GEOMETRIC QUANTITIES ARISING FROM BUBBLING ANALYSIS OF MEAN FIELD EQUATIONS

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ABSTRACT. Let $E = \mathbb{C}/\Lambda$ be a flat torus and G be its Green function with singularity at 0. Consider the multiple Green function G_n on E^n:

$$G_n(z_1, \cdots, z_n) := \sum_{i<j} G(z_i - z_j) - n \sum_{i=1}^{n} G(z_i).$$

A critical point $a = (a_1, \cdots, a_n)$ of G_n is called *trivial* if $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$. For such a point a, two geometric quantities $D(a)$ and $H(a)$ arising from bubbling analysis of mean field equations are introduced. $D(a)$ is a global quantity measuring asymptotic expansion and $H(a)$ is the Hessian of G_n at a. By way of geometry of Lamé curves developed in [3], we derive precise formulas to relate these two quantities.

1. INTRODUCTION

Let $E = E_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in \mathbb{H} = \{\tau \mid \operatorname{Im} \tau > 0\}$. We use the convention $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1 + \tau$. Consider the following mean field equation with singular strength $\rho > 0$:

$$\triangle u + e^u = \rho\, \delta_0 \quad \text{in } E, \tag{1.1}$$

where δ_0 is the Dirac measure at 0. Solutions to this simple looking equation (1.1) possess a rich structure from either the point of view of partial differential equations or of integrable systems. See [3, 4, 6].

Not surprisingly, (1.1) is related to various research areas. In conformal geometry, a solution $u(x)$ to (1.1) leads to a metric $ds^2 = \frac{1}{2}e^u\,(dx^2 + dy^2)$ with constant Gaussian curvature $+1$ acquiring a conic singularity at 0. It also appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model, hence its name. In the physical model of superconductivity, (1.1) is one of limiting equations of the well-known Chern–Simons–Higgs equation as the coupling parameter tends to 0. We refer the interested readers to [2, 5, 7, 8, 10, 11] and references therein for recent development on this equation.

One important feature of (1.1) is the so-called *bubbling phenomena*. Let u_k be a sequence of solutions to (1.1) with $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$, and $\max_E u_k(z) \to +\infty$ as $k \to +\infty$. Then it was proved in [5] that u_k has exactly n blowup points $\{a_1, \cdots, a_n\}$ in E and $a_i \neq 0$ for all i. The well-known *Pohozaev identity* says that the position of these blowup points are determined by the following system of equations:

$$n\nabla G(a_i) = \sum_{j\neq i}^{n} \nabla G(a_i - a_j), \qquad 1 \le i \le n. \tag{1.2}$$

Here $G(z,w) = G(z-w)$ is the Green function on E defined by

$$\begin{cases} -\triangle G = \delta_0 - \dfrac{1}{|E|} & \text{on } E, \\ \int_E G = 0, \end{cases}$$

and $|E|$ is the area of E.

If $\rho_k = 8\pi n$ for all k, then $\{u_k\}$ consists of *type II solutions* with explicit blowup behavior (cf. [3]). On the other hand, we have

Theorem A. [3, 4] *Let u_k be a sequence of bubbling solutions of equation* (1.1) *with $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$. If $\rho_k \neq 8\pi n$ for large k, then*

(1) *The blowup set $a = \{a_1, \cdots, a_n\}$ satisfies*

$$\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\} \quad \textit{in } E.$$

(2) *Let $\lambda_k := \max_E u_k(z)$, then there is a constant $D(a)$ such that*

$$\rho_k - 8\pi n = (D(a) + o(1))e^{-\lambda_k}. \tag{1.3}$$

From (1.3), the quantity $D(a)$ plays a fundamental role in controling the sign of $\rho_k - 8\pi n$. Thus it provides one of the key geometric messages for bubbling solutions u_k. The question is how to compute $D(a)$?

There exists a complicate expression for $D(a)$ which we will recall in (1.6) below. Define the regular part $\tilde{G}(z,w)$ of $G(z,w)$ by

$$\tilde{G}(z,w) := G(z,w) + \frac{1}{2\pi}\log|z-w|.$$

Given a blowup set $a = \{a_1, \cdots, a_n\}$ as in Theorem A (1), we set

$$\begin{aligned} f_{a_i}(z) = 8\pi\Bigg(&\tilde{G}(z,a_i) - \tilde{G}(a_i,a_i) + \sum_{j\neq i}(G(z,a_j) - G(a_i,a_j)) \\ &- n(G(z) - G(a_i))\Bigg), \end{aligned} \tag{1.4}$$

$$\mu_i := \exp\left(8\pi(\tilde{G}(a_i,a_i) + \sum_{j\neq i} G(a_i,a_j) - nG(a_i))\right). \tag{1.5}$$

Then $D(a)$ can be calculated by

$$D(a) = \lim_{r\to 0}\sum_{i=1}^{n}\mu_i\left(\int_{\Omega_i\setminus B_r(a_i)} \frac{e^{f_{a_i}(z)}-1}{|z-a_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i}\frac{1}{|z-a_i|^4}\right), \tag{1.6}$$

where Ω_i is any open neighborhood of a_i in E such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^n \bar{\Omega}_i = E$. The limit exists since $f_{a_i}(z) = O(|z-a_i|^3)$ plus a quadratic harmonic function for all i. For a proof, see [10].

Consider the divisor (complete diagonal) in $(E^\times)^n$:

$$\Delta_n = \{(z_1, \cdots, z_n) \in (E^\times)^n \mid z_i = z_j \text{ for some } i \neq j\}$$

and define the *multiple Green function* $G_n(z) = G_n(z;\tau)$ on $(E^\times)^n \setminus \Delta_n$ by

$$G_n(z) := \sum_{i<j} G(z_i - z_j) - n \sum_{i=1}^{n} G(z_i). \tag{1.7}$$

Notice that G_n is invariant under the permutation group S_n. It is clear that the system (1.2) gives the critical point equations of G_n. A critical point a is called *trivial* if $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$ in E. Theorem A (1) says that the blowup set of a sequence of bubbling solutions u_k of (1.1) with $\rho_k \neq 8\pi n$ for large k is a trivial critical point of G_n.

To proceed, it is crucial and natural to ask when is a trivial critical point a degenerate critical point? To answer this question, we need to study the Hessian $H(a)$ at a trivial critical point a:

$$H(a) := \det D^2 G_n(a). \tag{1.8}$$

The quantity $H(a)$ can be used to determine the local maximum points of u_k near a_i, $1 \leq i \leq n$, and to provide other useful geometric information for the bubbling solutions u_k (cf. [4]).

There are many potential applications of these two quantities. For example, $H(a)$ and $D(a)$ together imply *local uniqueness* of bubbling solutions, as described in the following theorem:

Theorem B. *Let $u_k(z)$ and $\tilde{u}_k(z)$ be two sequences of solutions to equation* (1.1) *with the same parameter $\rho_k \to 8\pi n$ and $\rho_k \neq 8\pi n$ for large k. If they have the same blowup set $a = \{a_1, \cdots, a_n\}$ and both $H(a)$ and $D(a)$ do not vanish, then $u_k(z) = \tilde{u}_k(z)$ for large k.*

The proof of Theorem B will be given in a forthcoming paper by the first author. It is unexpected since after some suitable scaling at each blowup point a_i, the solution $u_k(z)$ (resp. $\tilde{u}_k(z)$) converge to a solution of equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < \infty,$$

and it is easy to see that the linearized operator $\Delta + e^w$ has non-trivial kernel. To prove the uniqueness, we have to overcome the difficulty caused by the degeneracy of the operator $\Delta + e^w$.

Surprisingly, these two quantities $D(a)$ and $H(a)$ are related to each other as shown by the main result of this paper:

Theorem 1.1 (=Theorem 4.1)**.** *For fixed $n \in \mathbb{N}$ and any trivial critical point a of $G_n(z)$, there exists $c_a \geq 0$ such that*

$$H(a) = (-1)^n c_a D(a). \tag{1.9}$$

Moreover, $c_a > 0$ if and only if $B_a := (2n-1)\sum_{i=1}^n \wp(a_i)$ is not a multiple root of the Lamé polynomial $\ell_n(B)$.

Here is an outline of the proof, together with a brief description on the content of each section:

The mean field equation (1.1) is closely related to the *Lamé equation* $y'' = (n(n+1)\wp + B)y$. To prove (1.9), a key step is to express $D(a)$ in terms of quantities at a branch point of the hyperelliptic curve $Y_n \to \mathbb{C}$ associated to the Lamé equation. This *Lamé curve* Y_n can be represented by $C^2 = \ell_n(B)$ where the *Lamé polynomial* $\ell_n(B)$ has no multiple roots except for *finitely many* isomorphic classes of tori. This theory is well developed in [3] and the results we need will be reviewed in §2 (cf. Theorem 2.4).

In §3 we study the quantity $D(a)$ in details and derive the above mentioned expression of $D(a)$ in Theorem 3.4. In fact, the Lamé curve encodes the $n-1$ algebraic constraints of the system (1.2), with the remaining analytic constraint being $\sum_{i=1}^n \nabla G(a_i) = 0$. It is thus natural to study the map $a \mapsto \phi(a) := -4\pi \sum_{i=1}^n \nabla G(a_i)$ for $a \in Y_n$. It turns out that $D(a)$ is expressible in terms of the Jacobian of ϕ (Corollary 3.6).

The proof of Theorem 1.1 is completed in §4 by a process called *analytic adjunction*. The idea is simple: The quantity $H(a)$ is a (real) $2n$-dimensional Hessian on E^n/S_n while $D(a)$ can be regarded as a two dimensional Hessian on $Y_n \subset E^n/S_n$. To relate $H(a)$ with $D(a)$ it amounts to reducing the determinant by substituting the $n-1$ (complex) algebraic equations defining Y_n into it. We end this paper by investigating the case $n = 2$ in Example 4.2 where the value of c_a (given in (4.9)) is seem in more explicit terms.

2. Lamé equations and Lamé curves [3]

Let $\wp(z) = \wp(z;\tau)$ be the Weierstrass elliptic function with periods Λ_τ:

$$\wp(z;\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

which satisfies the well known cubic equation

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau).$$

Let $\zeta(z) = \zeta(z;\tau) := -\int^z \wp(\xi;\tau)\, d\xi$ be the Weierstrass zeta function with quasi-periods $\eta_1(\tau)$ and $\eta_2(\tau)$:

$$\eta_i(\tau) := \zeta(z+\omega_i;\tau) - \zeta(z;\tau), \quad i = 1, 2,$$

and $\sigma(z) = \sigma(z;\tau)$ be the Weierstrass sigma function defined by $\sigma(z) = \exp \int^z \zeta(\xi)\, d\xi$. $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ.

The Green function on E can be expressed in terms of elliptic functions. In [8], we proved that

$$-4\pi \frac{\partial G}{\partial z}(z) = \zeta(z) - r\eta_1 - s\eta_2 = \zeta(z) - z\eta_1 + 2\pi i s, \tag{2.1}$$

where $z = r + s\tau$ with $r, s \in \mathbb{R}$. Using (2.1), equations (1.2) can be translated into the following equivalent system: Consider $a = (a_1, \cdots, a_n) \in E^n$, subject to the constraint $a \in (E^\times)^n \setminus \Delta_n$, that is

$$a_i \neq 0, \quad a_i \neq a_j \text{ for } i \neq j. \tag{2.2}$$

Then

$$\sum_{j\neq i} (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \qquad 1 \leq i \leq n \tag{2.3}$$

(there are only $n-1$ independent equations), and

$$\sum_{i=1}^{n} \nabla G(a_i) = 0. \tag{2.4}$$

We will use (2.2)–(2.4) to connect a critical point of G_n defined in (1.7) with the classical Lamé equation. For the reader's convenience, we review some basics on it and refer the readers to [3, 12, 13] for further details.

Recall the Lamé equation

$$\mathcal{L}_{n,B}: \qquad y''(z) = (n(n+1)\wp(z) + B)y(z), \tag{2.5}$$

where $n \in \mathbb{R}_{\geq -1/2}$ and $B \in \mathbb{C}$ are its *index* and *accessory parameter* respectively. In general, a solution $y(z)$ is a multi-valued meromorphic function on $\mathbb{C}$ with branch points at Λ. Any lattice point is a regular singular point with local exponents $-n$ and $n+1$. In this paper we consider only $n \in \mathbb{N}$.

For $a = (a_1, \cdots, a_n)$, we consider the *Hermite–Halphen ansatz*:

$$y_a(z) := e^{z \sum_{i=1}^{n} \zeta(a_i)} \frac{\prod_{i=1}^{n} \sigma(z - a_i)}{\sigma(z)^n}. \tag{2.6}$$

Theorem 2.1 ([3, 13]). *Suppose that $a = (a_1, \cdots, a_n) \in (E^\times)^n \setminus \Delta_n$. Then $y_a(z)$ is a solution to $\mathcal{L}_{n,B}$ for some B if and only if a satisfies* (2.3) *and*

$$B = B_a := (2n-1) \sum_{i=1}^{n} \wp(a_i). \tag{2.7}$$

Note that if $a = (a_1, \cdots, a_n) \in (E^\times)^n \setminus \Delta_n$ satisfies (2.3), then so does $-a = (-a_1, \cdots, -a_n)$, and then $y_{-a}(z)$ is also a solution of the same Lamé equation because $B_a = B_{-a}$. Clearly $y_a(z)$ and $y_{-a}(z)$ are linearly independent if and only if $\{a_1, \cdots, a_n\} \neq \{-a_1, \cdots, -a_n\}$ in E. Furthermore, the condition actually implies that

$$\{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset \tag{2.8}$$

because $y_a(z)$ and $y_{-a}(z)$ can not have common zeros. For otherwise the Wronskian of $(y_a(z), y_{-a}(z))$ would be identically zero, which forces that $y_a(z), y_{-a}(z)$ are linearly dependent.

Definition 2.2. *Suppose that $a = (a_1, \cdots, a_n) \in (E^\times)^n \setminus \Delta_n$ satisfies* (2.3). *Then a is called a branch point if $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$ in E.*

Note that if a is *not* a branch point, then $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$. By the addition formula

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)},$$

the system (2.3) is equivalent to

$$\sum_{j\neq i} \frac{\wp'(a_i) + \wp'(a_j)}{\wp(a_i) - \wp(a_j)} = 0, \qquad 1 \leq i \leq n. \tag{2.9}$$

The following non-obvious equivalence is crucial for our purpose:

Proposition 2.3. [3, Proposition 5.8.3] *Suppose that* $a = (a_1, \cdots, a_n) \in (E^\times)^n$ *satisfies* $\wp(a_i) \neq \wp(a_j)$ *for* $i \neq j$. *Then* (2.9) *is equivalent to*

$$\sum_{i=1}^{n} \wp'(a_i)\wp(a_i)^l = 0, \qquad 0 \leq l \leq n-2. \tag{2.10}$$

Let $a \in (E^\times)^n \setminus \Delta_n$ satisfy (2.3) and suppose that it is not a branch point. Then (2.10) implies that

$$g_a(z) := \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{C(a)}{\prod_{i=1}^{n}(\wp(z) - \wp(a_i))} \tag{2.11}$$

for a constant $C(a) \neq 0$. Equivalently,

$$C(a) = \sum_{i=1}^{n} \wp'(a_i) \prod_{j\neq i} (\wp(z) - \wp(a_j)). \tag{2.12}$$

There are various ways to represent $C(a)$ by plugging in different values of z in (2.12). For example, for $z = a_i$ we get

$$C(a) = \wp'(a_i) \prod_{j\neq i} (\wp(a_i) - \wp(a_j)) \tag{2.13}$$

which is independent of the choices of i. Notice that if a is a branch point then $g_a(z) \equiv 0$ and so $C(a) = 0$.

Then we have the following important result:

Theorem 2.4. [3] *There exists a polynomial* $\ell_n(B) = \ell_n(B; g_2, g_3) \in \mathbb{Q}[g_2, g_3][B]$ *of degree* $2n+1$ *in* B *such that if* $a \in (E^\times)^n \setminus \Delta_n$ *satisfies* (2.3), *then* $C^2 = \ell_n(B)$, *where* $C = C(a)$ *and* $B = B_a$ *are given in* (2.13) *and* (2.7) *respectively.*

This polynomial $\ell_n(B)$ is called the *Lamé polynomial* in the literature.

Let $Y_n = Y_n(\tau) \subset \operatorname{Sym}^n E = E^n/S_n$ be the set of $a = \{a_1, \cdots, a_n\}$ which satisfies (2.2) and (2.3). Clearly $-a := \{-a_1, \cdots, -a_n\} \in Y_n$ if $a \in Y_n$, and $a \in Y_n$ is a *branch point* if $a = -a$ in E. Then the map $B : Y_n \to \mathbb{C}$ in (2.7) is a ramified covering of degree 2, and Theorem 2.4 implies that

$$Y_n \cong \left\{ (B, C) \mid C^2 = \ell_n(B) \right\},$$

(cf. [3, Theorem 7.4]). Therefore, Y_n is a hyperelliptic curve, known as the *Lamé curve*. Furthermore, Y_n *is singular at a trivial critical point a if and only if* B_a *is a multiple zero of* $\ell_n(B)$. For later usage, we denote

$$X_n := \{a \in Y_n \mid a \text{ is not a branch point}\} \subset Y_n.$$

Since a is a branch point of Y_n if and only if it is a trivial critical point of G_n. From now on we will switch these two notions freely.

There are several ways to compute the Lamé polynomial $\ell_n(B)$. A recursive construction can be found in [3, Theorem 7.4].

Example 2.5. [1, 3] $\ell_n(B)$ for $n = 1, 2$. Denote $e_k = \wp(\frac{\omega_k}{2})$ for $k = 1, 2, 3$.
(1) $n = 1$, $\bar{X}_1 \cong E$, $C^2 = \ell_1(B) = 4B^3 - g_2B - g_3 = 4\prod_{i=1}^3 (B - e_i)$.
(2) $n = 2$ (notice that $e_1 + e_2 + e_3 = 0$),

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81}B^5 - \tfrac{7}{27}g_2B^3 + \tfrac{1}{3}g_3B^2 + \tfrac{1}{3}g_2^2B - g_2g_3 \\ &= \frac{2^2}{3^4}(B^2 - 3g_2)\prod_{i=1}^3 (B + 3e_i). \end{aligned}$$

Consequently, $\ell_2(B;\tau)$ has multiple zeros if and only if $g_2(\tau) = 0$, that is τ is equivalent to $e^{\pi i/3}$ under the $\mathrm{SL}(2,\mathbb{Z})$ action.

If $a = \{a_1, a_2\}$ is a branch point of Y_2, then $\{a_1, a_2\} = \{-a_1, -a_2\}$ in E implies that either (1) $a = \{\frac{1}{2}\omega_i, \frac{1}{2}\omega_j\}$ with $\{i, j, k\} = \{1, 2, 3\}$, which corresponds to $B_a = 3(e_i + e_j) = -3e_k$, or (2) $a_1 = -a_2 \neq \frac{\omega_k}{2}$. Then $\pm\sqrt{3g_2} = B_a = 6\wp(a_1)$, i.e. $\wp(a_1) = \pm\sqrt{g_2/12}$. We conclude that the branch points of Y_2 are given by $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) \mid i \neq j\}$ and $\{(q_\pm, -q_\pm) \mid \wp(q_\pm) = \pm\sqrt{g_2/12}\}$.

3. The invariant $D(a)$ and its geometric meaning

The purpose of this section is to generalize the invariant $D(a)$ studied in [9] for $\rho = 8\pi$, where a is a half-period point, to the general case $\rho = 8\pi n$ for all $n \in \mathbb{N}$. $D(a)$ is fundamental in analyzing the bubbling behavior of a sequence u_k with $\rho_k \to 8\pi n$. By Theorem A, the bubbling loci $a = \{a_1, \cdots, a_n\}$ must be a branch point of Y_n if $\rho_k \neq 8\pi n$ for k large. Thus it is essential to study the geometric meaning of $D(a)$ at those $2n + 1$ branch points as in the case $n = 1$ in [9, Theorem 0.4].

For $a = (a_1, \cdots, a_n) \in (E^\times)^n \setminus \Delta_n$ a trivial critical point, we recall (1.6):

$$D(a) := \lim_{r\to 0}\sum_{i=1}^n \mu_i \left(\int_{\Omega_i \setminus B_r(a_i)} \frac{e^{f_{a_i}(z)} - 1}{|z - a_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i} \frac{1}{|z - a_i|^4}\right), \tag{3.1}$$

where $f_{a_i}(z)$, μ_i are defined in (1.4) and (1.5) respectively. Notice that the sum in the RHS of (3.1) can be written as

$$\sum_{i=1}^n \left(\int_{\Omega_i \setminus B_r(a_i)} \frac{\mu_i e^{f_{a_i}(z)}}{|z - a_i|^4} - \int_{\mathbb{R}^2\setminus B_r(a_i)} \frac{\mu_i}{|z - a_i|^4}\right),$$

where

$$K(z) := \frac{\mu_i e^{f_{a_i}(z)}}{|z-a_i|^4} = \exp\Big(8\pi \sum_{j=1}^n G(z,a_j) - 8\pi n G(z)\Big) \tag{3.2}$$

is independent of i. Hence (3.1) of is independent of the choices of Ω_i's.

From now on, we use notation $p = \{p_1, \cdots, p_n\}$ instead of $a = \{a_1, \cdots, a_n\}$ to denote branch points. Assume that $p = \{p_1, \cdots, p_n\} \in Y_n \setminus X_n$ is a branch point. Then $\{p_1, \cdots, p_n\} = \{-p_1, \cdots, -p_n\}$ and

$$\begin{aligned} K(z) &= \exp 4\pi \Big(\sum_{j=1}^n \big(G(z,p_j) + G(z,-p_j) - 2G(z)\big)\Big) \\ &= e^c \prod_{i=1}^n |\wp(z) - \wp(p_i)|^{-2} \end{aligned} \tag{3.3}$$

for some constant $c \in \mathbb{R}$. The last equality follows by the comparison of singularities. We remark here that, in comparison with [9, §2], for non-half period points the simultaneous appearance of $\pm p_i$ is essential to arrive at the above simple looking closed form.

For convenience, we define $\Lambda_2 = \{\, i \mid p_i \in E[2]\,\}$, the two-torsion part, and for $i \notin \Lambda_2$ we define $i^* \notin \Lambda_2$ to be the index so that $p_{i^*} = -p_i$.

Choose a sequence $a^k \in X_n$ with $\lim_{k\to\infty} a^k = p$. For ease of notations we drop the index k and simply denote $a = (a_1, \cdots, a_n) \to (p_1, \cdots, p_n)$.

In §2 we show that $a \in X_n$ is equivalent to the following equation:

$$g_a(z) := \sum_{i=1}^n \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{C(a)}{\prod_{i=1}^n (\wp(z) - \wp(a_i))} \tag{3.4}$$

(so that $\operatorname{ord}_{z=0} g_a(z) = 2n$) for a constant $C(a) \neq 0$ given by

$$C(a) = \wp'(a_i) \prod_{j\neq i} (\wp(a_i) - \wp(a_j)), \quad \text{for any } i = 1, \ldots, n.$$

For $a \in Y_n$, $C(a) = 0$ if and only if a is a branch point. It is easy to describe the behavior of the limit $C(a) \to C(p) = 0$ as $a \to p$:

Lemma 3.1. *Let $p \in Y_n \setminus X_n$ and $a \in X_n$ near p. If $i \in \Lambda_2$ then*

$$C(a) = \wp''(p_i) \prod_{j\neq i} (\wp(p_i) - \wp(p_j))(a_i - p_i) + o(|a_i - p_i|), \tag{3.5}$$

and if $i \notin \Lambda_2$ then

$$C(a) = \wp'(p_i)^2 \prod_{j\neq i, i^*} (\wp(p_i) - \wp(p_j))(a_i + a_{i^*}) + o(|a_i + a_{i^*}|). \tag{3.6}$$

Lemma 3.2. *For $p \in Y_n$ being a branch point, the residue for*

$$P_p(z) := \prod_{i=1}^n (\wp(z) - \wp(p_i))^{-1}$$

at p_i is zero for all $i = 1, \ldots, n$.

Proof. Choose $a \in X_n$ with $a \to p$ as above. We compute from (3.4) that

$$P_a(z) = \frac{g_a(z)}{C(a)} = \frac{1}{C(a)} \sum_{i\in\Lambda_2} \frac{\wp'(a_i)}{\wp(z)-\wp(a_i)} + \frac{1}{2C(a)} \sum_{i\notin\Lambda_2} \Big(\frac{\wp'(a_i)}{\wp(z)-\wp(a_i)} + \frac{\wp'(a_{i^*})}{\wp(z)-\wp(a_{i^*})}\Big).$$

By Lemma 3.1, the first sum has limit

$$\sum_{i\in\Lambda_2} \frac{\prod_{j\neq i}(\wp(p_i)-\wp(p_j))^{-1}}{\wp(z)-\wp(p_i)}$$

when $a \to p$, which obviously has zero residue at p_i because $i \in \Lambda_2$ means $p_i = \frac{1}{2}\omega_k$ in E for some $k \in \{1,2,3\}$.

For the second sum, we rewrite each i-th summand as

$$\frac{1}{2C}\frac{\wp'(a_i)-\wp'(-a_{i^*})}{\wp(z)-\wp(a_i)} - \frac{\wp'(a_{i^*})}{2C}\frac{\wp(a_i)-\wp(a_{i^*})}{(\wp(z)-\wp(a_i))(\wp(z)-\wp(-a_{i^*}))},$$

which has limit

$$\frac{1}{2}\left(\frac{\wp''(p_i)}{\wp'(p_i)^2}\frac{1}{\wp(z)-\wp(p_i)} + \frac{1}{(\wp(z)-\wp(p_i))^2}\right)\prod_{j\neq i,i^*}(\wp(p_i)-\wp(p_j))^{-1}.$$

A direct Taylor expansion shows that the residues of both terms at p_i ($i \notin \Lambda_2$) cancel out with each other. This proves the lemma. □

By Lemma 3.2, we may rewrite

$$P_p(z) = \prod_{i=1}^{n}(\wp(z)-\wp(p_i))^{-1} = \sum_{j=1}^{n} c_j\wp(z-p_j) + c_0. \tag{3.7}$$

Since the vanishing order of the LHS at $z = 0$ is $2n$, the coefficients must satisfy the constraints

$$\sum_{j=1}^{n} c_j\wp(p_j) + c_0 = 0, \tag{3.8}$$

$$\sum_{j=1}^{n} c_j\wp^{(k)}(-p_j) = 0, \quad \text{for } k = 1,\ldots,2n-1.$$

Also, it is easy to see from (3.7) that for $i \in \Lambda_2$,

$$c_i = 2\wp''(p_i)^{-1}\prod_{j\neq i}(\wp(p_i)-\wp(p_j))^{-1}, \tag{3.9}$$

and if $i \notin \Lambda_2$ then

$$c_i = \wp'(p_i)^{-2}\prod_{j\neq i,i^*}(\wp(p_i)-\wp(p_j))^{-1}. \tag{3.10}$$

In particular $c_{i^*} = c_i$.

This vector $\vec{c} = (c_1,\cdots,c_n)$ indeed has important geometric meaning:

Lemma 3.3. *By considering C as the local holomorphic coordinate of the hyperelliptic curve $Y_n \ni a(C)$ near a branch point p, then we have $a'(0) = \vec{c}/2$. Moreover,*

$$\frac{\partial a_j}{\partial C}(0) = \frac{c_j}{2} \notin \{0, \infty\}$$

for $j = 1, \cdots, n$.

Proof. We first show that if $i \notin \Lambda_2$ then

$$\frac{\partial a_i}{\partial C}(0) = \frac{\partial a_{i^*}}{\partial C}(0). \tag{3.11}$$

Suppose that $a(C) = (a_i(C))$ represents the point $(B, C) \in Y_n$ close to p, where $B = (2n-1)\sum_{i=1}^n \wp(a_i(C))$. Then $\tilde{a}(C) = (a_i(-C))$ represent the other point $(B, -C)$ with the same B. That is, $B = (2n-1)\sum_{i=1}^n \wp(a_i(-C))$ too. By the hyperelliptic structure on Y_n, we conclude that

$$\{a_1(-C), \cdots, a_n(-C)\} = \{-a_1(C), \cdots, -a_n(C)\}.$$

If $i \notin \Lambda_2$, then we must have $a_i(-C) = -a_{i^*}(C)$ and $a_{i^*}(-C) = -a_i(C)$. Therefore, $a_i(-C) + a_{i^*}(-C) = -(a_i(C) + a_{i^*}(C))$ and

$$a_i(-C) - a_{i^*}(-C) = a_i(C) - a_{i^*}(C).$$

That is, $a_i(C) - a_{i^*}(C)$ is even in C, which implies (3.11).

The lemma now follows from (3.5)-(3.6) in Lemma 3.1. For example, if $i \in \Lambda_2$, then (3.5) implies $\lim_{C\to 0} \frac{a_i(C) - p_i}{C} = \frac{c_i}{2}$. If $i \notin \Lambda_2$, then (3.11) and (3.6) imply

$$2\frac{\partial a_i}{\partial C}(0) = \frac{\partial a_i}{\partial C}(0) + \frac{\partial a_{i^*}}{\partial C}(0) = \lim_{C\to 0} \frac{a_i(C) + a_{i^*}(C)}{C} = c_i.$$

Notice that the property $c_j \neq 0, \infty$ for all j is clear from the expressions in (3.9) and (3.10) since (i) $p_i \notin \Lambda$ for all i and $\wp(p_i) \neq \wp(p_j)$ for all $i \neq j$, and (ii) $\wp''(p_i) \neq 0$ for $i \in \Lambda_2$ and $\wp'(p_i) \neq 0$ for $i \notin \Lambda_2$. □

Using the tangent vector $\vec{c}$, we may derive a simple formula for $D(p)$.

Theorem 3.4. *Let $p \in Y_n \setminus X_n$ be a branch point of the hyperelliptic curve Y_n defined by $C^2 = \ell_n(B)$. Consider the local parameter C near p and let $\vec{c} = 2a'(0) = 2\partial a/\partial C|_{C=0}$. Denote also by $s = \sum_{j=1}^n c_j$ and $c_0 = -\sum_{j=1}^n c_j \wp(p_j)$. Then*

$$\begin{aligned} D(p) &= \operatorname{Im}\tau \cdot e^c \left(|c_0 - s\eta_1|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Re} \bar{s}(c_0 - s\eta_1) \right) \\ &= \operatorname{Im}\tau \cdot e^c |s|^2 \left(\left| \frac{c_0}{s} - \eta_1 \right|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Re} \left(\frac{c_0}{s} - \eta_1 \right) \right). \end{aligned} \tag{3.12}$$

Proof. By Lemma 3.3, $\vec{c}$ coincides with the vector formed by the coefficients $c_1, \cdots, c_n$ appeared in the expansion formula of $P_p(z)$ in (3.7).

Let $T \subset \mathbb{R}^2$ be a fundamental domain of E_τ with $p \cap \partial T = \emptyset$. Then

$$D(p) = \lim_{r\to 0}\left(e^c \int_{T\setminus \cup_i B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^{n} \int_{\mathbb{R}^2\setminus B_r(p_i)} \frac{\mu_i}{|z-p_i|^4}\right) \tag{3.13}$$
$$= \lim_{r\to 0}\left(e^c \int_{T\setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^{n} \frac{\pi\mu_i}{r^2}\right).$$

Consider an anti-derivative of $P_p(z)$:

$$L_p(z) := \int_0^z P_p(w)\,dw = -\sum_{j=1}^{n} c_j\zeta(z-p_j) + c_0 z. \tag{3.14}$$

For $i = 1, 2$, we define the "quasi-periods" χ_i by

$$\chi_i = L_p(z+\omega_i) - L_p(z) = c_0\omega_i - s\eta_i. \tag{3.15}$$

To compute $D(p)$, we note from (3.7) that

$$P_p(z) = \frac{c_i}{(z-p_i)^2} + O(1)$$

and from (3.2), the definition (1.5) of μ_i that

$$K(z) = \frac{\mu_i}{|z-p_i|^4} + O(|z-p_i|^{-2}).$$

Inserting these into (3.3) leads to

$$\mu_i = e^c|c_i|^2, \quad \forall 1 \le i \le n. \tag{3.16}$$

Now we denote

$$L_p(z) = u + \sqrt{-1}v, \quad z = x + \sqrt{-1}y.$$

Then $P_p(z) = L_p'(z) = u_x - iu_y$, i.e.

$$|P_p(z)|^2 = u_x^2 + u_y^2 = (uu_x)_x + (uu_y)_y \text{ for } z \text{ outside } \{p_1, \cdots, p_n\},$$

so

$$\int_{T\setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2$$
$$= \int_{\partial T} \left(uu_x dy - uu_y dx\right) - \sum_{i=1}^{n} \int_{|z-p_i|=r} \left(uu_x dy - uu_y dx\right).$$

Applying (3.15) we obtain

$$\int_{\partial T} \left(uu_x dy - uu_y dx\right) = \int_{\partial T} u dv = -\frac{1}{2}\operatorname{Im}\int_{\partial T} L_p d\bar{L}_p \tag{3.17}$$
$$= \frac{1}{2}\operatorname{Im}(\bar{\chi}_1\chi_2 - \chi_1\bar{\chi}_2).$$

Since near p_i,

$$u + \sqrt{-1}v = L_p(z) = -\frac{c_i}{z-p_i} + f(z),$$

where $f(z)$ is holomorphic in a neighborhood of p_i, it is easy to prove that

$$-\int_{|z-p_i|=r} (uu_x dy - uu_y dx) = -\int_{|z-p_i|=r} u dv = \frac{\pi |c_i|^2}{r^2} + O(r).$$

Therefore, we conclude from (3.16) and (3.17) that

$$\begin{aligned} D(p) &= \lim_{r\to 0} \left(e^c \int_{T\setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^n \frac{\pi \mu_i}{r^2} \right) \\ &= \frac{e^c}{2} \operatorname{Im}(\bar\chi_1 \chi_2 - \chi_1 \bar\chi_2). \end{aligned} \tag{3.18}$$

By direct substitution, we compute

$$\begin{aligned} \bar\chi_1\chi_2 - \chi_1\bar\chi_2 &= |c_0|^2(\bar\omega_1\omega_2 - \omega_1\bar\omega_2) + |s|^2(\bar\eta_1\eta_2 - \eta_1\bar\eta_2) \\ &\quad + \bar c_0 s(\eta_1\bar\omega_2 - \eta_2\bar\omega_1) - c_0\bar s(\bar\eta_1\omega_2 - \bar\eta_2\omega_1). \end{aligned}$$

Now we plug in $\omega_1 = 1$, $\omega_2 = \tau = a + bi$, and use the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. Then

$$\begin{aligned} \bar\omega_1\omega_2 - \omega_1\bar\omega_2 &= 2ib, \\ \bar\eta_1\eta_2 - \eta_1\bar\eta_2 &= 2i(b|\eta_1|^2 - \pi(\eta_1 + \bar\eta_1)), \\ \eta_1\bar\omega_2 - \eta_2\bar\omega_1 &= 2i(\pi - \eta_1 b). \end{aligned}$$

Hence

$$\begin{aligned} D(p) &= e^c\Big(b(|c_0|^2 + |s\eta_1|^2) + 2\operatorname{Re}\big(c_0\bar s(\pi - \bar\eta_1 b) - |s|^2\pi\eta_1\big)\Big) \\ &= be^c\Big(|c_0 - s\eta_1|^2 + \frac{2\pi}{b}\operatorname{Re}\bar s(c_0 - s\eta_1)\Big). \end{aligned}$$

This proves the theorem. □

In fact there is a simple geometric interpretation of the expression appeared in the RHS of (3.12).

Proposition 3.5. *Consider the vector-valued map $(E^\times)^n \to \mathbb{R}^2$ defined by*

$$a \mapsto \phi(a) := -4\pi \sum_{i=1}^n \nabla G(a_i).$$

Let $C = u + iv \mapsto a(C) \in E^n$ be a local holomorphic parametrization of a Riemann surface $V \subset E^n$. Then the Jacobian $J(\phi\circ a)(u, v)$ is given by

$$\det\Big(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\Big) = -\Big(|c_0 - s\eta_1|^2 + \frac{2\pi}{\operatorname{Im}\tau}\operatorname{Re}\bar s(c_0 - s\eta_1)\Big), \tag{3.19}$$

where $\vec c = (c_i) := 2a'(C)$, $s := \sum_{i=1}^n c_i$, and $c_0 := -\sum_{i=1}^n c_i\wp(a_i)$.

Proof. Denote $a_j = x_j + \sqrt{-1}y_i$, $b = \operatorname{Im}\tau$ and $\phi = (\phi_1, \phi_2)^T$. By (2.1), we have

$$\begin{aligned} \phi_1 &= 2\operatorname{Re}\,(\sum_i \zeta(a_i) - \eta_1 a_i), \\ \phi_2 &= -2\operatorname{Im}\,(\sum_i \zeta(a_i) - \eta_1 a_i) - \frac{4\pi}{b}\sum y_i. \end{aligned} \tag{3.20}$$

The chain rule shows that

$$\begin{aligned} \partial_u\phi_1 &= -2\operatorname{Re}\Big[\sum_i(\wp(a_i)+\eta_1)\frac{\partial a_i}{\partial C}\Big] = \operatorname{Re}\,(c_0 - s\eta_1), \\ \partial_v\phi_1 &= -2\operatorname{Re}\Big[\sum_i(\wp(a_i)+\eta_1)\frac{\partial a_i}{\partial C}\sqrt{-1}\Big] = -\operatorname{Im}\,(c_0 - s\eta_1), \\ \partial_u\phi_2 &= 2\operatorname{Im}\Big[\sum_i(\wp(a_i)+\eta_1)\frac{\partial a_i}{\partial C}\Big] - \frac{4\pi}{b}\sum_i\frac{\partial y_i}{\partial u} \\ &= -\operatorname{Im}\,(c_0 - s\eta_1) - \frac{2\pi}{b}\operatorname{Im} s, \\ \partial_v\phi_2 &= 2\operatorname{Im}\Big[\sum_i(\wp(a_i)+\eta_1)\frac{\partial a_i}{\partial C}\sqrt{-1}\Big] - \frac{4\pi}{b}\sum_i\frac{\partial y_i}{\partial v} \\ &= -\operatorname{Re}\,(c_0 - s\eta_1) - \frac{2\pi}{b}\operatorname{Re} s. \end{aligned}$$

Hence the Jacobian is given by

$$\begin{aligned} &-|c_0 - s\eta_1|^2 - \frac{2\pi}{b}(\operatorname{Re}\,(c_0 - s\eta_1)\operatorname{Re} s + \operatorname{Im}\,(c_0 - s\eta_1)\operatorname{Im} s) \\ &= -\Big(|c_0 - s\eta_1|^2 + \frac{2\pi}{b}\operatorname{Re}\bar{s}(c_0 - s\eta_1)\Big) \end{aligned}$$

as expected. □

Corollary 3.6. *For $p \in Y_n \setminus X_n$ with local coordinate C, we have*

$$D(p) = -\operatorname{Im}\tau\, e^c J(\phi \circ a)(0) \tag{3.21}$$

for some constant c.

Proof. This follows from Theorem 3.4 and Proposition 3.5. □

Corollary 3.6 will play important role in our subsequent degeneration analysis of these branch points $p \in Y_n \setminus X_n$. One may also interpret the above proof of it as a stationary phase integral calculation.

Example 3.7. For $n = 1$, $c_0 = -c_1\wp(p_1) = -c_1 e_i$ if $p_1 = \frac{1}{2}\omega_i$, and $s = c_1$. The formula reduces to the one for $\det D^2G(p)$ first studied in [8]:

$$|e_i + \eta_1|^2 - \frac{2\pi}{\operatorname{Im}\tau}\operatorname{Re}(e_i + \eta_1).$$

4. PROOF OF THEOREM 1.1: ANALYTIC ADJUNCTION

It is elementary to see that for $\chi_1 = a_1 + b_1 i$ and $\chi_2 = a_2 + b_2 i$,

$$\bar{\chi}_1\chi_2 - \chi_1\bar{\chi}_2 = 2i(a_1b_2 - a_2b_1).$$

Hence the formula in (3.18) says that $D(p)$ is exactly e^c times the signed area spanned by χ_1 and χ_2 in $\mathbb{R}^2$. Indeed, $\chi_1 = c_0 - s\eta_1 = -\sum_{j=1}^n c_j(\wp(p_j) + \eta_1)$. So we may rewrite (3.12) as

$$D(p) = -\mathrm{Im}\,\tau\, e^c|s|^2 \begin{vmatrix} -\mathrm{Re}\,\chi_1 s^{-1} & +\mathrm{Im}\,\chi_1 s^{-1} \\ +\mathrm{Im}\,\chi_1 s^{-1} & \mathrm{Re}\,\chi_1 s^{-1} + \dfrac{2\pi}{\mathrm{Im}\,\tau} \end{vmatrix}. \tag{4.1}$$

Formula (4.1) suggests the possibility for interpreting $D(p)$ in terms of the determinant of the Hessian of some "Green function" for general $n \in \mathbb{N}$. To find such a Green function on $\bar{X}_n$ will require the search for a suitable conformal metric on it. Alternatively we consider the multiple Green function G_n defined in (1.7):

$$G_n(z_1, \cdots, z_n) := \sum_{i<j} G(z_i - z_j) - n\sum_{i=1}^n G(z_i)$$

for $z = (z_1, \cdots, z_n) \in (E^\times)^n \setminus \Delta_n$. Then G_n is a Green function on E^n with divisor D_n where $(E^\times)^n \setminus \Delta_n = E^n \setminus D_n$. Recall that *$p$ is a branch point of Y_n if and only if it is a trivial critical point of G_n.*

Theorem 4.1 (Analytic adjunction formula)**.** *For any fixed $n \in \mathbb{N}$ and any branch point $p = (p_1, \cdots, p_n) \in Y_n$, there is a constant $c_p \geq 0$ such that*

$$\det D^2 G_n(p) = (-1)^n c_p D(p).$$

Moreover, $c_p = 0$ precisely when the associated hyperelliptic curve $Y_n(\tau)$ for $E = E_\tau$ is singular at p. There are only finitely many such tori E_τ for each n.

For $n = 1$, this is [9, Theorem 0.4]. For $n = 2$, a direct check based on Theorem 3.4 is still possible (c.f. Example 4.2). For $n \geq 3$ the D^2G_n is a $2n \times 2n$ matrix and it is cumbersome to compute $\det D^2G_n(p)$ directly. The proof of Theorem 4.1 given below is based on Corollary 3.6.

Proof. It was proved in [3, §5.3] (recalled in (2.3)–(2.4)) that the system of equations (1.2) given by $-2\pi\nabla G_n(a) = 0$ is equivalent to holomorphic equations $g^1(a) = \cdots = g^{n-1}(a) = 0$ with

$$g^i(a) = \sum_{j\neq i}^n (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)), \quad 1 \leq i \leq n-1, \tag{4.2}$$

which defines Y_n, and the non-holomorphic equation $g^n(a) = 0$ with

$$g^n(a) = \tfrac{1}{2}\phi(a) = -2\pi\sum_{i=1}^n \nabla G(a_i). \tag{4.3}$$

By (2.1), we easily obtain for $1 \le i \le n-1$,

$$g^i(a) = -2\pi \left(\sum_{j \ne i} 2G_z(a_i - a_j) - 2nG_z(a_i) + \sum_{j=1}^{n} 2G_z(a_j) \right). \tag{4.4}$$

For any i, we have

$$\nabla_i G_n(a) = \sum_{j \ne i} \nabla G(a_i - a_j) - n\nabla G(a_i).$$

By taking into account that $\nabla G \mapsto 2G_z$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $g^n = \frac{2\pi}{n} \sum_{i=1}^{n} \nabla_i G_n$, the equivalence between the map $a \mapsto g(a) := (g^1(a), \cdots, g^n(a))^T$ and $-2\pi \nabla G_n$ is induced by a real $2n \times 2n$ matrix A given by

$$A = \begin{bmatrix} 1 & & & & & 1 & \\ & 1 & & & & & -1 \\ & & \ddots & & & \vdots & \vdots \\ & & & 1 & & 1 & \\ & & & & 1 & & -1 \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ \frac{-1}{n} & & \cdots & \frac{-1}{n} & & \frac{-1}{n} & & \\ & \frac{-1}{n} & \cdots & & \frac{-1}{n} & & \frac{-1}{n} & \end{bmatrix}.$$

In other words, by considering $g^k = (\mathrm{Re} g^k, \mathrm{Im} g^k)^T$ for $1 \le k \le n-1$ and $G_z = (\mathrm{Re} G_z, \mathrm{Im} G_z)^T$, we have $2G_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nabla G$. Inserting this into (4.4), it is easy to obtain $g(a) = -2\pi A \nabla G_n(a)$. Consequently,

$$J(g)(a) = J(-2\pi A \nabla G_n)(a) = \frac{(-1)^{n-1}}{n^2} (2\pi)^{2n} \det D^2 G_n(a), \tag{4.5}$$

so it suffices to compute (the real Jacobian) $J(g)$.

Let $p \in Y_n \setminus X_n$ and consider a holomorphic parametrization $C \mapsto a(C)$ of a branch of Y_n near p, where p corresponds to $C = 0$. Notice that if p is not a singular point of Y_n, i.e. B_p is a simple zero of $\ell_n(B) = 0$, then there is only one branch of Y_n near p and the map $C \to a(C)$ is unique.

We denote

$$C = u + \sqrt{-1}v, \quad a_k = x^k + \sqrt{-1}y^k, \quad g^k = U^k + \sqrt{-1}V^k, \quad 1 \le k \le n.$$

Along Y_n we have by chain rule (denote $g^i_j = \partial g^i / \partial a_j$)

$$0 = \frac{\partial g^i}{\partial C} = \sum_{j=1}^{n} g^i_j \frac{\partial a_j}{\partial C}, \qquad 1 \le i \le n-1. \tag{4.6}$$

If g^n is also holomorphic, then (4.6) can be used to evaluate the "complex determinant" $\det D^{\mathbb{C}} g = \det(g^i_j)$ by elementary column operations. For example, if $\partial a_k/\partial C \neq 0$ then we may eliminate all the entries of the k-th column except the last (n-th) one. The case $k = n$ reads as:

$$\det D^{\mathbb{C}} g = \det(g^i_j)_{i,j=1}^{n-1} \times \frac{\partial g^n}{\partial C} \times \Big(\frac{\partial a_n}{\partial C}\Big)^{-1}. \tag{4.7}$$

In the current case $g^n = \frac{1}{2}\phi$ is not holomorphic (see (3.20) for the additional linear term $-2\pi \sum_k y^k/b$ for $V^n = \frac{1}{2}\phi_2$). The same argument via implicit functions still applies if we work with the real components U^k, V^k and real variables x^k, y^k and u, v instead.

More precisely, (4.6) takes the real form: For $1 \le i \le n-1$,

$$0 = \begin{bmatrix} U^i_u & U^i_v \\ V^i_u & V^i_v \end{bmatrix} = \sum_{k=1}^n \begin{bmatrix} U^i_{x^k} & U^i_{y^k} \\ V^i_{x^k} & V^i_{y^k} \end{bmatrix} \begin{bmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{bmatrix}. \tag{4.8}$$

The two rows are equivalent by the Cauchy–Riemann equation.

The elementary column operation on the $2n \times 2n$ real jacobian matrix Dg is now replaced by the right multiplication with the matrix

$$R_n := \begin{bmatrix} 1 & & & x^1_u & x^1_v \\ & 1 & & y^1_u & y^1_v \\ & & \ddots & \vdots & \vdots \\ & & & x^n_u & x^n_v \\ & & & y^n_u & y^n_v \end{bmatrix}.$$

In fact we may do so for any $(2k-1, 2k)$-th pair of columns—since

$$\begin{vmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{vmatrix} = |a'_k(0)|^2 \neq 0, \infty$$

by Lemma 3.3, and get a similar matrix R_k. We take $R = R_n$ below.

Denote by $D'g$ the principal $2(n-1) \times 2(n-1)$ sub-matrix of Dg. Notice from (4.3) that

$$\tfrac{1}{2} D(\phi \circ a) = \begin{bmatrix} U^n_u & U^n_v \\ V^n_u & V^n_v \end{bmatrix} = \sum_{k=1}^n \begin{bmatrix} U^n_{x^k} & U^n_{y^k} \\ V^n_{x^k} & V^n_{y^k} \end{bmatrix} \begin{bmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{bmatrix},$$

which is precisely the right bottom 2×2 sub-matrix of $(Dg)R$. Hence it follows from (4.8) that

$$(Dg)R = \begin{bmatrix} D'g & 0 \\ * & \frac{1}{2} D(\phi \circ a) \end{bmatrix},$$

which can be used to calculate the determinant:

$$\det Dg \, \det R = \det((Dg)R) = \det D'g \, \det \tfrac{1}{2} D(\phi \circ a).$$

By (4.5) and Corollary 3.6, we get

$$\det D^2 G_n(p) = \frac{(-1)^n n^2 e^{-c}}{4 \mathrm{Im}\, \tau \, (2\pi)^{2n}} \frac{|\det D'^{\mathbb{C}} g(p)|^2}{|a'_n(0)|^2} D(p),$$

where $D'^{\mathbb{C}}g$ is the principal $(n-1)\times(n-1)$ sub-matrix of $D^{\mathbb{C}}g$. Here we used $\det D'g(p) = |\det D'^{\mathbb{C}}g(p)|^2$ because g^k is holomorphic for any $1 \le k \le n-1$. Thus

$$c_p = \frac{n^2 e^{-c}}{4\mathrm{Im}\,\tau\,(2\pi)^{2n}} \frac{|\det D'^{\mathbb{C}}g(p)|^2}{|a_n'(0)|^2} \ge 0. \tag{4.9}$$

To complete the proof of Theorem 4.1, we recall the standard Jacobian criterion for smoothness of the point $p \in Y_n$. Since $g^1 = 0, \cdots, g^{n-1} = 0$ are the defining equations for Y_n, $p \in Y_n$ is a non-singular point if and only if there is some $(n-1)\times(n-1)$ minor of the $(n-1)\times n$ matrix $D^{\mathbb{C}}\tilde{g}(p)$ which does not vanish at p, where $\tilde{g} := (g^1, \cdots, g^{n-1})^T$. Notice that (4.9) is valid for all choices of those minors (with $a_n'(0)$ being replaced by $a_k'(0)$), thus $p \in Y_n$ is non-singular is indeed equivalent to $\det D'^{\mathbb{C}}g(p) \ne 0$ (which actually implies that any $(n-1)\times(n-1)$ minor does not vanish at p). Since $p \in Y_n \setminus X_n$ is a branch point and Y_n is defined by the hyperelliptic equation $C^2 = \ell_n(B)$, this is precisely the case when B_p is a simple zero of $\ell_n(B) = 0$. The proof is complete. □

Example 4.2 (The case $n = 2$). For any flat torus E_τ and $p \in Y_2(\tau) \setminus X_2(\tau)$, we compute directly the constant $c_p = c_p(\tau) \ge 0$ such that

$$\det D^2 G_2(p) = c_p D(p).$$

To serve as a consistency check with (4.9) we will not follow the procedure used in the proof of Theorem 4.1. Instead we will compute $\det D^2 G_2(p)$ directly. It will be clear that $c_p(\tau) > 0$ if $\tau \not\equiv e^{\pi/3}$ under the $\mathrm{SL}(2,\mathbb{Z})$ action.

By Example 2.5 (2), we see that the five branch points in Y_n are given by $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) \mid i \ne j\}$ and $\{(q_\pm, -q_\pm) \mid \wp(q_\pm) = \pm\sqrt{g_2/12}\}$. Note that $\wp^2(q) = \frac{1}{12}g_2$ if and only if $\wp''(q) = 0$. The only case that these five points reduce to four points is when $g_2 = 0$. This happens precisely when $\tau \equiv e^{\pi/3}$ and then $\bar{Y}_n$ becomes a singular hyperelliptic curve.

To compute the Hessian of G_2, we recall the formulae [9, (2.4) and (2.5)] for the second partial derivatives of G. Namely,

$$\det D^2 G = \frac{-1}{4\pi^2}\Big(|(\log\vartheta)_{zz}|^2 + \frac{2\pi}{b}\mathrm{Re}\,(\log\vartheta)_{zz}\Big), \tag{4.10}$$

where $b = \mathrm{Im}\,\tau$ and in terms of the Weierstrass theory

$$(\log\vartheta)_{zz}(\tfrac{1}{2}\omega_i) = -\wp(\tfrac{1}{2}\omega_i) - \eta_1 = -(e_i + \eta_1), \tag{4.11}$$

where $(\log\vartheta)_z(z) = \zeta(z) - \eta_1 z$ is used.

First we compute $D^2G(p)$ for $p = (\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$. Denote by $\frac{1}{2}\omega_k$ the third remaining half period point. Notice that $\wp(\frac{1}{2}\omega_i - \frac{1}{2}\omega_j) = \wp(\frac{1}{2}\omega_k) = e_k$. For simplicity we write

$$w_k := (\log\vartheta)_{zz}(\tfrac{1}{2}\omega_k) = -(e_k + \eta_1) = u_k + v_k i,$$

and similarly for the indices i, j. Then we have

$$D^2G_2(p) = \frac{1}{2\pi}\begin{pmatrix} -u_k+2u_i & v_k-2v_i & u_k & -v_k \\ v_k-2v_i & u_k-2u_i-\frac{2\pi}{b} & -v_k & -u_k-\frac{2\pi}{b} \\ u_k & -v_k & -u_k+2u_j & v_k-2v_j \\ -v_k & -u_k-\frac{2\pi}{b} & v_k-2v_j & u_k-2u_j-\frac{2\pi}{b} \end{pmatrix}.$$

A lengthy yet straightforward calculation shows that

$$\det D^2G_2(p) = \frac{4}{(2\pi)^4}\Big(|2e_ie_j+e_k^2-3e_k\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}\,\big(3\bar{e}_k(2e_ie_j+e_k^2-3e_k\eta_1)\big)\Big). \tag{4.12}$$

The details will be omitted here. We only note that when $\tau \in i\mathbb{R}$, all e_l's and η_1 are real numbers. Thus all the imaginary parts vanish: $v_1 = v_2 = v_3 = 0$. In this case (4.12) can be verified easily.

By (3.9) and the fact that $\wp''(\frac{1}{2}\omega_i) = 2(e_i - e_j)(e_i - e_k)$, we compute

$$\begin{aligned} c_1 &= 2\wp''(\tfrac{1}{2}\omega_i)^{-1}(e_i-e_j)^{-1} = (e_i-e_j)^{-2}(e_i-e_k)^{-1}, \\ c_2 &= (e_j-e_i)^{-2}(e_j-e_k)^{-1}, \\ s &= c_1+c_2 = -3e_k(e_i-e_j)^{-2}(e_i-e_k)^{-1}(e_j-e_k)^{-1}, \\ c_0 &= -(c_1e_i+c_2e_j) = -(2e_ie_j+e_k^2)(e_i-e_j)^{-2}(e_i-e_k)^{-1}(e_j-e_k)^{-1}. \end{aligned}$$

By Theorem 3.4, we get

$$D(p) = c(p)\Big(|2e_ie_j+e_k^2-3e_k\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}\,\big(3\bar{e}_k(2e_ie_j+e_k^2-3e_k\eta_1)\big)\Big)$$

with $c(p) = be^c|e_i-e_j|^{-2}|e_i-e_k|^{-1}|e_j-e_k|^{-1}$. Thus

$$\det D^2G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j) = c_pD(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j)$$

with

$$c_p = (4\pi^4c(p))^{-1} = e^{-c}|e_i-e_j||e_i-e_k||e_j-e_k|/(4b\pi^4) > 0. \tag{4.13}$$

Next we consider $p = (q, -q)$ with $q \in \{q_+, q_-\}$. Let $\mu = \wp(q)$. Since $\wp''(q) = 0$, we have also $\wp(2q) = -2\wp(q) = -2\mu$ by the addition (duplication) formula. Denote by $\mu = u + iv$ and $\eta_1 = s + it$. Then we have

$$D^2G_2(p) = \frac{1}{2\pi}\begin{pmatrix} -4u-s & 4v+t & 2u-s & -2v+t \\ 4v+t & 4u+s-\frac{2\pi}{b} & -2v+t & -2u+s-\frac{2\pi}{b} \\ 2u-s & -2v+t & -4u-s & 4v+t \\ -2v+t & -2u+s-\frac{2\pi}{b} & 4v+t & 4u+s-\frac{2\pi}{b} \end{pmatrix}.$$

A straightforward calculation easier than the previous case shows that the determinant is given by

$$\begin{aligned} \det D^2G_2(p) &= \frac{144}{(2\pi)^4}(u^2+v^2)\Big((u+s)^2+(v+t)^2-\frac{2\pi}{b}(u+s)\Big) \\ &= \frac{9}{\pi^4}|\wp(q)|^2\Big(|\wp(q)+\eta_1|^2 - \frac{2\pi}{b}\mathrm{Re}(\wp(q)+\eta_1)\Big). \end{aligned} \tag{4.14}$$

By (3.10), we compute easily that $c_1 = c_2 = \wp'(q)^{-2}$, $c_0 = -2\wp(q)\wp'(q)^{-2}$, and $s = c_1 + c_2 = 2\wp'(q)^{-2}$. Hence by Theorem 3.4

$$\begin{aligned} D(p) &= 4be^c|\wp'(q)|^{-4}\Big(|-\wp(q)-\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}(-\wp(q)-\eta_1)\Big) \\ &= c_p^{-1}\det D^2G_2(p), \end{aligned}$$

where

$$c_p = \frac{9e^{-c}}{4b\pi^4}|\wp(q)|^2|\wp'(q)|^4 \geq 0. \tag{4.15}$$

Since $\wp'(q) \neq 0$, $c_p > 0$ unless $\wp(q) = \pm\sqrt{g_2/12} = 0$. This is the case precisely when τ is equivalent to $e^{\pi i/3}$. We leave the simple consistency check of (4.13) and (4.15) with the general formula (4.9) to the readers.

References

[1] F. Beukers and A. Waall; *Lamé equations with algebraic solutions*. J. Diff. Equ. **197** (2004), 1–25.

[2] L.A. Caffarelli and Y. Yang; *Vortex condensation in the Chern–Simons–Higgs model: An existence theorem.* Comm. Math. Phys. **168** (1995), 321–336.

[3] C.-L. Chai, C.-S. Lin and C.-L. Wang; *Mean field equations, hyperelliptic curves, and modular forms: I.* Cambridge J. of Math. **3** (2015), 127–274.

[4] C.-C. Chen and C.-S. Lin; *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*. Comm. Pure Appl. Math. **55** (2002), 728–771.

[5] C.-C. Chen and C.-S. Lin; *Mean field equation of Liouville type with singular data: Topological degree*. Comm. Pure Appl. Math. **68** (2015), 887–947.

[6] C.-C. Chen, C.-S. Lin and G. Wang; *Concentration phenomena of two-vortex solutions in a Chern–Simons model*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **3** (2004), 367–397.

[7] K. Choe; *Asymptotic behavior of condensate solutions in the Chern–Simons–Higgs theory.* J. Math. Phys. **48** (2007).

[8] C.-S. Lin and C.-L. Wang; *Elliptic functions, Green functions and the mean field equations on tori*. Annals of Math. **172** (2010), no. 2, 911–954.

[9] C.-S. Lin and C.-L. Wang; *On the minimality of extra critical points of Green functions on flat tori*, Int. Math. Res. Notices, to appear.

[10] C.-S. Lin and S. Yan; *Existence of bubbling solutions for Chern-Simons model on a torus.* Arch. Ration. Mech. Anal. **207** (2013), 353–392.

[11] M. Nolasco and G. Tarantello; *Double vortex condensates in the Chern-Simons-Higgs theory.* Calc. Var. PDE. **9** (1999), 31–94.

[12] E.G.C. Poole; *Introduction to the theory of linear differential equations*. Oxford University Press, 1936.

[13] E.T. Whittaker and G.N. Watson; *A course of modern analysis*, *4th edition*. Cambridge University Press, 1927.

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