned} \tag{1.9}$$

On $E\backslash B_\delta(q)$, by (1.8) and direct estimate we have

$$\begin{aligned}\int_{E\backslash B_\delta(q)} e^{-8\pi G+\varphi_\epsilon} &= \int_{E\backslash B_\delta(q)} e^{8\pi(G(z,q)-G(z))}\\ &= \int_{T(q)\backslash B_\delta(q)} \frac{e^{8\pi(\tilde{G}(z,q)-G(z))}}{|z-q|^4} \qquad (1.10)\\ &= \pi e^{8\pi(\gamma-G(q))}\frac{1}{\delta^2}+O(1),\end{aligned}$$

where $O(1)$ denotes a bounded number which is independent of δ and ϵ.

By taking into account of (1.6)–(1.10), we get for $0<\epsilon\ll\delta$

$$\begin{aligned}&J_{8\pi}(\varphi_\epsilon-c_\epsilon)\\ &=8\pi(\log(1+\delta^2/\epsilon^2)-1)+16\pi(-\log\delta+2\pi\gamma)\\ &\quad -8\pi\log\pi e^{8\pi(\gamma-G(q))}\Big(\frac{1}{\epsilon^2}+\frac{1}{\delta^2}-\frac{1}{\epsilon^2+\delta^2}\Big)+O(\delta)+O(\epsilon^2\log\epsilon)\\ &=64\pi^2G(q)-32\pi^2\gamma-8\pi(1+\log\pi)+O(\delta)+O(\epsilon^2\log\epsilon).\end{aligned}$$

Let $\epsilon\to 0$ and then let $\delta\to 0$. From (1.4) we conclude that

$$J_{8\pi}(v^\lambda)\le 64\pi^2G(q)-32\pi^2\gamma-8\pi(1+\log\pi). \qquad (1.11)$$

From (1.3), u^λ blows up at p as $\lambda\to+\infty$. By using the explicit expression (1.3), a similar calculation as the above shows that

$$\lim_{\lambda\to+\infty} J_{8\pi}(v^\lambda)=64\pi^2G(p)-32\pi^2\gamma-8\pi(1+\log\pi).$$

Therefore (1.11) implies

$$G(p)\le G(q),$$

which finishes the proof. □

Corollary 1.2. *Suppose that G has five critical points. Then any half-period is a saddle point of G.*

Proof. Since the extra critical point p (reps. $-p$) is a discrete minimal point, the index of ∇G at p (reps. $-p$) is 1. By the Hopf–Poincaré index theorem,

$$-1=\chi(E_\tau\backslash\{0\})=2+\sum_{i=1}^{3}\operatorname{ind}_{\frac{1}{2}\omega_i}\nabla G.$$

Since $\frac{1}{2}\omega_i$ is non-degenerate, ∇G has index ±1 at it. Hence the index must be -1 for all $i=1,2,3$. This implies that $\frac{1}{2}\omega_i$ is a saddle point for all i. □

2. Computations for $D(\frac{1}{2}\omega_j)$

Let u_k be a sequence of blowup solutions to

$$\triangle u_k+e^{u_k}=\rho_k\delta_0 \qquad (2.1)$$

in E_τ and $\rho_k \to 8\pi$. Suppose that $\rho_k \neq 8\pi$. In [2, Theorem 0.7.5], it was proved that u_k blows up at a half period q. Let

$$\lambda_k := \max_{E_\tau} u_k.$$

We recall a result in [6]:

Theorem 2.1. *Let $\tilde{G}(z,q)$ be the regular part of $G(z,q)$, namely $\tilde{G}(z,q) = G(z-q) + \frac{1}{2\pi}\log|z-q|$. Let $\phi(q) := \tilde{G}(q,q)$ and $h(z) = e^{-8\pi G(z)}$. Then*

$$\rho_k - 8\pi = (D(q) + o(1))e^{-\lambda_k},$$

where

$$D(q) := \int_{E_\tau} \frac{h(z)e^{8\pi(\tilde{G}(z,q)-\phi(q))} - h(q)}{|z-q|^4} - \int_{E_\tau^c} \frac{h(q)}{|z-q|^4}.$$

The quantity $D(q)$ is well defined for any critical point of $G(z,q)$. However, if q is not a half period then $D(q) = 0$ since such a blow-up can only occur for $\rho_k = 8\pi$. When q is a half period, $D(q)$ has a geometric interpretation. Indeed,

$$\begin{aligned} D(q) &= \lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(q)} \frac{e^{-8\pi G(z)}e^{8\pi(\tilde{G}(z,q)-\phi(q))}}{|z-q|^4} - \int_{\mathbb{R}^2\setminus B_r(q)} \frac{e^{-8\pi G(q)}}{|z-q|^4}\right) \\ &= \lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(q)} e^{-8\pi\phi(q)}e^{8\pi(G(z-q)-G(z))} - \int_{\mathbb{R}^2\setminus B_r(q)} \frac{e^{-8\pi G(q)}}{|z-q|^4}\right). \end{aligned}$$

Note that $8\pi(G(z-q) - G(z))$ is a doubly periodic harmonic function in $\mathbb{R}^2$ with singularities $-4\log|z-q|$ at $z = q$ and $4\log|z|$ at $z = 0$. Thus

$$8\pi(G(z-q) - G(z)) = 2\log|\wp(z-q) - \wp(q)| + C$$

for the constant $C = 8\pi(\phi(q) - G(q))$. (The identity does not hold if q is not a half period.) Therefore,

$$e^{8\pi(G(z-q)-G(z))} = e^{8\pi(\phi(q)-G(q))}|\wp(z-q) - \wp(q)|^2,$$

and

$$D(q) = e^{-8\pi G(q)} \lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(0)} |\wp(z) - \wp(q)|^2 - \int_{\mathbb{R}^2\setminus B_r(0)} \frac{1}{|z|^4}\right).$$

Let T be a fundamental domain of E_τ with $0 \notin \partial T$. Let γ be the image of $\Gamma := \partial T$ under the map

$$\Sigma(z) := -\zeta(z) - \wp(q)z.$$

Denote by $\Lambda_+(q)$ be the union of components bounded by γ and covered by T under Σ, and by $\Lambda_-(q)$ the union of components bounded by γ but not covered by T under Σ. Then obviously

$$|\Lambda_+(q)| - |\Lambda_-(q)| = \lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(0)} |\wp(z) - \wp(q)|^2 - \int_{\mathbb{R}^2\setminus B_r(0)} \frac{1}{|z|^4}\right), \tag{2.2}$$

and so

$$D(q) = e^{-8\pi G(q)}(|\Lambda_+(q)| - |\Lambda_-(q)|). \tag{2.3}$$

We will give another characterization of $D(q)$ in terms of the Hessian of G at q, hence establish a correspondence between the geometric interpretation and the degeneracy structure of the Green function. Recall [8, (7.7)]:

$$\det D^2 G = \frac{-1}{4\pi^2}\Big(|(\log\vartheta)_{zz}|^2 + \frac{2\pi}{b}\mathrm{Re}\,(\log\vartheta)_{zz}\Big). \tag{2.4}$$

To write it in the Weierstrass theory we use $(\log\vartheta)_z(z) = \zeta(z) - \eta_1 z$ and

$$(\log\vartheta)_{zz}(\tfrac{1}{2}\omega_i) = -\wp(\tfrac{1}{2}\omega_i) - \eta_1 = -(e_i + \eta_1). \tag{2.5}$$

Theorem 2.2. *For any half period q,*

$$|\Lambda_+(q)| - |\Lambda_-(q)| = -4\pi^2 b \det D^2 G(q).$$

Proof. Without loss of generality, we assume that $q = \frac{1}{2}\omega_1 = \frac{1}{2}$ and denote $\Lambda_+(q)$ and $\Lambda_-(q)$ by Λ_+ and Λ_- respectively. By (2.2), we have

$$\begin{aligned} |\Lambda_+| - |\Lambda_-| = {} & \lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(0)} |\wp(z)|^2 - \int_{\mathbb{R}^2\setminus B_r(0)} \frac{1}{|z|^4}\right) \\ & - \lim_{r\to 0}\int_{E_\tau\setminus B_r(0)} (\wp(z)\bar{e}_1 + \bar{\wp}(z)e_1) + b|e_1|^2, \end{aligned}$$

where $\tau = a + bi$.

To compute the first term, write the Weierstrass zeta function as $\zeta = u + iv$ and then $\wp = -\zeta' = -u_x - iv_x = -u_x + iu_y$. Hence

$$|\wp|^2 = u_x^2 + u_y^2 = \partial_x(uu_x) + \partial_y(uu_y).$$

Using integration by parts, and noticing that the singularity at $z = 0$ is cancelled out by the second integral, the first limit term then becomes

$$\int_\Gamma uu_x\,dy - uu_y\,dx = \int_\Gamma u(v_x\,dx + v_y\,dy) = \int_\Gamma u\,dv.$$

This can be calculated easily as

$$-\tfrac{1}{2}\mathrm{Im}\int_\Gamma \zeta\,d\bar{\zeta} = \tfrac{1}{2}\mathrm{Im}\,(\bar{\eta}_1\eta_2 - \eta_1\bar{\eta}_2).$$

Applying the Legendre relation $\eta_2 = \eta_1\tau - 2\pi i$, we get

$$\bar{\eta}_1\eta_2 - \eta_1\bar{\eta}_2 = \bar{\eta}_1(\eta_1\tau - 2\pi i) - \eta_1(\bar{\eta}_1\bar{\tau} + 2\pi i) = 2ib|\eta_1|^2 - 2\pi i(\eta_1 + \bar{\eta}_1).$$

Consequently,

$$\lim_{r\to 0}\left(\int_{E_\tau\setminus B_r(0)} |\wp(z)|^2 - \int_{\mathbb{R}^2\setminus B_r(0)} \frac{1}{|z|^4}\right) = b|\eta_1|^2 - \pi(\eta_1 + \bar{\eta}_1). \tag{2.6}$$

For the second limit term, we first compute

$$\begin{aligned}\int_{E_\tau\setminus B_r(0)} \wp(z) &= \frac{i}{2}\int_{T\setminus B_r(0)} \wp\, dz\wedge d\bar z = -\frac{i}{2}\int_{T\setminus B_r(0)} d(\zeta\, d\bar z)\\ &= -\frac{i}{2}\Big(\int_\Gamma \zeta\, d\bar z - \int_{\partial B_r(0)} \zeta\, d\bar z\Big).\end{aligned}$$

This first integral gives $\eta_1\bar\tau - \eta_2 = \eta_1\bar\tau - \eta_1\tau + 2\pi i = -2bi\eta_1 + 2\pi i$. For the second integral, in the limit $r \to 0$ it tends to $\int_0^{2\pi} e^{-i\theta}e^{-i\theta}(-i)\, d\theta = 0$. Hence

$$\lim_{r\to 0}\int_{E_\tau\setminus B_r(0)} \wp(z) = -\eta_1 b + \pi.$$

Putting everything together we get (c.f. (2.4) and (2.5))

$$\begin{aligned}&|\Lambda_+| - |\Lambda_-|\\ &= b|\eta_1|^2 - \pi(\eta_1 + \bar\eta_1) + (\eta_1 b - \pi)\bar e_1 + (\bar\eta_1 b - \pi)e_1 + b|e_1|^2\\ &= b|e_1+\eta_1|^2 - \pi((e_1+\eta_1) + (\overline{e_1+\eta_1}))\\ &= -4\pi^2 b \det D^2G(\tfrac12;\tau).\end{aligned}$$

The proof is completed. ∎

Corollary 2.3. *Let u_k be a sequence of blow-up solutions to (2.1) with $\rho_k \to 8\pi$ and q the blow-up point.*

(1) *q is a half period and a saddle point of $G(z;\tau)$ if and only if $\rho_k > 8\pi$.*
(2) *q is a half period and a minimal point of $G(z;\tau)$ if and only if $\rho_k < 8\pi$.*

3. Uniqueness of solutions

In this section we classify all solutions to

$$\triangle u + e^u = \rho\,\delta_0 \quad \text{on } E \tag{3.1}$$

for $0 < \rho \le 8\pi + \epsilon_0$ where ϵ_0 is a small positive number.

Recall in [8] we showed that equation (3.1) has a unique solution for $\rho = 4\pi$, and a unique *even* solution for $4\pi \le \rho \le 8\pi$. Here we prove the uniqueness result without the evenness assumption.

Lemma 3.1. *Equation (3.1) has a unique solution for $0 < \rho \le 4\pi$.*

Proof. We first show that for any solution u to (3.1) with $\rho \le 4\pi$, the linearized equation

$$\triangle\phi + e^u\phi = 0 \quad \text{on } E \tag{3.2}$$

has only trivial solution $\phi = 0$.

Suppose that ϕ is a solution to (3.2). Then a straightforward computation shows that $(\phi_{zz} - u_z\phi_z)_{\bar z} = 0$. Since

$$u(z) \sim \frac{\rho}{2\pi}\log|z|,$$

$\phi_{zz} - u_z\phi_z$ is an elliptic function on E whose only singularity is a pole of order one at 0. This forces that $\phi_z(0) = 0$ and

$$\phi_{zz} - u_z\phi_z = c_1 \quad \text{on } E$$

for some constant c_1, or equivalenly

$$(e^{-u}\phi_z)_z = c_1 e^{-u}.$$

Notice that

$$|e^{-u}\phi_z(z)| \leq c_2|z|^{1-\rho/2\pi} \tag{3.3}$$

for some constant $c_2 > 0$. Thus if $\rho < 4\pi$,

$$\lim_{r\to 0}\int_{E\setminus B_r(0)}(e^{-u}\phi_z)_z = \frac{1}{2}\lim_{r\to 0}\int_{\partial B_r(0)} e^{-u}\phi_z\frac{\bar{z}}{|z|}\,ds = 0,$$

and if $\rho = 4\pi$ the above limit is finite. If $c_1 \neq 0$, this implies that

$$\int_E e^{-u} = \begin{cases} 0 & \text{if } \rho < 4\pi, \\ < \infty & \text{if } \rho = 4\pi, \end{cases}$$

which leads to a contradiction. So we have $c_1 = 0$ and $e^{-u}\phi_{\bar{z}}$ is an elliptic function. By (3.3) this again implies that $e^{-u}\phi_{\bar{z}} = c_3$ is a constant.

If $\phi \not\equiv 0$ then ϕ has a maximum point p and a minimum point q with $p \neq q$. One of p, q is not a lattice point where $\phi_{\bar{z}} = 0$. This implies that $c_3 = 0$ and hence $\phi_{\bar{z}} \equiv 0$. This leads to $\phi \equiv 0$ which is a contradiction to $\phi \not\equiv 0$. Hence we must have $\phi \equiv 0$.

Now the uniqueness follows from the fact that (3.1) has only one solution at $\rho = 4\pi$. □

Remark 3.2. In [8] we showed that the unique even solution to (3.1) with $\rho \in [4\pi, 8\pi]$ is non-degenerate in the class of $H^1_{ev} = \{u \in H^1 \mid u(-z) = u(z)\}$. Now the proof of Lemma 3.1 allows us to remove the evenness assumption: u is non-degenerate in the whole space H^1, provided that $0 < \rho < 8\pi$.

To see this, we may assume that the solution ϕ is odd. Therefore $\phi_{zz} - u_z\phi_z$ is odd and by exactly the same calculation we have

$$\phi_{zz} - u_z\phi_z = c_1 = 0.$$

This implies that $e^{-u}\phi_{\bar{z}}$ is an elliptic function on E, with 0 being its only pole. However, since ϕ is odd, $\phi_{\bar{z}}$ is even and the estimate (3.3) can be improved to

$$|e^{-u}\phi_z(z)| \leq c_2|z|^{2-\rho/2\pi}.$$

If $\rho < 8\pi$, we find $2 - \rho/2\pi > -2$. This implies that $e^{-u}\phi_{\bar{z}}$ is a constant. If $\phi \not\equiv 0$, by evaluating it at a maximum or minimum point, with one of it not a lattice point, we conclude that $e^{-u}\phi_{\bar{z}} \equiv 0$, and then $\phi \equiv 0$ follows. (Notice that if $\rho = 8\pi$ then $e^{-u}\phi_{\bar{z}} = c\wp(z)$ for some constant $c \neq 0$.)

Now we may conclude that the unique even solution u is always a minimum point of the non-linear functional J_ρ in (1.1) for $0 < \rho \leq 8\pi$. In fact we can prove a stronger result, namely Theorem 0.5.

Lemma 3.3. *Let u be a solution to (3.1) with $\rho \notin 8\pi\mathbb{N}$. Then u is even.*

Proof. This was proved in [2] for $\rho = 4\pi l$ with l being a positive odd integer, so we assume that $\rho \notin 4\pi\mathbb{N}$.

Let $f(z)$ be a multi-valued developing map of u. The readers are referred to [2, §8] for the details to treat these multi-valued functions as global analytic functions $\mathbf{f}(\xi)$, which are defined on the universal cover $\xi \in \mathbb{H} \to E^\times$. In particular the cusp $\xi = 0$ is mapped to the cusp $z = 0$ in $E^\times$.

As in [2], we have $S(f) = 2(\eta(\eta+1)\wp + B)$ for $\eta = \rho/8\pi$ for some $B \in \mathbb{C}$. Thus $f = w_1/w_2$ and $\mathbf{f} = \mathbf{w}_1/\mathbf{w}_2$ for two linearly independent solutions w_1 and w_2 to the Lamé equation

$$w'' = (\eta(\eta+1)\wp + B)w. \tag{3.4}$$

Since $\tilde{w}_i(z) := w_i(-z)$ are also two linearly independent solutions to (3.4), $\tilde{f} := \tilde{w}_1/\tilde{w}_2$ also defines a global analytic function $\tilde{\mathbf{f}}$ and we have

$$\tilde{\mathbf{f}} = S\mathbf{f} = \frac{a\mathbf{f}+b}{c\mathbf{f}+d}, \quad \text{for some} \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

Consider the covering transformations on $\mathbb{H}$: $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$ determined by the two free generators of $\pi_1(E^\times) \cong \mathbb{Z} * \mathbb{Z}$. Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be the rank two free subgroup generated by g_1 and g_2, and $r : \Gamma \to \mathrm{PSU}(2)$ be the unitary representation associated to the solution u. The mapping $(-1) : z \mapsto -z$ on $E^\times$ lifts to a map ι on $\mathbb{H}$ which is not a covering map for $\mathbb{H} \to E^\times$. Nevertheless the composition $\iota \circ \iota$, namely we apply (-1) twice, does give a covering map for $\mathbb{H} \to E^\times$. That is, the matrix S^2 can be represented as an element generated by $S_1 := r(g_1)$ and $S_2 := r(g_2)$.

By considering the action of (-1) in a simply connected neighborhood U of $0 \in E$, we see that $S^2 f = f(e^{2\pi i}z) = \beta f(z)$ for some $\beta \in \mathrm{PSU}(2,\mathbb{C})$. Indeed, $\beta = r(g_2^{-1}g_1^{-1}g_2g_1) = S_2^{-1}S_1^{-1}S_2S_1$. Under some normalization on $\mathbf{f}$, the matrix β is calculated in [2, Lemma 8.3.4, p.262] (suppress all index k in the formula in the bottom of p.262) as

$$\beta = \begin{pmatrix} |p|^2\alpha + |q|^2\bar{\alpha} & -\bar{p}q(\alpha - \bar{\alpha}) \\ p\bar{q}(\bar{\alpha} - \alpha) & |p|^2\bar{\alpha} + |q|^2\alpha \end{pmatrix},$$

where $\mathbf{f}(0) := \lim_{\xi \to 0} \mathbf{f}(\xi) = q/p$ with $|p|^2 + |q|^2 = 1$, $p, q \neq 0$, and $\alpha = e^{2\pi i \eta}$. Clearly $\alpha \neq \bar{\alpha}$ since $\eta \notin \frac{1}{2}\mathbb{Z}$. In particular $\beta \neq \pm I_2$.

We claim that $S \in \mathrm{PSU}(2,\mathbb{C})$. To prove it, we choose a new unitary basis to diagonalize β to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $e^{i\theta} \neq \pm 1$. Since

$$S^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix},$$

$S^2 = \beta$ implies that either $a + d = 0$ or both $b = 0$ and $c = 0$. If $a + d = 0$ then $a^2 + bc = -ad + bc = -1$ and $d^2 + bc = -1$, which leads to $e^{i\theta} = -1$, a contradiction. Hence $b = c = 0$ and $ad = 1$, $a^2 = e^{i\theta}$, $d^2 = e^{-i\theta}$. Therefore,

$S \in \mathrm{PSU}(2, \mathbb{C})$ and $\tilde{f}(z) = f(-z)$ gives rise to the same solution u. Hence $u(-z) = u(z)$ and the the lemma follows. □

Proof of Theorem 0.5. By Lemma 3.1 and 3.3, the uniqueness of solution holds for $0 < \rho < 8\pi$. The statements for $\rho = 8\pi$ was proved in [8].

For $\tau \in \Omega_3$, the unique solutions u_ρ blows up as $\rho \nearrow 8\pi$ (since equation (3.1) has no solutions at $\rho = 8\pi$). The blow-up point of u_ρ must be the minimum point which is one of the half periods. The other two half periods q_1 and q_2 are saddle critical point of G. By Theorem 0.4 and Remark 0.3, we have $\det D^2 G(q_i) < 0$ and then $D(q_i) > 0$. Under these conditions, by the method in [4] we can construct a bubbling sequence of solutions $u_{\rho,i}$ to (3.1), for each $i = 1, 2$, with $\rho > 8\pi$ which blows up at q_i.

Remark 3.4. In [4] the non-degenerate condition $D(q_i) \neq 0$ was replaced by some other non-degenerate condition. Nevertheless the similar process as there still works in our current case (see e.g. the remark in [6] concerning with the degree counting formula).

Indeed, for the Chern–Simons–Higgs equation, the same non-degenerate conditions $D(q) < 0$ and $\det D^2 G(q) \neq 0$ were recently used to construct such kind of bubbling solutions [10].

Now we need the following uniqueness theorem:

Theorem 3.5. *Suppose that u_k and $\tilde{u}_k$ are two sequences of solutions to (3.1) with $\rho_k \to 8\pi$, and both sequences have the same blow-up point q.*

If $D(q) \neq 0$, i.e. q is a non-degenerate critical point of G by Theorem 0.4, then $u_k = \tilde{u}_k$ for large k.

This is recently proved in [11] for the Chern–Simons-Higgs equation

$$\triangle u + \frac{1}{\epsilon} e^u (1 - e^u) = 8\pi \delta_0,$$

but the proof given there also works for (3.1).

By Theorem 3.5, $u_{\rho,i}$ are exactly all the solutions to equation (3.1) for $8\pi < \rho < 8\pi + \epsilon_0$. This proves (i).

For $\tau \in \Omega_5$, all the three half periods are saddle points of G. By Theorem 3.5 again, we must have three bubbling solutions. On the other hand, (3.1) has a unique even solution u for $\rho = 8\pi$ whose linearized equation in the class of even functions is non-degenerate. Therefore for $8\pi < \rho < 8\pi + \epsilon_0$ there is a unique even solution u_ρ which converges to to u as $\rho \searrow 8\pi$.

By Lemma 3.3, (3.1) has only even solutions for $8\pi < \rho < 8\pi + \epsilon_0$, we conclude that (3.1) has the only one even solution u_ρ which converges to u as $\rho \searrow 8\pi$. Hence there are four solutions in total. This proves (ii) and thus completes the proof Theorem 0.5. □

REFERENCES

[1] S.-Y. Chang, C.-C. Chen and C.-S. Lin; *Extremal functions for a mean field equation in two dimension*, Lectures on PDE, 61–93, New Stud. Adv. Math. **2**, Int. Press, 2003.

[2] C.-L. Chai, C.-S. Lin and C.-L. Wang; *Mean field equations, hyperelliptic curves, and modular forms: I*, Cambridge J. of Math. **3** (2015), no. 1–2, 127–274.

[3] C.-C. Chen and C.-S. Lin; *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55** (2002), 728–771.

[4] ——; *Topological degree for a mean field equation on a Riemann surface*, Comm. Pure Appl. Math. **56** (2003), 1667–1727.

[5] ——; *Topological degree for a mean field equation with singular sources*, Comm. Pure Appl. Math. **68** (2015), 887–947.

[6] C.-C. Chen, C.-S. Lin and G. Wang; *Concentration phenomenon of two-vortex solutions in a Chern–Simons model*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (5) Vol. III (2004), 367–379.

[7] Z. Chen, K.-J. Kuo, C.-S. Lin and C.-L. Wang; *Green function, Painlevé VI equation, and Eisentein series of weight one*, preprint 2015.

[8] C.-S. Lin and C.-L. Wang; *Elliptic functions, Green functions and the mean field equations on tori*, Annals of Math. **172** (2010), no.2, 911–954.

[9] ——; *A function theoretic view of the mean field equations on tori*, in "Recent advances in geometric analysis", 173–193, Adv. Lect. Math. 11, Int. Press, Somerville MA, 2010.

[10] C.-S. Lin and S.-S. Yan; Existence of bubbling solutions for Chern–Simons model on a torus, Arch. Ration. Mech. Anal. **207** (2013), no. 2, 352–392.

[11] ——; *On the self-dual condensate of the Chern–Simons–Higgs model, Part II: local uniqueness and applications*, preprint 2014.

[12] M. Nolasco and G. Tarantello; *Double vortex condensation in the Chern–Simons–Higgs theory*, Calc. Var. Partial Diff. Equ. **9** (1999), 31–94.

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