INVARIANCE OF QUANTUM RINGS UNDER ORDINARY FLOPS

YUAN-PIN LEE, HUI-WEN LIN, AND CHIN-LUNG WANG

ABSTRACT. For ordinary flops over a smooth base, we determine the defect of the cup product under the canonical correspondence and show that it is corrected by the small quantum product attached to the extremal ray. If the flop is of splitting type, the big quantum cohomology ring is also shown to be invariant after an analytic continuation in the Kähler moduli space.

Viewed from the context of the K-equivalence (crepant transformation) conjecture, there are two new features of our results. First, there is no semipositivity assumption on the varieties. Second, the local structure of the exceptional loci can not be deformed to any explicit (e.g. toric) geometry and the analytic continuation is nontrivial. This excludes the possibility of an *ad hoc* comparison by explicit computation of both sides.

To achieve that, we have to clear a few technical hurdles. One technical breakthrough is a *quantum Leray–Hirsch theorem* for the local models (a certain toric bundle) which extends the quantum $\mathscr{D}$ modules of Dubrovin connection on the base by a Picard–Fuchs system of the toric fibers.

Nonsplit flops as well as further applications of the quantum Leray–Hirsch theorem will be discussed in subsequent papers.

CONTENTS

0. INTRODUCTION

0.1. **Background review.** Two complex manifolds X and X' are *K-equivalent*, denoted by $X =_K X'$, if there are proper birational morphisms $(\phi, \phi') : Y \to$

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$X \times X'$ such that $\phi^* K_X = \phi'^* K_{X'}$. Major examples come from *birational minimal models* in Mori theory and especially from *birational Calabi–Yau manifolds* in the mathematical study of string theory. K-equivalent projective manifolds share the same Betti and Hodge numbers. It has been conjectured that a *canonical correspondence* $T \in A(X \times X')$ exists which induces isomorphisms of cohomology groups and preserves the *Poincaré pairing*. For a survey, see [22].

However, simple examples shows that the classical cup product is generally not preserved under $\mathscr{F}$, and this leads to new directions of study in higher dimensional birational geometry. On the other hand, according to the philosophy of *crepant transformation conjecture* and string theory, the *quantum product* should be more natural and display certain functoriality not available to the cup product among K-equivalent manifolds.

Flops are typical examples of K-equivalent birational maps:

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\psi"'] & & X' \arrow[dl, "\psi'"] \\ & \bar{X} & \end{tikzcd}$$

In fact they form the building blocks to connect birational minimal models [8]. The simplest flop is the simple P^1 flop (Atiyah flop) in dimension 3. It is known that up to deformations it generates, *locally* or *symplectically*, all K-equivalent maps for threefolds. The quantum corrections by extremal ray invariants to the cup product in the local 3 dimensional case was first noticed by Aspinwall–Morrison and Witten [26] and later globalized by Li–Ruan through the degeneration formula [17].

The higher dimensional generalizations are known as *ordinary P^r flops* (also abbreviated as "ordinary flops" or "P^r flops"). The local geometry is encoded in a triple (S, F, F') where S is a smooth variety and F, F' are two rank $r+1$ vector bundles over S. If $Z \subset X$ is the f-exceptional loci, then $\bar{\psi} : Z \cong P(F) \to S \subset \bar{X}$ with fibers spanned by the flopped curves $C \cong P^1$ and $N_{Z/X} = \bar{\psi}^* F' \otimes \mathscr{O}_Z(-1)$. Similar structure holds for $Z' \subset X'$, with F and F' exchanged. See Section 1.1 for details. (We note that the Atiyah flop corresponds to $S = \mathrm{pt}$ and $r = 1$.) Thus it is reasonable to expect that ordinary flops will also play vital roles in the study of K-equivalent maps. For example, up to complex cobordism, any K-equivalent map can be decomposed into P^1 flops [23].

The study of invariance of quantum product under ordinary flops in higher dimensions was started in [11]. The canonical correspondence is given by the graph closure $[\bar{\Gamma}_f]$ and the quantum invariance under $\mathscr{F} = [\bar{\Gamma}_f]_* : QH(X) \to QH(X')$ is proved for all *simple P^r flops*, i.e. with $S = \mathrm{pt}$. The crucial idea is to interpret *$\mathscr{F}$-invariance* in terms of *analytic continuations in Gromov–Witten theory*.

Let us explain this point in a little more details. We use [3] as our general reference for early developments in Gromov–Witten invariants. Let $\overline{M}_{g,n}(X,\beta)$ be the moduli space of stable maps from genus g nodal curves with n marked points to X, and let $e_i : \overline{M}_{g,n}(X,\beta) \to X$ be the evaluation maps. The Gromov–Witten potential

$$F_g^X(t) = \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}^X = \sum_{n\geq 0,\, \beta\in NE(X)} \frac{q^\beta}{n!} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n e_i^* t$$

is a formal function in $t \in H(X)$ and Novikov variables q^β, with $\beta \in NE(X)$, the Mori cone of effective classes of one cycles. Modulo convergence issues, it is a function on the *complexified Kähler cone* $\omega \in \mathcal{K}_X^{\mathbb{C}} := H_{\mathbb{R}}^{1,1} + i\mathcal{K}_X$ via

$$q^\beta = e^{2\pi i(\beta.\omega)}.$$

Under the canonical correspondence $\mathscr{F}$, F_g^X and $F_g^{X'}$ share the same variable $t \in H \cong H(X,\mathbb{C}) \cong H(X',\mathbb{C})$. However, $\mathscr{F}$ does not identify $NE(X)$ with $NE(X')$. Indeed, for the flopped curve classes $\ell = [C]$ (resp. $\ell' = [C']$), we have

$$\mathscr{F}\ell = -\ell' \notin NE(X').$$

By duality this implies that $\mathcal{K}_X^{\mathbb{C}} \cap \mathcal{K}_{X'}^{\mathbb{C}} = \emptyset$ in $H_{\mathbb{C}}^2$. Hence F_g^X and $F_g^{X'}$ have different *domains* and comparison can only make sense after analytic continuations over a certain compactification of $\mathcal{K}_X^{\mathbb{C}} \cup \mathcal{K}_{X'}^{\mathbb{C}} \subset H_{\mathbb{C}}^2$. (Thus the *naive Kähler moduli* $\mathcal{K}$ is usually regarded as the closure of the union of all $\mathcal{K}_{X'}^{\mathbb{C}}$'s with $X' =_K X$.) In other words, we set $\mathscr{F}q^\beta = q^{\mathscr{F}\beta}$. Then $\mathscr{F}F_g^X$ can not be a formal GW potential of X'.

In this paper, we will focus on genus zero theory, which carries a quantum product structure, or equivalently a Frobenius structure [19]. Let $\{T_\mu\}$ be a basis of H and $\{T^\mu := \sum g^{\mu\nu} T_\nu\}$ the dual basis with respect to the Poincaré pairing, where $g_{\mu\nu} = (T_\mu.T_\nu)$ and $(g^{\mu\nu}) = (g_{\mu\nu})^{-1}$ is the inverse matrix. Denote $t = \sum t^\mu T_\mu$ a general element in H. The *big quantum ring* $(QH(X), *)$ uses only the genus zero potential with 3 or more marked points:

$$T_\mu *_t T_\nu = \sum_\kappa \frac{\partial^3 F_0^X}{\partial t^\mu \partial t^\nu \partial t^\kappa}(t) T^\kappa = \sum_{\kappa,\, n\geq 0,\, \beta\in NE(X)} \frac{q^\beta}{n!} \langle T_\mu, T_\nu, T_\kappa, t^n \rangle_{0,n+3,\beta}^X T^\kappa.$$

The *Witten–Dijkgraff–Verlinde–Verlinde equation* (WDVV) guarantees that $*_t$ is a family of associative products on H parameterized by $t \in H$. Equivalently, for, it equips H a structure of *formal Frobenius manifold* H_X with a family (in $z \in \mathbb{C}^\times$) of integrable (= flat) *Dubrovin connections*

$$\nabla^z = d - z^{-1} \sum_\mu dt^\mu \otimes T_\mu *_t$$

on the tangent bundle $TH = H \times H$.

There is a natural embedding of $\mathcal{K}_X^{\mathbb{C}}$ in H. With suitable choice of coordinates we have $q^\ell = e^{2\pi i t_\ell}$ with the Kähler constraint $\operatorname{Im} t_\ell > 0$. Since now $\mathscr{F} q^\ell = q^{-\ell'}$, $\{q^\ell, q^{\ell'}\}$ serve as an atlas for P^1, the compactification of $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$. This gives the formal H an analytic P^1 direction. In [11], for simple flops the structural constants $\partial^3_{\mu\nu\kappa} F_0^X(t)$ for big quantum product are shown to be analytic (in fact algebraic) in q^ℓ. Moreover, $\mathscr{F}$ identifies H_X and $H_{X'}$ through analytic continuations over this P^1. Based on this, in [9] the Frobenius structure is further exploited to conclude analytic continuations from F_g^X to $F_g^{X'}$ for all simple flops and for all $g \geq 0$.

0.2. **Introduction to the main results.** This paper studies Gromov–Witten theory, mostly in $g = 0$, under flops over a non-trivial base. The first three sections inherit the basic structure developed in [11] for the simple-flop case, with various theoretical and technical improvements to handle the complexity arising from the geometry of (S, F, F'). The last four sections contain a number of new techniques which could be useful for later developments. They enable us to give the first result on the K-equivalence (crepant transformation) conjecture where the local structure of the exceptional loci can not be deformed to any explicit (e.g. toric) geometry and the analytic continuation is nontrivial. As far as we know, this is also the first result for which the analytic continuation is established for nontrivial Birkhoff factorization.

Since the proof is quite technical and involves many aspects of GW theory, it might be helpful to outline the major steps below. Roughly, each step below corresponds to a section in the main text.

Conventions. Throughout this paper, we work on the even cohomology $H = H^{even}$ to avoid the complications on signs. In particular, the degree always means the Chow degree. Nevertheless all our discussions and results work for the full cohomology spaces.

0.2.1. *Defect of cup product under the canonical correspondence.* Let $\{\bar{T}_i\}$ be a basis of $H(S)$ with dual basis $\{\check{\bar{T}}_i\}$. Let $h = c_1(\mathscr{O}_Z(1))$ and $H_k = c_k(Q_F)$ where $Q_F \to Z = P(F)$ is the universal quotient bundle. Similarly we define h' and H'_k on the X' side. The H_k's are of fundamental importance since

$$\mathscr{F} H_k = (-1)^{r-k} H'_k$$

and the dual basis of $\{\bar{T}_i h^j\}$ in $H(Z)$ is given by $\{\check{\bar{T}}_i H_{r-j}\}$.

Theorem 0.1 (Topological defect)**.** *Let $a_1, a_2, a_3 \in H(X)$ with $\sum \deg a_i = \dim X$. Then*

$$\begin{aligned}
&(\mathscr{F} a_1.\mathscr{F} a_2.\mathscr{F} a_3)^{X'} - (a_1.a_2.a_3)^X \\
&\quad = (-1)^r \times \textstyle\sum_{i_*, j_*} (a_1.\check{\bar{T}}_{i_1} H_{r-j_1})^X (a_2.\check{\bar{T}}_{i_2} H_{r-j_2})^X (a_3.\check{\bar{T}}_{i_3} H_{r-j_3})^X \\
&\qquad\qquad \times (s_{j_1+j_2+j_3-(2r+1)}(F + F'^*) \bar{T}_{i_1} \bar{T}_{i_2} \bar{T}_{i_3})^S,
\end{aligned}$$

where s_i is the i-th Segre class.

0.2.2. *Quantum corrections attached to the flopping extremal rays.* We then proceed to calculate the *quantum corrections* attached to the flopping extremal ray $\mathbb{N}\ell$. Using the calculation, we demonstrate that the "quantum corrected product", combining the classical product and the quantum deformation attached to the extremal ray, is $\mathscr{F}$-invariant after the analytic continuation.

The stable map moduli for the extremal ray has a bundle structure over S:

$$\begin{tikzcd} \overline{M}_{0,n}(P^r, d\ell) \arrow[r] & \overline{M}_{0,n}(Z, d\ell) \arrow[r, "e_i"] \arrow[d, "\Psi_n"'] & Z \arrow[dl, "\bar{\psi}"] \\ & S & \end{tikzcd}$$

In this case, the GW invariants on X are reduced to *twisted invariants* on Z by certain obstruction bundles. We define the fiber integral

$$\Big\langle \prod\nolimits_{i=1}^n h^{j_i} \Big\rangle_d^{/S} := \Psi_{n*}\Big(\prod\nolimits_{i=1}^n e_i^* h^{j_i}\Big) \in A^\nu(S)$$

as a *$\bar{\psi}$-relative invariant over S*, a cycle of codimension $\nu := \sum j_i - (2r + 1 + n - 3)$. The absolute invariant is obtained by the pairing on S: For $\bar{t}_i \in H(S)$,

$$\langle \bar{t}_1 h^{j_1}, \cdots, \bar{t}_n h^{j_n} \rangle_d^X = \Big(\langle h^{j_1}, \cdots, h^{j_n} \rangle_d^{/S} . \prod\nolimits_{i=1}^n \bar{t}_i \Big)^S.$$

If $\nu = 0$ then the invariant reduces to the simple case. This happens for $n = 2$ since then $j_1 = j_2 = r$. Thus we may calculate *extremal functions* based on the 2-point case by (divisorial) reconstruction. To state the result, let

$$\mathbf{f}(q) := \frac{q}{1 - (-1)^{r+1} q}$$

which satisfies the functional equation $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$.

For 3-point functions, we show that $W_\nu := \sum_{d \in \mathbb{N}} \langle h^{j_1}, h^{j_2}, h^{j_3} \rangle_d^{/S} q^d$ with $1 \le j_i \le r$ lies in $A^\nu(S)[\mathbf{f}]$ and is independent of the choices of j_i's.

Theorem 0.2 (Quantum corrections)**.** *The function W_ν is the action on $\mathbf{f}$ by a Chern classes valued polynomial in the operator $\delta = qd/dq$. It satisfies*

$$W_\nu - (-1)^{\nu+1} W'_\nu = (-1)^r s_\nu(F + F'^*).$$

This implies that the topological defect is corrected by the 3-point extremal functions. The analytic continuation for $n \ge 4$ points follows by reconstruction.

0.2.3. *Degeneration analysis.* The next step is to prove that the big quantum ring, involving all curve classes, are $\mathscr{F}$-invariant. As a first step, this statement is reduced to a corresponding one on f-special descendent invariants on the *projective local models*

$$X_{loc} := \tilde{E} = P(N_{Z/X} \oplus \mathscr{O}) \xrightarrow{p} S$$

and

$$X'_{loc} := \tilde{E}' = P(N_{Z'/X'} \oplus \mathscr{O}) \xrightarrow{p'} S$$

by *degeneration analysis*.

To compare GW invariants of non-extremal classes, the application of *degeneration formula* and *deformation to the normal cone* are well suited for ordinary flops with base S. It reduces the problem to local models with induced flop $f : \tilde{E} \dashrightarrow \tilde{E}'$. The reduction has two steps. The first reduces the problem to *relative local* invariants $\langle A \mid \varepsilon, \mu \rangle^{(\tilde{E},E)}$ where $E \subset \tilde{E}$ is the infinity divisor. The second is a further reduction back to *absolute* local invariants, with possibly descendent insertions coupled to E, called *f-special type*.

The local model $\bar{p} := \bar{\psi} \circ p : \tilde{E} \to S$ and the flop f are all over S, with simple case as fibers. In particular, the kernel of $\bar{p}_* : N_1(\tilde{E}) \to N_1(S)$ is spanned by the p-fiber line class γ and $\bar{\psi}$-fiber line class ℓ. $\mathscr{F}$ is compatible with $\bar{p}$. Namely

$$\begin{tikzcd} N_1(\tilde{E}) \arrow[rr, "\mathscr{F}"] \arrow[dr, "\bar{p}_* \oplus d_2"'] & & N_1(\tilde{E}') \arrow[dl, "\bar{p}'_* \oplus d'_2"] \\ & N_1(S) \oplus \mathbb{Z} & \end{tikzcd}$$

is commutative. Here we write a class β in $N_1(\tilde{E})$ as $\beta_S + d\ell + d_2\gamma$ with some β_S in $N_1(S)$ and $d, d_2 \in \mathbb{Z}$. Thus the functional equation of a generating series $\langle A \rangle$ is equivalent to those of its various subseries (fiber series) $\langle A \rangle_{\beta_S, d_2}$ labeled by $NE(S) \oplus \mathbb{Z}$.

Theorem 0.3 (Degeneration reduction)**.** *To prove $\mathscr{F}\langle \alpha \rangle_g^X \cong \langle \mathscr{F}\alpha \rangle_g^{X'}$ for all $\alpha \in H(X)^{\oplus n}$, $g \le g_0$, it is enough to prove the local case $f : \tilde{E} \to \tilde{E}'$ for descendent invariants of f-special type:*

$$\mathscr{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle_{g,\beta_S,d_2}^{\tilde{E}} \cong \langle \mathscr{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle_{g,\beta_S,d_2}^{\tilde{E}'}$$

for any $A \in H(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$, $\varepsilon_j \in H(E) \subset H(\tilde{E})$, $g \le g_0$, $\beta_S \in NE(S)$ and $d_2 \ge 0$.

0.2.4. *Further reduction to the big quantum ring/quasi-linearity on the local models.* While the degeneration reduction works for higher genera, for $g = 0$ more can be said. Using the *topological recursion relation* (TRR) and the *divisor axiom* (for descendent invariants), the $\mathscr{F}$-invariance for f-special invariants can be completely reduced to the $\mathscr{F}$-invariance of big quantum rings for local models.

We then employ the *divisorial reconstruction* [13] and the WDVV equation to make a further reduction to an $\mathscr{F}$-invariance statement about *elementary* f-special invariants with at most one special insertion.

To state the result, we assume now $X = X_{loc} = \tilde{E}$. Since $X \to S$ is a double projective bundle, $H(X)$ is generated by $H(S)$ and the relative hyperplane classes h for $Z \to S$ and ξ for $X \to Z$. This leads to another useful reduction: By moving all the classes h, ξ and ψ into the last insertion (divisorial reconstruction), the problem is reduced to the case

$$\langle \bar{t}_1, \ldots, \bar{t}_{n-1}, \bar{t}_n \tau_k h^j \xi^i \rangle^X_{\beta_S, d_2}$$

with $\bar{t}_l \in H(S)$, $d_2 \in \mathbb{Z}$, where $k \neq 0$ only if $i \neq 0$.

By a further application of WDVV equations, the $\mathscr{F}$-invariance can always be reduced to the case $i \neq 0$ even if $k = 0$. Since ξ is the class of infinity divisor which is within the isomorphism loci of the flop, such an $\mathscr{F}$-invariance statement is intuitively plausible. We call it the *type I quasi-linearity* property (c.f. Theorem 4.5).

The above steps furnish a complete reduction to projective local models X_{loc}, which works for any F and F'.

To proceed, notice that these descendent invariants are encoded by their generating function, i.e. the so called (big) J function: For $\tau \in H(X)$,

$$J^X(\tau, z^{-1}) := 1 + \frac{\tau}{z} + \sum_{\beta, n, \mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau, \cdots, \tau \right\rangle^X_{0, n+1, \beta}.$$

The determination of J usually relies on the existence of $\mathbb{C}^\times$ actions. Certain localization data I_β coming from the stable map moduli are of hypergeometric type. For "good" cases, say $c_1(X)$ is semipositive and $H(X)$ is generated by H^2, $I(t) = \sum I_\beta q^\beta$ determines $J(\tau)$ on the small parameter space $H^0 \oplus H^2$ through the "classical" *mirror transform* $\tau = \tau(t)$. For a simple flop, $X = X_{loc}$ is indeed semi-Fano toric and the classical Mirror Theorem (of Lian–Liu–Yau and Givental) is sufficient [11]. (It turns out that $\tau = t$ and $I = J$ on $H^0 \oplus H^2$.)

For general base S with given $QH(S)$, the determination of $QH(P)$ for a projective bundle $P \to S$ is far more involved. To allow *fiberwise localization* to determine the structure of GW invariants of X_{loc}, the bundles F and F' are then assumed to be split bundles.

0.2.5. *Birkhoff factorization and generalized mirror transformation.* The second half of this paper considers ordinary flops of *splitting type*, namely $F \cong \bigoplus_{i=0}^r L_i$ and $F' \cong \bigoplus_{i=0}^r L'_i$ for some line bundles L_i and L'_i on S. The splitting assumption allows us to apply the $\mathbb{C}^\times$ localizations along the fibers of the toric bundle $X \to S$. Using this and other sophisticated technical tools, J. Brown (and A. Givental) [1] proved that the *hypergeometric modification*

$$I^X(D, \bar{t}, z, z^{-1}) := \sum_\beta q^\beta e^{\frac{D}{z} + (D.\beta)} I^{X/S}_\beta(z, z^{-1}) \bar{\psi}^* J^S_{\beta_S}(\bar{t}, z^{-1})$$

lies in Givental's *Lagrangian cone* generated by $J^X(\tau, z^{-1})$. Here $D = t^1 h + t^2 \xi$, $\bar{t} \in H(S)$, $\beta_S = \bar{\psi}_* \beta$, and the explicit form of $I_\beta^{X/S}$ is given in Section 6.2.

Based on Brown's theorem, we prove the following. (See § 5 for notations on higher derivatives $\partial^{z\mathbf{e}}$'s.)

Theorem 0.4 (BF/GMT). *There is an unique matrix factorization*

$$\partial^{z\mathbf{e}} I(z, z^{-1}) = z\nabla J(z^{-1}) B(z),$$

called the Birkhoff factorization *(BF) of I, valid along $\tau = \tau(D, \bar{t}, q)$.*

BF can be stated in another way. There is a recursively defined polynomial differential operator $P(z, q; \partial) = 1 + O(z)$ in t^1, t^2 and $\bar{t}$ such that

$$J(z^{-1}) = P(z, q; \partial) I(z, z^{-1}).$$

In other words, P removes the z-polynomial part of I in the $NE(X)$-adic topology. In this form, the *generalized mirror transform* (GMT)

$$\tau(D, \bar{t}, q) = D + \bar{t} + \sum_{\beta \neq 0} q^\beta \tau_\beta(D, \bar{t})$$

is the coefficient of z^{-1} in $J = PI$.

0.2.6. *Hypergeometric modification and $\mathscr{D}$ modules.* In principle, knowing BF, GMT and GW invariants on S allows us to calculate all $g = 0$ invariants on X and X' by reconstruction. These data are in turn encoded in the I-functions. One might be tempted to prove the $\mathscr{F}$-invariance by comparing I^X and $I^{X'}$. While they are rather symmetric-looking, the defect of cup product implies $\mathscr{F} I^X \neq I^{X'}$ and the comparison via tracking the defects of ring isomorphism becomes hopelessly complicated. This can be overcome by studying a more "intrinsic" object: the cyclic $\mathscr{D}$ module $\mathscr{M}_J = \mathscr{D} J$, where $\mathscr{D}$ denotes the ring of differential operators on H with suitable coefficients.

It is well known (by TRR) that $(z\partial_\mu J)$ forms a *fundamental solution matrix* of the Dubrovin connection: Namely we have the *quantum differential equations* (QDE)

$$z\partial_\mu z\partial_\nu J = \sum_\kappa \tilde{C}_{\mu\nu}^\kappa(t)\, z\partial_\kappa J,$$

where $\tilde{C}_{\mu\nu}^\kappa(t) = \sum_\iota g^{\kappa\iota} \partial^3_{\mu\nu\iota} F_0(t)$ are the structural constants of $*_t$. This implies that $\mathscr{M}$ is a *holonomic* $\mathscr{D}$ module of length $N = \dim H$. For I we consider a similar $\mathscr{D}$ module $\mathscr{M}_I = \mathscr{D} I$. The BF/GMT theorem furnishes a change of basis which implies that $\mathscr{M}_I$ is also holonomic of length N.

The idea is to go backward: To find $\mathscr{M}_I$ first and then transform it to $\mathscr{M}_J$. We do not have similar QDE since I does not have enough variables. Instead we construct higher order *Picard–Fuchs equations* $\Box_\ell I = 0$, $\Box_\gamma I = 0$ in divisor variables, with the nice property that "up to analytic continuations" they generate $\mathscr{F}$-invariant ideals:

$$\mathscr{F} \langle \Box_\ell^X, \Box_\gamma^X \rangle \cong \langle \Box_{\ell'}^{X'}, \Box_{\gamma'}^{X'} \rangle.$$

0.2.7. *Quantum Leray–Hirsch and the conclusion of the proof.* Now we want to determine $\mathscr{M}_I$. While the derivatives along the fiber directions are determined by the Picard–Fuchs equations, we need to find the derivatives along the base direction. Write $\bar{t} = \sum \bar{t}^i \bar{T}_i$. This is achieved by *lifting* the QDE on $QH(S)$, namely

$$z\partial_i z\partial_j J^S = \sum_k \bar{C}_{ij}^k(\bar{t})\, z\partial_k J^S,$$

to a differential system on $H(X)$. A key concept needed for such a lifting is the I-minimal lift of a curve class $\beta_S \in NE(S)$ to $\beta_S^I \in NE(X)$. Various lifts of curve classes are discussed in Section 6. See in particular Definition 6.7.

Using Picard–Fuchs and the lifted QDE, we show that $\mathscr{F}\mathscr{M}_{I^X} \cong \mathscr{M}_{I^{X'}}$.

Theorem 0.5 (Quantum Leray–Hirsch)**.**

(1) *(I-Lifting) The quantum differential equation on $QH(S)$ can be lifted to $H(X)$ as*

$$z\partial_i\, z\partial_j I = \sum_{k,\beta_S} q^{\beta_S^I} e^{(D.\beta_S^I)} \bar{C}_{ij,\beta_S}^k(\bar{t})\, z\partial_k D_{\beta_S^I}(z) I,$$

where $D_{\beta_S^I}(z)$ is an operator depending only on β_S^I. Any other lifting is related to it modulo the Picard–Fuchs system.

(2) *Together with the Picard–Fuchs $\Box_\ell$ and $\Box_\gamma$, they determine a first order matrix system under the naive quantization $\partial^{z\mathbf{e}}$ (Definition 7.7) of canonical basis (Notations 7.1) $T_{\mathbf{e}}$'s of $H(X)$:*

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a(z,q), \qquad \text{where } t^a = t^1,\, t^2 \text{ or } \bar{t}^i.$$

(3) *The system has the property that for any fixed $\beta_S \in NE(S)$, the coefficients are formal functions in $\bar{t}$ and polynomial functions in $q^\gamma e^{t^2}$, $q^\ell e^{t^1}$ and $\mathbf{f}(q^\ell e^{t^1})$.*

(4) *The system is $\mathscr{F}$-invariant.*

The final step is to go from $\mathscr{M}_I$ to $\mathscr{M}_J$. From the perspective of $\mathscr{D}$ modules, the BF can be considered as a *gauge transformation*. The defining property $(\partial^{z\mathbf{e}} I) = (z\nabla J)B$ of B can be rephrased as

$$z\partial_a(z\nabla J) = (z\nabla J)\tilde{C}_a$$

such that

$$\tilde{C}_a = (-z\partial_a B + BC_a)B^{-1} \tag{0.1}$$

is independent of z.

This formulation has the advantage that all objects in (0.1) are $\mathscr{F}$-invariant (while I and J are not). It is therefore easier to first establish the $\mathscr{F}$-invariance of C_a's and use it to derive the $\mathscr{F}$-invariance of BF and GMT.

Theorem 0.6 (Quantum invariance)**.** *For ordinary flops of splitting type, the big quantum cohomology ring is invariant up to analytic continuations.*

By the reduction above, this is equivalent to the quasi-linearity property of the local models. This completes the outline.

Results in this paper had been announced, in increasing degree of generalities, by the authors in various conferences during 2008-2010; see e.g. [18, 24, 12] where more example-studies can be found.

In a subsequent work, we will apply ideas in *algebraic cobordism* of bundles on varieties [14] to remove the splitting assumption.

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1. DEFECT OF THE CLASSICAL PRODUCT

1.1. **Cohomology correspondence for P^r flops.** We recall the construction of ordinary flops in [11] to fix notations.

Let X be a smooth complex projective manifold and $\psi : X \to \bar{X}$ a flopping contraction in the sense of minimal model theory, with $\bar{\psi} : Z \to S$ the restriction map on the exceptional loci. Assume that

(i) $\bar{\psi}$ equips Z with a P^r-bundle structure $\bar{\psi} : Z = P(F) \to S$ for some rank $r+1$ vector bundle F over a smooth base S,

(ii) $N_{Z/X}|_{Z_s} \cong \mathscr{O}_{P^r}(-1)^{\oplus(r+1)}$ for each $\bar{\psi}$-fiber Z_s, $s \in S$.

Then there is another rank $r+1$ vector bundle F' over S such that

$$N_{Z/X} \cong \mathscr{O}_{P(F)}(-1) \otimes \bar{\psi}^* F'.$$

We may blow up X along Z to get $\phi : Y \to X$. The exceptional divisor

$$E = P(N_{Z/X}) \cong P(\bar{\psi}^* F') = \bar{\psi}^* P(F') = P(F) \times_S P(F')$$

is a $P^r \times P^r$-bundle over S. We may then blow down E along another fiber direction $\phi' : Y \to X'$ to get another contraction $\psi' : X' \to \bar{X}$, with exceptional loci $\bar{\psi}' : Z' = P(F') \to S$ and $N_{Z'/X'}|_{\bar{\psi}'-\text{fiber}} \cong \mathscr{O}_{P^r}(-1)^{\oplus(r+1)}$.

We call the $f : X \dashrightarrow X'$ an *ordinary P^r flop*. The various sets and maps are summarized in the following commutative diagram.

$$\begin{tikzcd}
 & E \arrow[r, hook, "j"] \arrow[dl, "\bar{\phi}"'] \arrow[dr, "\bar{\phi}'"] & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\
Z \arrow[r, hook, "i"] \arrow[dr, "\bar{\psi}"'] & X \arrow[dr, "\psi"] & Z' \arrow[r, hook, "i'"] \arrow[dl, "\bar{\psi}'"'] & X' \arrow[dl, "\psi'"] \\
 & S \arrow[r, hook, "j'"] & \overline{X} &
\end{tikzcd}$$

where the normal bundle of E in Y is

$$N_{E/Y} = \bar{\phi}^* \mathscr{O}_{P(F)}(-1) \otimes \bar{\phi}'^* \mathscr{O}_{P(F')}(-1).$$

First of all, we have found a *canonical* correspondence between the cohomology groups of X and X'.

Theorem 1.1. [11] *For an ordinary P^r flop $f : X \dashrightarrow X'$, the graph closure $T := [\bar{\Gamma}_f] \in A(X \times X')$ identifies the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X', i.e. $\hat{X} \cong \hat{X}'$ via $T^t \circ T = \Delta_X$ and $T \circ T^t = \Delta_{X'}$. In particular, $\mathscr{F} := T_* : H(X) \to H(X')$ preserves the Poincaré pairing on cohomology groups.*

In practice, the correspondence T associates a map on Chow groups:

$$\mathscr{F} : A(X) \to A(X'); \quad W \mapsto p'_*(\bar{\Gamma}_f.p^*W) = \phi'_*\phi^*W$$

where p (resp. p') is the projection map from $X \times X'$ to X (resp. X').

Secondly, parallel to the procedure in [11], we need to determine the explicit formulae for the associated map $\mathscr{F}$ restricted to $A(Z)$. The Leray–Hirsch theorem says that

$$A(Z) = \bar{\psi}^*A(S)[h]/f_F(h)$$

where $f_F(\lambda) = \lambda^{r+1} + \bar{\psi}^*c_1(F)\lambda^r + \cdots + \bar{\psi}^*c_{r+1}(F)$ is the Chern polynomial of F and $h = c_1(\mathscr{O}_{P(F)}(1))$. Thus a class $\alpha \in A(Z)$ has the form $\alpha = \sum_{i=0}^r h^i\bar{\psi}^*a_i$ for some $a_i \in A(S)$.

By the formulae for pull-back from the intersection theory, it is easy to see that if $a \in A_k(Z)$ then

$$\phi^*(i_*a) = j_*\Big(c_r(\mathscr{E}).\bar{\phi}^*a\Big)$$

in $A_k(Y)$ where $\mathscr{E}$ is the excess normal bundle defined by

$$0 \to N_{E/Y} \to \phi^*N_{Z/X} \to \mathscr{E} \to 0.$$

By the functoriality of pull-back and push-forward together with the above formula, we can conclude from $\mathscr{F}(i_*(\sum h^i\bar{\psi}^*a_i)) = \sum \mathscr{F}(i_*(h^i))i'_*\bar{\psi}'^*a_i$ that $\mathscr{F}$ restricted to $A(Z)$ is $A(S)$-linear. Here we identify the ring $A(S)$ with its isomorphic images in $A(Z)$ and $A(Z')$ via $\bar{\psi}^*$ and $\bar{\psi}'^*$ respectively.

Under such an identification, we will abuse notations to denote $c_i(F)$, $\bar{\psi}^*c_i(F)$ and $\bar{\psi}'^*c_i(F)$ by the same symbol c_i. Similarly we denote $c_i(F')$, $\bar{\psi}^*c_i(F')$ and $\bar{\psi}'^*c_i(F')$ by c'_i. We use this abbreviation for any class in $A(S)$. And for $\alpha \in A(Z)$ we often omit i_* from $i_*\alpha$ when α is regarded as a class in $A(X)$, unless possible confusion should arise. Similarly, we do these for $\alpha' \in A(Z') \hookrightarrow A(X')$.

The $A(S)$-linearity of $\mathscr{F}$ restricted to $A(Z)$ allows us to focus on the study of a basis for $A(Z)$ over $A(S)$. Recall that for a simple P^r flop we have the basic transformation formula $\mathscr{F}(h^k) = (-1)^{r-k}h'^k$. Unfortunately, for a general P^r flop, this does not hold any more, so a better candidate has to be sought out.

Note that the key ingredient in the pull-back formula is $c_r(\mathscr{E})$. From the Euler sequence

$$0 \to \mathscr{O}_{Z'}(-1) \to \bar{\psi}'^*F' \to Q_{F'} \to 0$$

and the short exact sequence defining the excess normal bundle $\mathscr{E}$, we get $\mathscr{E} = \bar{\phi}^* \mathscr{O}_{P(F)}(-1) \otimes \bar{\phi}'^* Q_{F'}$. A simple computation leads to

$$c_r(\mathscr{E}) = (-1)^r (\bar{\phi}^* h^r - \bar{\phi}'^* H'_1 \bar{\phi}^* h^{r-1} + \bar{\phi}'^* H'_2 \bar{\phi}^* h^{r-2} + \cdots + (-1)^r \bar{\phi}'^* H'_r),$$

where $H'_k = c_k(Q_{F'})$. Explicitly,

$$H'_k = h'^k + c'_1 h'^{k-1} + \cdots + c'_k$$

where $h' = c_1(\mathscr{O}_{P(F')}(1))$. Similarly, we denote

$$H_k = c_k(Q_F) = h^k + c_1 h^{k-1} + \cdots + c_k.$$

Notice that $H_k = 0 = H'_k$ for $k > r$. Finally, we find that H_k, H'_k turn out to be the correct choice.

Proposition 1.2. *For all positive integers $k \le r$,*

$$\mathscr{F}(H_k) = (-1)^{r-k} H'_k.$$

Proof. First of all, we have the basic identities: $h^{r+1} + c_1 h^r + \cdots + c_{r+1} = 0$, $\bar{\phi}'_* \bar{\phi}^* h^i = 0$ for all $i < r$ and $\bar{\phi}'_* \bar{\phi}^* h^r = [Z']$. The latter two follow from the definitions and dimension consideration.

In order to determine $\mathscr{F}(H_k) = \bar{\phi}'_*(c_r(\mathcal{E}).\bar{\phi}^* H_k)$, we need to take care of the class $\bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^i . \bar{\phi}^* H_k)$ with $0 \le i \le r$, here $H'_0 := 1$.

If $i > r - k$, then

$$\begin{aligned} \bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^i . \bar{\phi}^* H_k) &= \bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* (h^{k+i} + c_1 h^{k+i-1} + \cdots + c_k h^i)) \\ &= -\bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* (c_{k+1} h^{i-1} + c_{k+2} h^{i-2} + \cdots + c_{r+1} h^{i+k-r-1})) = 0 \end{aligned}$$

since the power in h is at most $i - 1 < r$.

If $i < r - k$, then again $\bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^i . \bar{\phi}^* H_k) = 0$ since the power in h is at most $i + k < r$.

For the remaining case $i = r - k$,

$$\bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^i . \bar{\phi}^* H_k) = \bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^r) = H'_{r-i} = H'_k.$$

We conclude that

$$\mathscr{F}(H_k) = (-1)^r \sum_{i=0}^{r} (-1)^{r-i} \bar{\phi}'_*(\bar{\phi}'^* H'_{r-i} \bar{\phi}^* h^i . \bar{\phi}^* H_k) = (-1)^{r-k} H'_k.$$

□

Remark 1.3. Unlike simple P^r flops, here the image class of h^k under $\mathscr{F}$ looks more complicated. As a simple corollary of the above proposition, we may show, by induction on k, that for all $k \in \mathbb{N}$,

$$\mathscr{F}(h^k) = (-1)^{r-k} (a_0 h'^k + a_1 h'^{k-1} + \cdots + a_k) \in A(Z')$$

where $a_0 = 1$ and $a_k \in A(S)$ are determined by the recursive relations:

$$c'_k = a_k - c_1 a_{k-1} + c_2 a_{k-2} + \cdots + (-1)^k c_k.$$

And symmetrically
$$\mathscr{F}^*(h'^k) = (-1)^{r-k}(a'_0 h^k + a'_1 h^{k-1} + \cdots + a'_k) \in A(Z)$$
with $a'_0 = 1,\ \ a'_k = c'_1 a'_{k-1} - c'_2 a'_{k-2} + \cdots + (-1)^{k-1} c'_k + c_k$.

To put these formulae into perspective, we consider the virtual bundles
$$A := F' - F^*; \qquad A' := F - F'^*.$$
Then $a_k = c_k(A)$ and $a'_k = c_k(A')$. Notice that since a_k and a'_k are Chern classes of virtual bundles, they may survive even for $k \geq r+1$.

It is also interesting to notice that the explicit formula reduces to
$$\mathscr{F}(h^k) = (-1)^{r-k} h'^k$$
without lower order terms precisely when $F' = F^*$, the dual of F.

1.2. **Triple product.** Let $\{\bar{T}_i^k\}$ be a basis of $H^{2k}(S)$ and $\{\check{\bar{T}}_i^k\} \subset H^{2(s-k)}(S)$ be its dual basis where $s = \dim S$. It is an easy but quite crucial discovery that the dual basis of the canonical basis $\{\bar{T}_i^k h^j\}$ in $H(Z)$ can be expressed in terms of $\{H_k\}_{k\geq 0}$.

Lemma 1.4. *The dual basis of* $\{\bar{T}_i^{k-j} h^j\}_{j\leq \min\{k,r\}}$ *in* $H^{2k}(Z)$ *is* $\{\check{\bar{T}}_i^{k-j} H_{r-j}\}_{j\leq \min\{k,r\}}$ *in* $H^{2(r+s-k)}(Z)$.

Proof. We have to check that $(\bar{T}_i^{k-j} h^j . \check{\bar{T}}_i^{k-j} H_{r-j}) = 1$ and $(\bar{T}_i^{k-j} h^j . \check{\bar{T}}_i^{k-j'} H_{r-j'}) = 0$ for any $j \neq j'$. Indeed,
$$(\bar{T}_i^{k-j} h^j . \check{\bar{T}}_i^{k-j} H_{r-j}) = \bar{T}^s(h^r + c_1 h^{r-1} + \cdots) = \bar{T}^s h^r = 1$$
since $\bar{T}^s c_i = 0$ for all $i \geq 1$ by degree consideration.

Notations 1.5. When X is a bundle over S, classes in $H(S)$ may be considered as classes in $H(X)$ by the obvious pullback, which we often omit. To avoid confusion, we consistently employ the notation $\check{\bar{T}}_i$ as the dual class of $\bar{T}_i \in H(S)$ with respect to the Poincaré pairing in S. The "raised" index form, e.g. T^μ as the dual of $T_\mu \in H(X)$, is reserved for duality with respect to Poincaré pairing in X.

Now if $j' > j$ then
$$k - j + (s - (k - j')) = s + (j' - j) > s,$$
which implies that $\bar{T}_i^{k-j} \check{\bar{T}}_i^{k-j'} = 0$. Conversely, if $j' < j$ then $\bar{T}_i^{k-j} \check{\bar{T}}_i^{k-j'} \in H^{2(s-(j-j'))}(S)$ and
$$\begin{aligned} h^j H_{r-j'} &= h^{r+(j-j')} + c_1 h^{r+(j-j')-1} + \cdots + c_{r-j'} h^j \\ &= -c_{r-j'+1} h^{j-1} - \cdots - c_{r+1} h^{j-j'-1}. \end{aligned}$$
Again since
$$(s - (j - j')) + (r - j' + z) = s + (r + z - j) > s$$
for $z \geq 1$, we have $\bar{T}_i^{k-j} \check{\bar{T}}_i^{k-j'} c_{r-j'+z} h^{j-z-1} = 0$. The result follows. $\square$

Now we can determine the difference of the pullback classes of a and $\mathscr{F}a$ as follows.

Proposition 1.6. *For a class $a \in H^{2k}(X)$, let $a' = \mathscr{F}a$ in X'. Then*

$$\phi'^* a' = \phi^* a + j_* \sum_i \sum_{1 \le j \le \min\{k,r\}} (a.\check{\bar{T}}_i^{k-j} H_{r-j}) \bar{T}_i^{k-j} \frac{x^j - (-y)^j}{x+y}$$

where $x = \bar{\phi}^* h$, $y = \bar{\phi}'^* h'$.

Proof. Recall that

$$N_{E/Y} = \bar{\phi}^* \mathscr{O}_Z(-1) \otimes \bar{\phi}'^* \mathscr{O}_{Z'}(-1)$$

and hence $c_1(N_{E/Y}) = -(x+y)$. Since the difference $\phi'^* a' - \phi^* a$ has support in E, we may write $\phi'^* a' - \phi^* a = j_* \lambda$ for some $\lambda \in H^{2(k-1)}(E)$. Then

$$(\phi'^* a' - \phi^* a)|_E = j^* j_* \lambda = c_1(N_{E/Y})\lambda = -(x+y)\lambda.$$

Notice that while the inclusion-restriction map $j^* j_*$ on $H(E)$ may have non-trivial kernel, elements in the kernel never occur in $\phi'^* a' - \phi^* a$ by the Chow moving lemma. Indeed if $j^* j_* \lambda \equiv j_* \lambda|_E = 0$ then $j_* \lambda$ is rationally equivalent to a cycle λ' disjoint from E. Applying ϕ'_* to the equation

$$\phi'^* a' - \phi^* a = j_* \lambda \sim \lambda'$$

gives rise to

$$\phi'_* \lambda' \sim \phi'_* \phi'^* a' - \phi'_* \phi^* a = a' - a' = 0.$$

This leads to $\lambda' \sim 0$ on Y.

Hence

$$\lambda = -\frac{1}{x+y}((\phi'^* a')|_E - (\phi^* a)|_E) = -\frac{1}{x+y}(\bar{\phi}'^*(a'|_{Z'}) - \bar{\phi}^*(a|_Z)).$$

By the above lemma, we get

$$\begin{aligned} \bar{\phi}^*(a|_Z) &= \bar{\phi}^*(\sum_i \sum_{j \le \min\{k,r\}} (a.\check{\bar{T}}_i^{k-j} H_{r-j}) \bar{T}_i^{k-j} h^j) \\ &= \sum_i \sum_{j \le \min\{k,r\}} (a.\check{\bar{T}}_i^{k-j} H_{r-j}) \bar{T}_i^{k-j} x^j. \end{aligned}$$

Similarly, we have

$$\bar{\phi}'^*(a'|_{Z'}) = \sum_i \sum_{j \le \min\{k,r\}} (a'.\check{\bar{T}}_i^{k-j} H'_{r-j}) \bar{T}_i^{k-j} y^j.$$

Since $\mathscr{F}$ preserves the Poincaré pairing,

$$(a'.\check{\bar{T}}_i^{k-j} H'_{r-j}) = (\mathscr{F}a.\mathscr{F}((-1)^{r-(r-j)} \check{\bar{T}}_i^{k-j} H_{r-j})) = (-1)^j (a.\check{\bar{T}}_i^{k-j} H_{r-j}).$$

Putting these together, we obtain

$$\lambda = \sum_i \sum_{1 \le j \le \min\{k,r\}} (a.\check{\bar{T}}_i^{k-j} H_{r-j}) \bar{T}_i^{k-j} \frac{x^j - (-y)^j}{x+y}.$$

□

Remark 1.7. Notice that since the power in x (and in y) is at most $r-1$, the class λ clearly contains non-trivial $\bar{\phi}$ and $\bar{\phi}'$ fiber directions. Thus this proposition in particular gives rise to an alternative proof of equivalence of Chow motives under ordinary flops (Theorem 0.2). Indeed this is precisely the quantitative version of the original proof in [11].

Now we may compare the triple products of classes in X and X'.

Theorem 1.8 (= Theorem 0.1)**.** *Let $a_i \in H^{2k_i}(X)$ for $i = 1,2,3$ with $k_1+k_2+k_3 = \dim X = s+2r+1$. Then*

$$(\mathscr{F}a_1.\mathscr{F}a_2.\mathscr{F}a_3) = (a_1.a_2.a_3) + (-1)^r \times \sum (a_1.\check{\bar{T}}_{i_1}^{k_1-j_1} H_{r-j_1})(a_2.\check{\bar{T}}_{i_2}^{k_2-j_2} H_{r-j_2})(a_3.\check{\bar{T}}_{i_3}^{k_3-j_3} H_{r-j_3}) \times (\tilde{s}_{j_1+j_2+j_3-2r-1} \bar{T}_{i_1}^{k_1-j_1} \bar{T}_{i_2}^{k_2-j_2} \bar{T}_{i_3}^{k_3-j_3}),$$

where the sum is over all possible i_1, i_2, i_3 and j_1, j_2, j_3 subject to constraint: $1 \le j_p \le \min\{r, k_p\}$ for $p = 1,2,3$ and $j_1+j_2+j_3 \ge 2r+1$. Here

$$\tilde{s}_i := s_i(F + F'^*)$$

is the ith Segre class of $F + F'^*$.

Proof. First of all, $\phi'^*\mathscr{F}a_i = \phi^* a_i + j_*\lambda_i$ for some $\lambda_i \in H^{2(k_i-1)}(E)$ which contains both fiber directions of $\bar{\phi}$ and $\bar{\phi}'$. Hence

$$\begin{aligned}(\mathscr{F}a_1.\mathscr{F}a_2.\mathscr{F}a_3) &= (\phi'^*\mathscr{F}a_1.\phi'^*\mathscr{F}a_2.(\phi^*a_3 + j_*\lambda_3)) \\ &= (\phi'^*\mathscr{F}a_1.\phi'^*\mathscr{F}a_2.\phi^*a_3) = ((\phi^*a_1 + j_*\lambda_1).(\phi^*a_2 + j_*\lambda_2).\phi^*a_3).\end{aligned}$$

Among the resulting terms, the first term is clearly equal to $(a_1.a_2.a_3)$.

For those terms with two pull-backs like $\phi^*a_1.\phi^*a_2$, the intersection values are zero since the remaining part necessarily contains nontrivial $\bar{\phi}$ fiber direction.

The terms with ϕ^*a_3 and two exceptional parts contribute

$$\begin{aligned}&\phi^*a_3.j_*\bar{T}_{i_1}^{k_1-j_1}\left(\frac{x^{j_1}-(-y)^{j_1}}{x+y}\right).j_*\bar{T}_{i_2}^{k_2-j_2}\left(\frac{x^{j_2}-(-y)^{j_2}}{x+y}\right) \\ &= -\phi^*a_3.j_*\big(\bar{T}_{i_1}^{k_1-j_1}\bar{T}_{i_2}^{k_2-j_2}(x^{j_1}-(-y)^{j_1})(x^{j_2-1}+x^{j_2-2}(-y)+\cdots+(-y)^{j_2-1})\big)\end{aligned}$$

times $(a_1.\check{\bar{T}}_{i_1}^{k_1-j_1} H_{r-j_1})(a_2.\check{\bar{T}}_{i_2}^{k_2-j_2} H_{r-j_2})$. The terms with non-trivial contribution must contain y^q with $q \ge r$ which implies $j_1+j_2-1 \ge r$, hence such terms are

$$-(-y)^{j_1}(x^{j_2-1-(r-j_1)}(-y)^{r-j_1} + x^{j_2-1-(r-j_1)-1}(-y)^{r-j_1+1} + \cdots + (-y)^{j_2-1})$$

and the contribution after taking ϕ_* is

$$(-1)^{r+1}(h^{j_1+j_2-r-1} - h^{j_1+j_2-r-2}s'_1 + \cdots + (-1)^{j_1+j_2-r-1}s'_{j_1+j_2-r-1})$$

where $s'_i := s_i(F')$ is the ith Segre class of F'. Here we use the property of Segre classes to obtain $\phi_* y^q = s'_{q-r}$ for $q \ge r+1$.

In terms of bundle-theoretic formulation,

$$
\begin{aligned}
&h^{j_1+j_2-r-1} - h^{j_1+j_2-r-2}s'_1 + \cdots + (-1)^{j_1+j_2-r-1}s'_{j_1+j_2-r-1}\\
&= \big((1-s'_1+s'_2+\cdots)(1+h+h^2+\cdots)\big)_{j_1+j_2-r-1}\\
&= \left(s(F'^*)\frac{1}{(1-h)}\right)_{j_1+j_2-r-1} = \left(\frac{c(F)}{(1-h)}s(F)s(F'^*)\right)_{j_1+j_2-r-1}\\
&= \big(c(Q_F).s(F+F'^*)\big)_{j_1+j_2-r-1}\\
&= H_{j_1+j_2-r-1} + H_{j_1+j_2-r-2}\tilde{s}_1 + \cdots + \tilde{s}_{j_1+j_2-r-1}.
\end{aligned}
$$

With respect to the basis $\{\check{\bar{T}}^k_i\}$, $\tilde{s}_p\bar{T}^{k_1-j_1}_{i_1}\bar{T}^{k_2-j_2}_{i_2}$ is of the form

$$
\sum_{i_3}\big(\tilde{s}_p\bar{T}^{k_1-j_1}_{i_1}\bar{T}^{k_2-j_2}_{i_2}\bar{T}^{k_3-(2r+1+p-j_1-j_2)}_{i_3}\big)\check{\bar{T}}^{k_3-(2r+1+p-j_1-j_2)}_{i_3}.
$$

We define the new index $j_3 = 2r+1+p-j_1-j_2$ and thus $j_1+j_2+j_3 \geq 2r+1$, also $p = j_1+j_2+j_3-2r-1$.

By summing all together, we get the result. □

There is a particularly simple case where no H_i or Segre classes $\tilde{s}_i$ are needed in the defect formula, namely the P^1 flops.

Corollary 1.9. *For P^1 flops over any smooth base S of dimension s, let $a_i \in H^{2k_i}(X)$ for $i = 1,2,3$ with $k_1+k_2+k_3 = \dim X = s+3$. Then*

$$
(\mathscr{F}a_1.\mathscr{F}a_2.\mathscr{F}a_3) = (a_1.a_2.a_3) - \sum(a_1.\check{\bar{T}}_1)(a_2.\check{\bar{T}}_2)(a_3.\check{\bar{T}}_3)(\bar{T}_1\bar{T}_2\bar{T}_3)
$$

with $\bar{T}_i$ running over all basis classes in $H^{2(k_i-1)}(S)$.

There is a trivial but useful observation on when the product is preserved:

Corollary 1.10. *For a P^r flop $f : X \dashrightarrow X'$, $a_1 \in H^{2k_1}(X)$, $a_2 \in H^{2k_2}(X)$ with $k_1+k_2 \leq r$, then $\mathscr{F}(a_1.a_2) = \mathscr{F}a_1.\mathscr{F}a_2$.*

This follows from Theorem 1.8 since all the correction terms vanish for any a_3. In fact it is a consequence of dimension count.

2. Quantum corrections attached to the extremal ray

2.1. **The set-up with nontrivial base.** Let $a_i \in H^{2k_i}(X)$, $i = 1,\ldots,n$, with

$$
\sum_{i=1}^n k_i = 2r+1+s+(n-3).
$$

Since

$$
a_i|_Z = \sum_{s_i}\sum_{j_i\leq\min\{k_i,r\}}(a_i.\check{\bar{T}}^{k_i-j_i}_{s_i}H_{r-j_i})\bar{T}^{k_i-j_i}_{s_i}h^{j_i},
$$

we compute

$$
\begin{aligned}
&\langle a_1, \dots, a_n \rangle^X_{0,n,d\ell} \\
&= \sum_{\vec{s},\vec{j}} \int_{M_{0,n}(Z,d\ell)} \prod_{i=1}^{n} \Big((a_i.\check{\bar{T}}_{s_i}^{k_i-j_i} H_{r-j_i})\ e_i^*(\bar{\psi}^* \bar{T}_{s_i}^{k_i-j_i}.h^{j_i}) \Big).e(R^1 ft_* e_{n+1}^* N) \\
&= \sum_{\vec{s},\vec{j}} \prod_{i=1}^{n} (a_i.\check{\bar{T}}_{s_i}^{k_i-j_i} H_{r-j_i}) \left[\prod_{i=1}^{n} \bar{T}_{s_i}^{k_i-j_i}.\Psi_{n*}\Big(\prod_{i=1}^{n} e_i^* h^{j_i}.e(R^1 ft_* e_{n+1}^* N) \Big) \right]^S,
\end{aligned}
$$

with the sum over all $\vec{s} = (s_1 \dots, s_n)$ and admissible $\vec{j} = (j_1, \dots, j_n)$. By the fundamental class axiom, we must have $j_i \ge 1$ for all i.

Here we make use of

$$[M_{0,n}(X, d\ell)]^{virt} = [M_{0,n}(Z, d\ell)] \cap e(R^1 ft_* e_{n+1}^* N)$$

and the fiber bundle diagram over S

$$
\begin{tikzcd}
 & M_{0,n+1}(Z,d\ell) \arrow[d, "ft"'] \arrow[dr, "e_{n+1}"] & N = N_{Z/X} \arrow[d] \\
M_{0,n}(P^r, d\ell) \arrow[r] & M_{0,n}(Z,d\ell) \arrow[r, "e_i"] \arrow[d, "\Psi_n"'] & Z \arrow[dl, "\bar{\psi}"] \\
 & S &
\end{tikzcd}
$$

as well as the fact that classes in S are constants among bundle morphisms (by the projection formula applying to $\Psi_n = \bar{\psi} \circ e_i$ for each i).

We must have $\sum (k_i - j_i) \le s$ to get nontrivial invariants. That is,

$$\sum_{i=1}^{n} j_i \ge 2r + 1 + n - 3.$$

If the equality holds, then $\prod_{i=1}^{n} \bar{T}_{s_i}^{k_i-j_i}$ is a zero dimensional cycle in S and the invariant readily reduces to the corresponding one on any fiber, namely the simple case, which is completely determined in [11]:

$$(\bar{T}_{s_1}^{k_1-j_1} \cdots \bar{T}_{s_n}^{k_n-j_n})^S \langle h^{j_1}, \dots, h^{j_n} \rangle^{\text{simple}}_{0,n,d\ell} = (\prod \bar{T}_{s_i})^S N_{\vec{j}}\, d^{n-3}.$$

On the contrary, if the strict inequality holds, by the dimension counting in the simple case, the restriction of the fiber integral $\Psi_{n*}(\cdot)$ to points in S vanishes. In fact the fiber integral is represented by a cycle $S_{\vec{j}} \subset S$ with codimension

$$\nu := \sum j_i - (2r + 1 + n - 3).$$

The structure of $S_{\vec{j}}$ necessarily depends on the bundles F and F'.

One would expect the end formula for $\Psi_{n*}(\cdot)$ to be

$$s_\nu(F + F'^*)\, N_{\vec{j}}\, d^{n-3}$$

with $N_{\vec{j}} = 1$ for $n \le 3$ so that the difference of the corresponding generating functions on X and X' cancels out with the classical defect on cup product. Unfortunately the actual behavior of these Gromov–Witten invariants with base dimension $s > 0$ is more delicate than this.

Notice that the new phenomenon does not occur for $n = 2$. In that case, $k_1 + k_2 = 2r + s$, $j_1 = j_2 = r$ and we may assume that $\bar{T}_{s_2}$ is running through the dual basis of $\bar{T}_{s_1}$. Since then the nontrivial terms only appear when $\bar{T}_{s_1}$ and $\bar{T}_{s_2}$ are dual to each other, we get

$$\begin{aligned}\langle a_1, a_2\rangle^X_{0,2,d\ell} &= \sum_s (a_1.\bar{T}_s)(a_2.\check{\bar{T}}_s)\langle h^r, h^r\rangle^{\text{simple}}_d \\ &= (-1)^{(d-1)(r+1)}\frac{1}{d}\sum_s (a_1.\bar{T}_s)(a_2.\check{\bar{T}}_s).\end{aligned}$$

It is also clear that the new phenomenon does not occur for P^1 flops over an arbitrary smooth base S. Thus before dealing with the general cases, we will work out the first (simplest) new case to demonstrate the general picture that will occur.

2.2. **Twisted relative invariants for** $\nu = 1$**.** Consider P^r flops with $n = 3$ and $j_1 + j_2 + j_3 = (2r + 1) + 1 = 2r + 2$, namely with one more degree (i.e. $\nu = 1$) than the old case. We start with $(j_1, j_2, j_3) = (2, r, r)$. Since classes from S can be merged into any marked point, the invariant to be taken care is

$$\langle h^2, h^r, \bar{t}h^r\rangle^X_d$$

for some $\bar{t} \in H^{2(s-1)}(S)$. Equivalently we define the fiber integral

$$\Big\langle \prod_{i=1}^n h^{j_i}\Big\rangle^{/S}_d := \Psi_{n*}\Big(\prod_{i=1}^n e_i^* h^{j_i}\Big) \in A(S)$$

to be a *$\bar{\psi}$-relative invariant over S* and we are computing

$$\langle h^2, h^r, \bar{t}h^r\rangle^X_d = (\langle h^2, h^r, h^r\rangle^{/S}_d.\bar{t})^S$$

now. Notice that for $r = 2$, $6 \ge j_1 + j_2 + j_3 > 5$ hence $(2, 2, 2)$ is precisely the only new case to compute.

The basic idea is to use the *divisor relation* [13] (for $n \ge 3$ points invariants)

$$e_i^* h = e_j^* h + \sum_{d'+d''=d} (d''[D_{ik,d'|j,d''}]^{virt} - d'[D_{i,d'|jk,d''}]^{virt}) \tag{2.1}$$

to move various h's into the same marked point. This type of process is also referred as *divisorial reconstruction* in this paper. Once the power exceeds r, the Chern polynomial relation reduces h^{r+1} into lower degree ones coupled with (Chern) classes from the base S. This will eventually reduce the *new invariants* to *old cases*. While this procedure is well known as the *reconstruction principle* in Gromov–Witten theory, the moral here is to show that this reconstruction transforms perfectly under flops.

Let $\Delta(X) = \sum_\mu T_\mu \otimes T^\mu$ be a diagonal splitting of $\Delta(X) \subset X \times X$. That is, $\{T_\mu\}$ is a cohomology basis of $H(X)$ with dual basis $\{T^\mu\}$. Apply the divisor relation (2.1)we get

$$\begin{aligned}\langle h^2, h^r, \bar{t}h^r\rangle_d &= \langle h, h^{r+1}, \bar{t}h^r\rangle_d \\ &\quad + \sum_{d'+d''=d} \sum_\mu d''\langle h, \bar{t}h^r, T_\mu\rangle_{d'} \langle T^\mu, h^r\rangle_{d''} - d'\langle h, T_\mu\rangle_{d'} \langle T^\mu, h^r, \bar{t}h^r\rangle_{d''}.\end{aligned}$$

The last terms vanish since there are no (non-trivial) two point invariants of the form $\langle h, T_\mu\rangle_{d'}$.

Since $h^{r+1} = -c_1 h^r - c_2 h^{r-1} - \cdots - c_{r+1}$, the first term clearly equals

$$-(c_1.\bar{t})^S \langle h, h^r, h^r\rangle_d^{\text{simple}} = -(-1)^{(d-1)(r+1)}(c_1.\bar{t})^S.$$

For the second terms, notice that the only degree zero invariant is given by 3-point classical cup product. Hence if $d' = 0$ then we may select $\{T^\mu\}$ in the way that $h.\bar{t}h^r$ appears as one of the basis elements, say $T^0 = \bar{t}h^{r+1}$ (this is not part of the canonical basis). Thus $d'' = d$ and the term equals

$$\begin{aligned}& d\langle h, \bar{t}h^r, T_0\rangle_0 \langle \bar{t}h^{r+1}, h^r\rangle_d \\ &= -d(c_1.\bar{t})^S \langle h^r, h^r\rangle_d^{\text{simple}} = -(-1)^{(d-1)(r+1)}(c_1.\bar{t})^S.\end{aligned}$$

It remains to consider $1 \le d'' \le d-1$. In this case we may assume that $T_0 = \check{\bar{t}}h^r$ since no lower power in h is allowed. To compute T^0 explicitly, since we are considering extremal rays, we may work on the projective local model $X_{loc} = P(N_{Z/X} \oplus \mathscr{O})$ of X along Z.

By applying Lemma 1.4 to $H(X_{loc})$, we get

Lemma 2.1. *Let $\{z_i\}$ be a basis of $H(Z)$ and $\xi = c_1(\mathscr{O}_{P(N\oplus\mathscr{O})}(1))$ be the class of the infinity divisor E. The dual basis for $\{z_i\xi^{r+1-j}\}_{j\le r+1}$ is given by $\{\check{z}_i\Theta_j\}_{j\le r+1}$ where*

$$\Theta_j := c_j(Q_N) = \xi^j + c_1(N)\xi^{j-1} + \cdots + c_j(N).$$

In particular, $\Theta_j|_Z = c_j(N)$. Moreover, since $N = \bar{\psi}^* F' \otimes \mathscr{O}(-1)$, we have

$$c_{r+1}(N) = (-1)^{r+1}(h^{r+1} - c_1' h^r + \cdots + (-1)^{r+1}c'_{r+1}).$$

Now if $z_0 = \check{\bar{t}}h^r$ and $T_0 = z_0\xi^0 = \check{\bar{t}}h^r$, then $T^0 = \bar{t}\Theta_{r+1}$ and the invariants become

$$\begin{aligned}& d''\langle h, \bar{t}h^r, \check{\bar{t}}h^r\rangle_{d'} \langle \bar{t}c_{r+1}(N), h^r\rangle_{d''} \\ &= -(-1)^{(d'-1)(r+1)}(-1)^{r+1}d''(\bar{t}.(c_1 + c_1'))^S \langle h^r, h^r\rangle_{d''}^{\text{simple}} \\ &= -(-1)^{(d'-1+d''-1+1)(r+1)}((c_1 + c_1').\bar{t})^S \\ &= -(-1)^{(d-1)(r+1)}((c_1 + c_1').\bar{t})^S.\end{aligned}$$

Summing together, we get

$$\langle h^2, h^r, \bar{t}h^r\rangle_d = (-1)^{(d-1)(r+1)}\Big(((-c_1 + c_1').\bar{t})^S - d((c_1 + c_1').\bar{t})^S\Big).$$

By exactly the same procedure, as long as $j_2 < r$ or $j_3 < r$, the boundary terms in the divisor relation necessarily vanish by the exact knowledge on 2-point invariants, hence

$$\langle h^{j_1}, h^{j_2}, \bar{t}h^{j_3}\rangle_d = \langle h^{j_1-1}, h^{j_2+1}, \bar{t}h^{j_3}\rangle_d.$$

In particular, any invariant with $j_1 + j_2 + j_3 = 2r + 2$ may be inductively switched into $\langle h^2, h^r, \bar{t}h^r\rangle_d$. Hence we have shown

Proposition 2.2 ($n = 3$, $\nu = 1$)**.** *For $\sum_{i=1}^3 j_i = 2r + 2$ and $\bar{t} \in H^{2(s-1)}(S)$,*

$$\langle h^{j_1}, h^{j_2}, \bar{t}h^{j_3}\rangle_d = (-1)^{(d-1)(r+1)}\Big((\tilde{s}_1.\bar{t})^S - d(c_1(F + F').\bar{t})^S\Big).$$

As in [11], this implies that the 3-point *extremal quantum corrections* for X and X' remedy the defect of classical cup product for the cases $\nu = 1$.

To see this, it is convenient to consider the basic rational function

$$\mathbf{f}(q) := \frac{q}{1 - (-1)^{r+1}q} = \sum_{d\geq 1} (-1)^{(d-1)(r+1)} q^d, \tag{2.2}$$

which is the 3-point extremal correction for the case $\nu = 0$. It is clear that

$$\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r.$$

Since $\mathscr{F}(\bar{t}h^j) = (-1)^j \bar{t}h'^j$ for $j \leq r$, the geometric series on X

$$\sum_{d\geq 1} (-1)^{(d-1)(r+1)} (\tilde{s}_1.\bar{t})^S\, q^{d\ell} = (\tilde{s}_1.\bar{t})^S \mathbf{f}(q^\ell)$$

together with its counterpart on X' *exactly* correct the classical term via

$$\begin{aligned}&(\tilde{s}_1.\bar{t})^S \mathbf{f}(q^\ell) - (-1)^{j_1+j_2+j_3} (\tilde{s}'_1.\bar{t})^S \mathbf{f}(q^{\ell'})\\ &= (\tilde{s}_1.\bar{t})^S (\mathbf{f}(q^\ell) + \mathbf{f}(q^{-\ell})) = (-1)^r (\tilde{s}_1.\bar{t})^S.\end{aligned}$$

The new feature for $\nu = 1$ is that we also have contributions involving the differential operator $\delta_h = q^\ell\, d/dq^\ell$, namely

$$-(c_1(F + F').\bar{t})^S \sum_{d\geq 1} (-1)^{(d-1)(r+1)} d q^{d\ell} = -(c_1(F + F').\bar{t})^S\, \delta_h \mathbf{f}(q^\ell).$$

This higher order series *does not* occur as corrections to the classical defect, though it is still derived from the $\nu = 0$ information together with the classical (bundle-theoretic) data. Of course it is invariant under P^r flops in terms of analytic continuation.

Remark 2.3. It is helpful to comment on $\bar{t}h^j$ and $\mathscr{F}(\bar{t}h^j)$ to avoid confusion. Since the Gromov–Witten theory of extremal curve classes localizes to Z, $\bar{t}h^j$ is regarded as $a|_Z$ for some $a \in H(X)$. If $j \leq r$, the familiar formula $\mathscr{F}a|_{Z'} = (-1)^j \bar{t}h'^j$ follows from Lemma 1.2, Lemma 1.4 and the invariance of Poincaré pairing. However this formula is not true for $j > r$. Instead, by the Segre relation $\bar{\psi}_* h^{r+\nu} = s_\nu$, we find that $h^{r+\nu} = s_\nu h^r +$ (lower order terms). This observation will be useful later.

2.3. **Twisted relative invariants for general ν.** We will show that when $\sum_{i=1}^3 j_i = 2r+1+\nu$ ($\nu \le r-1$), there is a degree ν cohomology valued polynomial $W_\nu^{F,F'}(d) = \sum_{i=0}^{\nu} w_{\nu,i}(F,F')\, d^i$ with coefficients $w_{\nu,i}(F,F') \in H^{2\nu}(S,\mathbb{Q})$ such that for any class $\bar{t} \in H^{2(r-\nu)}(S)$,

$$\begin{aligned}\langle h^{j_1}, h^{j_2}, \bar{t}h^{j_3}\rangle_d &= (-1)^{(d-1)(r+1)}(W_\nu^{F,F'}.\bar{t})^S(d)\\ &:= (-1)^{(d-1)(r+1)}\sum_{i=0}^{\nu}(w_{\nu,i}(F,F').\bar{t})^S\, d^i.\end{aligned}$$

Hence the 3-point extremal correction is given by

$$\langle h^{j_1}, h^{j_2}, \bar{t}h^{j_3}\rangle_+ := \sum_{d\ge 1}\langle h^{j_1}, h^{j_2}, \bar{t}h^{j_3}\rangle_d\, q^{d\ell} = (W_\nu^{F,F'}.\bar{t})^S(\delta_h)\mathbf{f}(q^\ell).$$

and the corresponding $\bar{\psi}$-relative invariant is equal to

$$\langle h^{j_1}, h^{j_2}, h^{j_3}\rangle_+^{/S} = W_\nu^{F,F'}(\delta_h)\mathbf{f}(q^\ell).$$

The constant term of $W_\nu^{F,F'}$ is the νth Segre class of $F + F'^*$. This is what we need because (as in the $\nu = 1$ case)

$$\tilde{s}_\nu\mathbf{f}(q^\ell) - (-1)^{j_1+j_2+j_3}\tilde{s}'_\nu\mathbf{f}(q^{\ell'}) = (-1)^r\tilde{s}_\nu.$$

That is, the classical defect is corrected.

Similarly, for the d^i component with $i \ge 1$,

$$w_{\nu,i}\,\delta_h^i\mathbf{f}(q^\ell) = w_{\nu,i}\,(-\delta_{h'})^i((-1)^r - \mathbf{f}(q^{\ell'})) = (-1)^{i+1}w_{\nu,i}\,\delta_{h'}^i\mathbf{f}(q^{\ell'}).$$

This is expected to agree with $(-1)^{j_1+j_2+j_3}w'_{\nu,i}\,\delta_{h'}^i\mathbf{f}(q^{\ell'})$. Hence we require the alternating nature of W:

$$w_{\nu,i}(F',F) = (-1)^{\nu+i}w_{\nu,i}(F,F').$$

Remark 2.4. We ignore the degree zero (classical) invariants in the formulation since they depends on the global geometry of X and X' and could not be expressed by local universal formula (only their difference could be).

Recall that for $1 \le \nu \le r-1$, any 3-point invariant $\langle \bar{t}_1h^{j_1}, \bar{t}_2h^{j_2}, \bar{t}_3h^{j_3}\rangle_d$ with $1 \le j_i \le r$ and $\sum j_i = (2r+1)+\nu$ is equal to the standard form $\langle h^{\nu+1}, h^r, \bar{t}h^r\rangle_d$ where $\bar{t} = \bar{t}_1\bar{t}_2\bar{t}_3 \in H^{2(s-\nu)}(S)$. The study of it is based on the recursive formula on extremal corrections $W_\nu := \langle h^{\nu+1}, h^r, h^r\rangle_+^{/S}$:

Proposition 2.5.

$$W_\nu = s_\nu\mathbf{f} + \sum_{j=1}^{\nu} W_{\nu-j}\big((-1)^r c_j\mathbf{f} - (-1)^{r+j}c'_j\mathbf{f} - c_j\big).$$

Proof. As in [11], by using the operator δ_h, the divisor relation can be used to obtain splitting relation of generating series

$$\langle h^{\nu+1}, h^r, \bar{t}h^r\rangle_+ = \langle h^\nu, h^{r+1}, \bar{t}h^r\rangle_+ + \sum_i\langle h^\nu, \bar{t}h^r, T_\mu\rangle_+\delta_h\langle T^\mu, h^r\rangle_+ + (s_\nu.\bar{t})^S\mathbf{f}.$$

The last term is coming from the case with $d_1 = 0$:

$$\sum_\mu \langle h^\nu, \bar{t}h^r, T_\mu\rangle_0 \delta_h \langle T^\mu, h^r\rangle_+ = \delta_h \langle \bar{t}h^{\nu+r}, h^r\rangle_+ = (s_\nu.\bar{t})^S \mathbf{f}.$$

Here the Segre relation $h^{r+\nu} = s_\nu h^r +$ (lower order terms) and the complete knowledge of 2-point invariants is used.

By the Chern polynomial relation, the first term equals

$$-\sum_{j=1}^{\nu} \langle h^\nu, c_j h^{r+1-j}, \bar{t}h^r\rangle_+ = -\sum_{j=1}^{\nu} \langle h^{\nu-j+1}, h^r, c_j \bar{t} h^r\rangle_+ = -\sum_{j=1}^{\nu} (W_{\nu-j}.c_j\bar{t})^S.$$

For the second sum, we take the degree $r+1$ part of T_μ's being of the form $\{\bar{t}_j h^{r+1-j}\}_{j=1}^{\nu}$ with $\bar{t}_j \in H^{2j}(S)$ to be determined later. Then as in the previous calculation, using local models, the corresponding dual basis T^μ's are given by $\{\check{\bar{t}}_j H_{j-1}\Theta_{r+1}\}_{j=1}^{\nu}$. We need the h^r part of

$$\begin{aligned} &H_{j-1}\Theta_{r+1} \\ &= (-1)^{r+1}(h^{j-1} + c_1 h^{j-2} + \cdots + c_{j-1})(h^{r+1} - c'_1 h^r + \cdots + (-1)^{r+1} c'_{r+1}) \end{aligned}$$

in the standard presentation of $H(Z)$. By $\tilde{c} := c(F + F'^*) = c(F)c(F'^*)$, it is $(-1)^{r+1}$ times the h^r part of

$$h^r(\tilde{c}_j - c_j) + h^{r+1}\tilde{c}_{j-1} + h^{r+2}\tilde{c}_{j-2} + \cdots + h^{r+j}.$$

By the Segre relation and $c(F'^*) = s(F)c(F + F'^*)$, the term is

$$h^r(\tilde{c}_j + s_1\tilde{c}_{j-1} + s_2\tilde{c}_{j-2} + \cdots + s_{j-1}\tilde{c}_1 + s_j - c_j) = h^r((-1)^j c'_j - c_j).$$

Now we let $\bar{t}_j = (-1)^j c'_j - c_j$, and then the sum becomes

$$(-1)^{r+1}\sum_{j=1}^{\nu} \langle h^\nu, \bar{t}h^r, \bar{t}_j h^{r+1-j}\rangle_+ \mathbf{f} = (-1)^{r+1}\sum_{j=1}^{\nu} (W_{\nu-j}((-1)^j c'_j - c_j)\mathbf{f}.\bar{t})^S.$$

The result follows by putting the three parts together. □

Theorem 2.6 (= Theorem 0.2)**.** *The $\bar{\psi}$-relative invariant over S*

$$W_\nu = \langle h^{j_1}, h^{j_2}, h^{j_3}\rangle_+^{/S}$$

with $1 \le j_i \le r$, $\nu = \sum j_i - (2r+1) \le r-1$ is the action on $\mathbf{f}$ by a universal (in $c(F)$ and $c(F')$) rational cohomology valued polynomial of degree ν in δ_h, which is independent of the choices of j_i's and satisfies the functional equation

$$W_\nu - (-1)^{\nu+1} W'_\nu = (-1)^r \tilde{s}_\nu$$

for $0 \le \nu \le r-1$.

Proof. Since $W_0 = \mathbf{f}$, by Proposition 2.5, it is clear that W_ν is recursively and uniquely determined, which is a degree $\nu + 1$ polynomial in $\mathbf{f}$ with coefficients being universal polynomial in $c(F)$ and $c(F')$ of pure degree ν.

Denote by $\delta = \delta_h = q\,d/dq$. In order to rewrite W_ν as a degree ν polynomial in $\delta\mathbf{f}$, we start with the basic relation

$$\delta\mathbf{f} = \mathbf{f} + (-1)^{r+1}\mathbf{f}^2.$$

Since $\delta(fg) = (\delta f)g + f\delta g$, it follows inductively that $\delta^m\mathbf{f}$ can be expressed as $P_m(\mathbf{f}) = \mathbf{f} + \cdots + (-1)^{m(r+1)}m!\mathbf{f}^{m+1}$ with P_m being an integral universal polynomial of degree $m+1$. Solving the upper triangular system between $\delta^m\mathbf{f}$'s and $\mathbf{f}^{m+1}$'s gives $\mathbf{f}^{\nu+1} = (-1)^{m(r+1)}\delta^\nu\mathbf{f}/\nu! + \cdots = Q_\nu(\delta)\mathbf{f}$ with Q_ν being a rational polynomial. Clearly W_ν then admits a corresponding rational cohomology valued expression as expected.

It remains to check that W_ν satisfies the required functional equation

$$W_\nu - (-1)^{\nu+1}W'_\nu = (-1)^r\tilde{s}_\nu.$$

We will prove it by induction. The case $\nu = 0$ goes back to $\mathbf{f} + \mathbf{f}' = (-1)^r$ where $\mathbf{f} := \mathbf{f}(q^\ell)$ and $\mathbf{f}' := \mathbf{f}(q^{\ell'}) \equiv \mathbf{f}(q^{-\ell})$ under the correspondence $\mathscr{F}$.

Assume the functional equation holds for all $j < \nu$. Then

$$W_\nu = s_\nu\mathbf{f} + \sum_{j=1}^{\nu} W_{\nu-j}\big((-1)^r c_j\mathbf{f} - (-1)^{r+j}c'_j\mathbf{f} - c_j\big),$$

$$W'_\nu = s'_\nu\mathbf{f}' + \sum_{j=1}^{\nu} W'_{\nu-j}\big((-1)^r c'_j\mathbf{f}' - (-1)^{r+j}c_j\mathbf{f}' - c'_j\big).$$

By substituting

$$W'_{\nu-j} = (-1)^{\nu-j+1}W_{\nu-j} + (-1)^{r+\nu-j}\tilde{s}_{\nu-j}$$

into W'_ν, we compute, after cancellations,

$$\begin{aligned}
&W_\nu - (-1)^{\nu+1}W'_\nu \\
&= s_\nu\mathbf{f} + (-1)^\nu s'_\nu\mathbf{f}' + \sum_{j=1}^{\nu}\big((-1)^j\tilde{s}_{\nu-j}c'_j\mathbf{f}' - \tilde{s}_{\nu-j}c_j\mathbf{f}' - (-1)^{r-j}\tilde{s}_{\nu-j}c'_j\big) \\
&= s_\nu\mathbf{f} + (-1)^\nu s'_\nu\mathbf{f}' + (s_\nu - \tilde{s}_\nu)\mathbf{f}' - ((-1)^\nu s'_\nu - \tilde{s}_\nu)\mathbf{f}' - (-1)^r(s_\nu - \tilde{s}_\nu) \\
&= s_\nu(\mathbf{f} + \mathbf{f}') - (-1)^r s_\nu + (-1)^r\tilde{s}_\nu \\
&= (-1)^r\tilde{s}_\nu,
\end{aligned}$$

where both directions of the Whitney sum relations

$$s(F) = s(F + F'^*)c(F'^*); \qquad s(F'^*) = s(F + F'^*)c(F)$$

are used. The proof is completed. □

Corollary 2.7. *For any ordinary flop over a smooth base, we have*

$$\mathscr{F}\langle a_1, a_2, a_3\rangle^X \cong \langle \mathscr{F}a_1, \mathscr{F}a_2, \mathscr{F}a_3\rangle^{X'}$$

modulo non-extremal curve classes.

2.4. **Functional equations for $n \geq 3$ point extremal functions.** For ordinary flops over any smooth base, we will show that Corollary 2.7 extends to all $n \geq 4$. Namely

$$\mathscr{F}\langle a_1, \cdots, a_n\rangle^X \cong \langle \mathscr{F}a_1, \cdots, \mathscr{F}a_n\rangle^{X'}$$

modulo non-extremal curve classes.

By restricting to Z and Z', it is equivalent to the nice looking formula

$$\mathscr{F}\langle h^{j_1}, \cdots, \bar{t}h^{j_n}\rangle \cong (-1)^{\sum j_i}\langle h'^{j_1}, \cdots, \bar{t}h'^{j_n}\rangle$$

for all $1 \leq j_l \leq r$, where for notational simplicity the n-point functions in this section refer to *extremal functions*, that is, the sum is only over $\mathbb{Z}_+\ell$.

Notices that $\mathscr{F}(\bar{t}h^j) = (-1)^j\bar{t}h'^j$ only for $j \leq r$ and it fails in general for $j > r$ if the base S is non-trivial. In fact, we have

Lemma 2.8.

$$\mathscr{F}(h^{r+1}) - (\mathscr{F}h)^{r+1} = (-1)^{r+1}\mathscr{F}\Theta_{r+1}$$

along Z'

Proof. This is simply a reformulation of Lemma 2.1. □

It is easy to see that $\mathscr{F}\langle h^{j_1}, \cdots, \bar{t}h^{j_n}\rangle \not\cong (-1)^{\sum j_i}\langle h'^{j_1}, \cdots, \bar{t}h'^{j_n}\rangle$ if some $j_l > r$. This appears as the subtle point in proving the functional equations for $n \geq 4$ points. The above lemma plays a crucial role in analyzing this.

Theorem 2.9. *Let $f : X \dashrightarrow X'$ be an ordinary P^r flop with exceptional loci $Z = P(F) \to S$ and $Z' = P(F') \to S$. Then for $n \geq 3$,*

$$\mathscr{F}\langle h^{j_1}, \cdots, \bar{t}h^{j_n}\rangle^X \cong \langle \mathscr{F}h^{j_1}, \cdots, \mathscr{F}\bar{t}h^{j_n}\rangle^{X'}$$

for all j_l's and $\bar{t} \in H^{2(s-\nu)}(S)$ with $\nu = \sum_{l=1}^n j_l - (2r+1+n-3)$.

Proof. This holds for $n = 3$ by Corollary 2.7. Suppose this has been proven up to some $n \geq 3$. The basic idea is that an iterated application of the divisor relation using the operator δ_h should allow us to reduce an $n+1$ point extremal function to ones with fewer marked points. The technical details however should be traced carefully.

The first point to make is on the diagonal splitting $\Delta(X) = \sum T_\mu \otimes T^\mu$. Since the Poincaré pairing is preserved, $\mathscr{F}T^\mu$ is still the dual basis of $\mathscr{F}T_\mu$ in $H(X')$. Thus we may take the diagonal splitting on the X' side to be $\Delta(X') = \sum \mathscr{F}T_\mu \otimes \mathscr{F}T^\mu$.

We only need to prove the case that all $j_l \leq r$. The P^1 flops always have $\nu = 0$ and the proof is reduced to the simple case. So we assume that $r \geq 2$.

We will prove the functional equation by further induction on j_1. The case $j_1 = 1$ holds by the divisor axiom and induction, so we assume that $j_1 \geq 2$. By applying the divisor relation to $(i, j, k) = (1, 2, 3)$, we get

$$\begin{aligned}&\langle h^{j_1}, h^{j_2}, h^{j_3}, \cdots\rangle = \langle h^{j_1-1}, h^{j_2+1}, h^{j_3}, \cdots\rangle\\ &+ \sum_\mu \langle h^{j_1-1}, h^{j_3}, \cdots, T_\mu\rangle\delta_h\langle h^{j_2}, \cdots, T^\mu\rangle - \delta_h\langle h^{j_1-1}, \cdots, T_\mu\rangle\langle h^{j_2}, h^{j_3}, \cdots, T^\mu\rangle.\end{aligned}$$

Since $j_1 - 1 < r$, $\langle h^{j_1-1}, \cdots, T_\mu \rangle$ can not be a 2-point invariant unless it is trivial. Hence we may assume that $\langle h^{j_2}, h^{j_3}, \cdots, T^\mu \rangle$ has fewer points.

The term $\langle h^{j_1-1}, h^{j_2+1}, h^{j_3}, \cdots \rangle$ is also handled by induction since $j_1 - 1 < j_1$. Thus we may apply $\mathscr{F}$ to the equation and apply induction to get

$$\begin{aligned}\mathscr{F}\langle h^{j_1}, h^{j_2}, h^{j_3}, \cdots \rangle = {} & \langle \mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_2+1}, \mathscr{F}h^{j_3}, \cdots \rangle \\ & + \sum_\mu \langle \mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_3}, \cdots, \mathscr{F}T_\mu \rangle \delta_{\mathscr{F}h} \mathscr{F}\langle h^{j_2}, \cdots, T^\mu \rangle \\ & - \delta_{\mathscr{F}h} \langle \mathscr{F}h^{j_1-1}, \cdots, \mathscr{F}T_\mu \rangle \langle \mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots, \mathscr{F}T^\mu \rangle,\end{aligned}$$

where $\mathscr{F} \circ \delta_h = \delta_{\mathscr{F}h} \circ \mathscr{F}$ by [11], Lemma 5.5.

Notice that in the first summand,

$$\mathscr{F}\langle h^{j_2}, \cdots, T^\mu \rangle = \langle \mathscr{F}h^{j_2}, \cdots, \mathscr{F}T^\mu \rangle$$

if it is not a 2-point invariant. Also the 2-point case survives precisely when $j_2 = r$ and $T^\mu = \mathrm{pt}.h^r$. In that case, by the invariance of 3-point extremal functions in the $\nu = 0$ (simple) case, the corresponding term becomes

$$\begin{aligned}\mathscr{F}\delta_h \langle h^r, T^\mu \rangle &= \mathscr{F}\langle h, h^r, T^\mu \rangle_+ \\ &= \langle \mathscr{F}h, \mathscr{F}h^r, \mathscr{F}T^\mu \rangle_+ + (-1)^r = \delta_{\mathscr{F}h} \langle \mathscr{F}h^r, \mathscr{F}T^\mu \rangle + (-1)^r.\end{aligned}$$

Also $T_\mu|_Z = \Theta_{r+1}|_Z$. Hence by Lemma 2.8 the extra $(-1)^r$ contributes

$$-\langle \mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_3}, \cdots, \mathscr{F}h^{r+1} \rangle - \langle \mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_3}, \cdots, (\mathscr{F}h)^{r+1} \rangle.$$

Since $j_2 = r$, the LHS cancels with the first term in the divisor relation and we end up with the RHS as the main term.

Now we compare it with the similar divisor relation for

$$\langle \mathscr{F}h^{j_1}, \mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots \rangle = \langle \mathscr{F}h.\mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots \rangle$$

under the diagonal splitting $\Delta(X') = \sum_\mu \mathscr{F}T_\mu \otimes \mathscr{F}T^\mu$. Namely

$$\begin{aligned}&\langle \mathscr{F}h^{j_1}, \mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots \rangle \\ &= \langle \mathscr{F}h^{j_1-1}, \mathscr{F}h.\mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots \rangle \\ &\quad + \sum_\mu \langle \mathscr{F}h^{j_1-1}, \mathscr{F}h^{j_3}, \cdots, \mathscr{F}T_\mu \rangle \delta_{\mathscr{F}h} \langle \mathscr{F}h^{j_2}, \cdots, \mathscr{F}T^\mu \rangle \\ &\quad - \delta_{\mathscr{F}h} \langle \mathscr{F}h^{j_1-1}, \cdots, \mathscr{F}T_\mu \rangle \langle \mathscr{F}h^{j_2}, \mathscr{F}h^{j_3}, \cdots, \mathscr{F}T^\mu \rangle.\end{aligned}$$

If $j_2 < r$ then there is no 2-point splitting and $\mathscr{F}h.\mathscr{F}h^{j_2} = \mathscr{F}h^{j_2+1}$, hence the functional equation holds. If $j_2 = r$ then $\mathscr{F}h.\mathscr{F}h^r = (\mathscr{F}h)^{r+1}$. This again agrees with the main term obtained above. Hence the proof of functional equations is complete by induction. □

Formula for $W_{\vec{j}} := \langle h^{j_1}, \cdots, h^{j_n} \rangle^{/S}$ can be achieved by a similar process as in Lemma 2.5, whose exact form would not be pursued here. In general it depends on the vector $\vec{j}$ instead of $\sum j_i$.

Remark 2.10. Theorem 0.2 and 2.9 (for the special case $F' = F^*$) have been applied in [4] to study stratified Mukai flops. In particular they provide non-trivial quantum corrections to flops of type $A_{n,2}$, D_5 and $E_{6,I}$.

3. Degeneration analysis revisited

Our next task is to *compare* the Gromov–Witten invariants of X and X' for all genera and for curve classes other than the flopped curve. As in [11], we use the degeneration formula [17, 16] to reduce the problem to local models. This has been achieved for *simple* ordinary flops in [11] for *genus zero* invariants. In this section we extend the argument to the general case and establish Theorem 0.3 (= Proposition 3.3 + 3.7) in the introduction.

3.1. **The degeneration formula.** We start by reviewing the basic setup. Details can be found in the above references.

Consider a pair (Y, E) with $E \hookrightarrow Y$ a smooth divisor. Let $\Gamma = (g, n, \beta, \rho, \mu)$ with $\mu = (\mu_1, \ldots, \mu_\rho) \in \mathbb{N}^\rho$ a partition of the intersection number $(\beta.E) = |\mu| := \sum_{i=1}^{\rho} \mu_i$. For $A \in H(Y)^{\otimes n}$ and $\varepsilon \in H(E)^{\otimes \rho}$, the relative invariant of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is

$$\langle A \mid \varepsilon, \mu\rangle_\Gamma^{(Y,E)} := \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} e_Y^* A \cup e_E^* \varepsilon$$

where $e_Y : \overline{M}_\Gamma(Y,E) \to Y^n$, $e_E : \overline{M}_\Gamma(Y,E) \to E^\rho$ are evaluation maps on marked points and contact points respectively. If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariant with disconnected domain curve is defined by the product rule:

$$\langle A \mid \varepsilon, \mu\rangle_\Gamma^{\bullet(Y,E)} := \prod_\pi \langle A \mid \varepsilon, \mu\rangle_{\Gamma^\pi}^{(Y,E)}.$$

We apply the degeneration formula to the following situation. Let X be a smooth variety and $Z \subset X$ be a smooth subvariety. Let $\Phi : W \to \mathscr{X}$ be its *degeneration to the normal cone*, the blow-up of $X \times \mathbb{A}^1$ along $Z \times \{0\}$. Let $t \in \mathbb{A}^1$. Then $W_t \cong X$ for all $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ with

$$\phi = \Phi|_{Y_1} : Y_1 \to X$$

the blow-up along Z and

$$p = \Phi|_{Y_2} : Y_2 := P(N_{Z/X} \oplus \mathscr{O}) \to Z \subset X$$

the projective completion of the normal bundle. $Y_1 \cap Y_2 =: E = P(N_{Z/X})$ is the ϕ-exceptional divisor which consists of the infinity part.

The family $W \to \mathbb{A}^1$ is a degeneration of a trivial family, so all cohomology classes $\alpha \in H(X, \mathbb{Z})^{\oplus n}$ have global liftings and the restriction $\alpha(t)$ on W_t is defined for all t. Let $j_i : Y_i \hookrightarrow W_0$ be the inclusion maps for $i = 1, 2$. Let $\{\mathbf{e}_i\}$ be a basis of $H(E)$ with $\{\mathbf{e}^i\}$ its dual basis. $\{\mathbf{e}_I\}$ forms a basis of

$H(E^\rho)$ with dual basis $\{\mathbf{e}^I\}$ where $|I| = \rho$, $\mathbf{e}_I = \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_\rho}$. The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

$$\langle \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \left\langle j_1^* \alpha(0) \,\middle|\, \mathbf{e}_I, \mu \right\rangle^{\bullet (Y_1,E)}_{\Gamma_1} \left\langle j_2^* \alpha(0) \,\middle|\, \mathbf{e}^I, \mu \right\rangle^{\bullet (Y_2,E)}_{\Gamma_2}.$$

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types

$$\Gamma_i = \textstyle\coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$$

with the same partition μ of contact order under the identification I_ρ of contact points. The gluing $\Gamma_1 +_{I_\rho} \Gamma_2$ has type (g, n, β) and is connected. In particular, $\rho = 0$ if and only if that one of the Γ_i is empty. The total genus g_i, total number of marked points n_i and the total degree $\beta_i \in NE(Y_i)$ satisfy the splitting relations

$$\begin{aligned} g - 1 &= \textstyle\sum_{\pi=1}^{|\Gamma_1|} (g_1(\pi) - 1) + \sum_{\pi=1}^{|\Gamma_2|} (g_2(\pi) - 1) + \rho \\ &= g_1 + g_2 - |\Gamma_1| - |\Gamma_2| + \rho, \\ n &= n_1 + n_2, \\ \beta &= \phi_* \beta_1 + p_* \beta_2. \end{aligned}$$

(The first one is the arithmetic genus relation for nodal curves.)

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. We denote by Ω the set of equivalence classes of all admissible triples; by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

Given an ordinary flop $f : X \dashrightarrow X'$, we apply degeneration to the normal cone to both X and X'. Then $Y_1 \cong Y_1'$ and $E = E'$ by the definition of ordinary flops. The following notations will be used

$$Y := \mathrm{Bl}_Z X \cong Y_1 \cong Y_1', \quad \tilde{E} := P(N_{Z/X} \oplus \mathscr{O}), \quad \tilde{E}' := P(N_{Z'/X'} \oplus \mathscr{O}).$$

Next we discuss the presentation of $\alpha(0)$. Denote by $\iota_1 \equiv j : E \hookrightarrow Y_1 = Y$ and $\iota_2 : E \hookrightarrow Y_2 = \tilde{E}$ the natural inclusions. The class $\alpha(0)$ can be represented by $(j_1^* \alpha(0), j_2^* \alpha(0)) = (\alpha_1, \alpha_2)$ with $\alpha_i \in H(Y_i)$ such that

$$\iota_1^* \alpha_1 = \iota_2^* \alpha_2 \quad \text{and} \quad \phi_* \alpha_1 + p_* \alpha_2 = \alpha.$$

Such representatives are called *liftings*, which are not unique.

The standard choice of lifting is

$$\alpha_1 = \phi^* \alpha \quad \text{and} \quad \alpha_2 = p^*(\alpha|_Z).$$

Other liftings can be obtained from the standard one by the following way.

Lemma 3.1 ([11]). *Let $\alpha(0) = (\alpha_1, \alpha_2)$ be a choice of lifting. Then*

$$\alpha(0) = (\alpha_1 - \iota_{1*} e, \alpha_2 + \iota_{2*} e)$$

is also a lifting for any class e in E of the same dimension as α. Moreover, any two liftings are related in this manner.

For an ordinary flop $f : X \dashrightarrow X'$, we compare the degeneration expressions of X and X'. For a given admissible triple $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ on the degeneration of X, one may pick the corresponding $\eta' = (\Gamma_1', \Gamma_2', I_\rho')$ on the degeneration of X' such that $\Gamma_1 = \Gamma_1'$. Since

$$\phi^*\alpha - \phi'^*\mathscr{F}\alpha \in \iota_{1*}H(E) \subset H(Y),$$

Lemma 3.1 implies that one can choose $\alpha_1 = \alpha_1'$. This procedure identifies relative invariants on the $Y_1 = Y = Y_1'$ from both sides, and we are left with the comparison of the corresponding relative invariants on $\tilde{E}$ and $\tilde{E}'$.

The ordinary flop f induces an ordinary flop

$$\tilde{f} : \tilde{E} \dashrightarrow \tilde{E}'$$

on the local model. Denote again by $\mathscr{F}$ the cohomology correspondence induced by the graph closure. Then

Lemma 3.2 ([11]). *Let $f : X \dashrightarrow X'$ be an ordinary flop. Let $\alpha \in H(X)$ with liftings $\alpha(0) = (\alpha_1, \alpha_2)$ and $\mathscr{F}\alpha(0) = (\alpha_1', \alpha_2')$. Then*

$$\alpha_1 = \alpha_1' \quad \Longleftrightarrow \quad \mathscr{F}\alpha_2 = \alpha_2'.$$

Now we are in a position to apply the degeneration formula to reduce the problem to relative invariants of local models.

Notice that $A_1(\tilde{E}) = \iota_{2*}A_1(E)$ since both are projective bundles over Z. We then have

$$\phi^*\beta = \beta_1 + \beta_2$$

by regarding β_2 as a class in $E \subset Y$ (c.f. [11]).

Define the generating series for genus g (connected) invariants

$$\langle A \mid \varepsilon, \mu\rangle_g^{(\tilde{E},E)} := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu\rangle_{g,\beta_2}^{(\tilde{E},E)} q^{\beta_2}.$$

and the similar one with possibly disconnected domain curves

$$\langle A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)} := \sum_{\Gamma;\, \mu_\Gamma = \mu} \frac{1}{|\mathrm{Aut}\,\Gamma|} \langle A \mid \varepsilon, \mu\rangle_\Gamma^{\bullet(\tilde{E},E)} q^{\beta^\Gamma} \kappa^{g^\Gamma - |\Gamma|}.$$

For connected invariants of genus g we assign the κ-weight κ^{g-1}, while for disconnected ones we simply assign the product weights.

Proposition 3.3. *To prove $\mathscr{F}\langle\alpha\rangle_g^X \cong \langle\mathscr{F}\alpha\rangle_g^{X'}$ for all α up to genus $g \le g_0$, it is enough to show that*

$$\mathscr{F}\langle A \mid \varepsilon, \mu\rangle_g^{(\tilde{E},E)} \cong \langle \mathscr{F}A \mid \varepsilon, \mu\rangle_g^{(\tilde{E}',E)}$$

for all A, ε, μ and $g \le g_0$.

Proof. For the n-point function

$$\langle\alpha\rangle^X = \sum_g \langle\alpha\rangle_g^X\,\kappa^{g-1} = \sum_{g;\,\beta\in NE(X)} \langle\alpha\rangle_{g,\beta}^X\, q^\beta\,\kappa^{g-1},$$

the degeneration formula gives

$$\begin{aligned}\langle\alpha\rangle^X &= \sum_{g;\beta\in NE(X)}\sum_{\eta\in\Omega_\beta}\sum_I C_\eta\,\langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle_{\Gamma_1}^{\bullet(Y_1,E)}\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle_{\Gamma_2}^{\bullet(Y_2,E)}\, q^{\phi^*\beta}\,\kappa^{g-1}\\ &= \sum_\mu\sum_I\sum_{\eta\in\Omega_\mu} C_\eta\times\\ &\quad\Big(\langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle_{\Gamma_1}^{\bullet(Y_1,E)}\, q^{\beta_1}\,\kappa^{g^{\Gamma_1}-|\Gamma_1|}\Big)\Big(\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle_{\Gamma_2}^{\bullet(Y_2,E)}\, q^{\beta_2}\,\kappa^{g^{\Gamma_2}-|\Gamma_2|}\Big)\,\kappa^\rho.\end{aligned}$$

(Notice that ρ is determined by μ.) In this formula, the variable q^{β_1} on Y_1 (resp. q^{β_2} on Y_2) is identified with $q^{\phi_*\beta_1}$ (resp. $q^{p_*\beta_2}$) on X.

To simplify the generating series, we consider also absolute invariants $\langle\alpha\rangle^{\bullet X}$ with possibly disconnected domain curves as in the relative case (with product weights in κ). Then by comparing the order of automorphisms,

$$\langle\alpha\rangle^{\bullet X} = \sum_\mu m(\mu)\sum_I \langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle^{\bullet(Y_1,E)}\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(Y_2,E)}\,\kappa^\rho.$$

To compare $\mathscr{F}\langle\alpha\rangle^{\bullet X}$ and $\langle\mathscr{F}\alpha\rangle^{\bullet X'}$, by Lemma 3.2 we may assume that $\alpha_1 = \alpha_1'$ and $\alpha_2' = \mathscr{F}\alpha_2$. This choice of cohomology liftings identifies the relative invariants of (Y_1, E) and those of (Y_1', E) with the same topological types. It remains to compare (c.f. Remark 3.4 below)

$$\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E},E)} \quad\text{and}\quad \langle\mathscr{F}\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E}',E)}.$$

We further split the sum into connected invariants. Let Γ^π be a connected part with the contact order μ^π induced from μ. Denote $P : \mu = \sum_{\pi\in P}\mu^\pi$ a partition of μ and $P(\mu)$ the set of all such partitions. Then

$$\langle A \mid \varepsilon,\mu\rangle^{\bullet(\tilde{E},E)} = \sum_{P\in P(\mu)}\prod_{\pi\in P}\sum_{\Gamma^\pi}\frac{1}{|\operatorname{Aut}\mu^\pi|}\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle_{\Gamma^\pi}^{(\tilde{E},E)}\, q^{\beta^{\Gamma^\pi}}\,\kappa^{g^{\Gamma^\pi}-1}.$$

In the summation over Γ^π, the only index to be summed over is β^{Γ^π} on $\tilde{E}$ and the genus. This reduces the problem to $\langle A^\pi \mid \varepsilon^\pi,\mu^\pi\rangle_g^{(\tilde{E},E)}$.

Instead of working with all genera, the proposition follows from the same argument by reduction modulo κ^{g_0}. $\square$

Remark 3.4. Notice that there is natural compatibility on our identifications of the curve classes which keeps track on the contact weight $|\mu|$. Namely, the identity $\langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle^{\bullet(Y_1,E)} = \langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle^{\bullet(Y_1',E)}$ leads to

$$\mathscr{F}\phi_*\langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle^{\bullet(Y_1,E)} = q^{|\mu|\ell'}\phi'_*\langle\alpha_1 \mid \mathbf{e}_I,\mu\rangle^{\bullet(Y_1',E)},$$

while $\mathscr{F}\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E},E)} \cong \langle\mathscr{F}\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E}',E)}$ leads to

$$\mathscr{F}p_*\langle\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E},E)} \cong q^{-|\mu|\ell'}p'_*\langle\mathscr{F}\alpha_2 \mid \mathbf{e}^I,\mu\rangle^{\bullet(\tilde{E}',E)}.$$

Thus we may ignore the issue of contact weights in our discussion.

3.2. **Relative local back to absolute local.** Now let $X = \tilde{E}$. We shall further reduce the relative cases to the absolute cases with at most descendent insertions along E. This has been done in [11] for genus zero invariants under simple flops. Here we extend the argument to ordinary flops over any smooth base S and to all genera.

The local model

$$\bar{p} := \bar{\psi} \circ p : \tilde{E} \xrightarrow{p} Z \xrightarrow{\bar{\psi}} S$$

as well as the flop $f : \tilde{E} \dashrightarrow \tilde{E}'$ are all over S, with each fiber isomorphic to the simple case. Thus the map on numerical one cycles

$$\bar{p}_* : N_1(\tilde{E}) \to N_1(S)$$

has kernel spanned by the p-fiber line class γ and $\bar{\psi}$-fiber line class ℓ, which is the flopping log-extremal ray.

Notice that for general S the structure of $NE(Z)$ could be complicated and $NE(\tilde{E})$ is in general larger than $i_*NE(Z) \oplus \mathbb{Z}^+\gamma$. For $\beta = \beta_Z + d_2(\beta)\gamma \in NE(\tilde{E})$, while $\beta_Z = p_*\beta$ is necessarily effective, $d_2(\beta)$ could possibly be negative if (and only if) $\beta_Z \neq 0$. Nevertheless we have the following:

Lemma 3.5. *The correspondence $\mathscr{F}$ is compatible with $N_1(S)$. Namely*

$$\begin{tikzcd} N_1(\tilde{E}) \arrow[rr, "\mathscr{F}"] \arrow[dr, "\bar{p}_* \oplus d_2"'] & & N_1(\tilde{E}') \arrow[dl, "\bar{p}'_* \oplus d'_2"] \\ & N_1(S) \oplus \mathbb{Z} & \end{tikzcd}$$

is commutative.

Proof. Since $N_1(\tilde{E}) = i_*N_1(Z) \oplus \mathbb{Z}\gamma$ and $\mathscr{F}\gamma = \gamma' + \ell'$, we see that $d_2 = d'_2 \circ \mathscr{F}$ and it is enough to consider $\beta \in N_1(Z)$. Also $\mathscr{F}\ell = -\ell'$, so the remaining cases are of the form $\beta = \bar{\psi}^*\beta_S.H_r$ for $\beta_S \in N_1(S)$. Then $\mathscr{F}\beta = \bar{\psi}'^*\beta_S.H'_r$ and it is clear that both β and $\mathscr{F}\beta$ project to β_S. $\square$

This leads to the following key observation, which applies to both absolute and relative invariants:

Proposition 3.6. *Functional equation of a generating series $\langle A \rangle$ over Mori cone on local models $f : \tilde{E} \dashrightarrow \tilde{E}'$ is equivalent to functional equations of its various subseries (fiber series) $\langle A \rangle_{\beta_S, d_2}$ labeled by $NE(S) \oplus \mathbb{Z}$. The fiber series is a sum over the affine ray $\beta \in (d_2\gamma + \bar{\psi}^*\beta_S.H_r + \mathbb{Z}\ell) \cap NE(\tilde{E})$.*

To analyze these fiber series $\langle A \rangle_{\beta_S, d_2}$ with $(\beta_S, d_2) \in NE(S) \oplus \mathbb{Z}$, we consider the partial order of effectivity (weight) of the quotient Mori cone

$$W := NE(\tilde{E})/\sim, \quad a \sim b \text{ if and only if } a - b \in \mathbb{Z}\ell.$$

Notice that $a > b$ and $b > a$ lead to $a \sim b$ since ℓ is an extremal ray. Under the natural identification, W can be regarded as a subset of $NE(S) \oplus \mathbb{Z}$. This partial order is equivalent to the alphabetical partial order of $NE(S) \oplus \mathbb{Z}$. For the ease of notations we also use

$$[\beta] \equiv (\beta_S, d_2) := (\bar{p}_*(\beta), d_2(\beta)) \in W$$

to denote the class of β modulo extremal rays.

Given insertions

$$A = (a_1, \ldots, a_n) \in H(\tilde{E})^{\oplus n}$$

and weighted partition

$$(\varepsilon, \mu) = \{(\varepsilon_1, \mu_1), \ldots, (\varepsilon_\rho, \mu_\rho)\},$$

the genus g relative invariant $\langle A \mid \varepsilon, \mu \rangle_g$ is summing over classes $\beta = \beta_Z + d_2\gamma \in NE(\tilde{E})$ with

$$\sum_{j=1}^{n} \deg a_j + \sum_{j=1}^{\rho} \deg \varepsilon_j = (c_1(\tilde{E}).\beta) + (\dim \tilde{E} - 3)(1-g) + n + \rho - |\mu|.$$

In this case, $d_2 = (E.\beta) = |\mu|$ is already fixed and non-negative.

Proposition 3.7. *For an ordinary flop $\tilde{E} \dashrightarrow \tilde{E}'$, to prove*

$$\mathscr{F}\langle A \mid \varepsilon, \mu \rangle_{g,\beta_S} \cong \langle \mathscr{F}A \mid \varepsilon, \mu \rangle_{g,\beta_S}$$

for any $A \in H(\tilde{E})^{\oplus n}$, $\beta_S \in NE(S)$ and (ε, μ) up to genus $g \le g_0$, it is enough to prove the $\mathscr{F}$-invariance for descendent invariants of f-special type. Namely,

$$\mathscr{F}\langle A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}}_{g,\beta_S,d_2} \cong \langle \mathscr{F}A, \tau_{k_1}\varepsilon_1, \cdots, \tau_{k_\rho}\varepsilon_\rho \rangle^{\tilde{E}'}_{g,\beta_S,d_2}$$

for any $A \in H(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$, $\varepsilon_j \in H(E)$ and $\beta_S \in NE(S)$, $d_2 \ge 0$ up to genus $g \le g_0$.

Proof. The proof proceeds inductively on the 5-tuple

$$(g, \beta_S, |\mu| = d_2, n, \rho)$$

in the lexicographical order, with ρ in the reverse order.

Given $\langle a_1, \cdots, a_n \mid \varepsilon, \mu \rangle_{g,\beta_S}$, since $\rho \le |\mu|$, there are only finitely many 5-tuples of lower order. The proposition holds for those cases by the induction hypothesis.

We apply degeneration to the normal cone for $Z \hookrightarrow \tilde{E}$ to get $W \to \mathbb{A}^1$. Then $W_0 = Y_1 \cup Y_2$ with $\pi : Y_1 \cong P(\mathscr{O}_E(-1,-1) \oplus \mathscr{O}) \to E$ a P^1 bundle and $Y_2 \cong \tilde{E}$. Denote by $E_0 = E = Y_1 \cap Y_2$ and $E_\infty \cong E$ the zero and infinity divisors of Y_1 respectively.

The idea is to analyze the degeneration formula for

$$\langle a_1, \cdots, a_n, \tau_{\mu_1 - 1}\varepsilon_1, \cdots, \tau_{\mu_\rho - 1}\varepsilon_\rho \rangle^{\tilde{E}}_{g,\beta_S,d_2}$$

since formally it sums over the same curve classes β as those in $\langle a_1, \cdots, a_n \mid \varepsilon, \mu\rangle_{g,\beta_S}$ such that

$$\begin{aligned}&\sum_{j=1}^{n} \deg a_j + |\mu| - \rho + \sum_{j=1}^{\rho}(\deg \varepsilon_j + 1)\\ =&(c_1(\tilde{E}).\beta) + (\dim \tilde{E} - 3)(1-g) + n + \rho.\end{aligned}$$

As in the proof of Proposition 3.3, we consider the generating series of invariants with possibly disconnected domain curves while keeping the total contact order $d_2 = |\mu|$. Then we degenerate the series according to the contact order.

We first analyze the splitting of curve classes. Under $N_1(\tilde{E}) = i_* N_1(Z) \oplus \mathbb{Z}\gamma$, $\beta = \beta_Z + d_2\gamma$ may be split into

$$\beta^1 \in NE(Y_1) \subset NE(E) \oplus \mathbb{Z}\bar{\gamma}, \quad \beta^2 \in NE(Y_2) \equiv NE(\tilde{E}),$$

such that

$$(\beta^1, \beta^2) = (\beta^1_E + c\bar{\gamma}, \beta^2_Z + e\gamma)$$

is subject to the condition $\phi_*\beta^1 + p_*\beta^2 = \beta$, i.e.

$$\bar{\phi}_*\beta^1_E + \beta^2_Z = \beta_Z, \quad c = d_2 \geq 0,$$

and the contact order relation

$$e = (E.\beta^2)^{\tilde{E}} = (E.\beta^1)^{Y_1} = c + (E.\beta^1_E)^{Y_1} = d_2 - (E.\beta^1_E)^{\tilde{E}}.$$

As an effective class in E, β^1_E is also effective in $\tilde{E}$, hence $\beta^1_E = \zeta + m\gamma$ with $\zeta \in NE(Z)$ and $m \in \mathbb{Z}$. It is clear that $\zeta = \bar{\phi}_*\beta^1_E$ and $m = (E.\beta^1_E)^{\tilde{E}}$. It should be noticed that

$$e = d_2 - m$$

is not necessarily smaller than d_2 since m maybe negative. This causes no trouble since we always have that

$$\beta - \beta^2 = (\beta_Z + d_2\gamma) - (\beta^2_Z + e\gamma) = \bar{\phi}_*\beta^1_E + m\gamma = \beta^1_E \geq 0.$$

The equality holds if and only if $\beta^1_E = 0$ and in that case we arrive at fiber class integrals on (Y_1, E) with $\beta^1 = d_2\bar{\gamma}$.

In fact, more is true. It is automatic that $[\beta] > [\beta^2]$ under the curve class splitting. The equality $[\beta] = [\beta^2]$ occurs if and only if β^1_E consists of extremal rays $d_1\ell$. But extremal rays must stay inside Z, hence we again conclude that $\beta^1_E = 0$ and get fiber integrals on (Y_1, E). No summation over extremal rays is needed for these integrals.

Next we analyze the splitting of cohomology insertions. It is sufficient to consider $(\varepsilon_1, \ldots, \varepsilon_\rho) = \mathbf{e}_I = (\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_\rho})$. Since $\varepsilon_i|_Z = 0$, one may choose the cohomology lifting $\varepsilon_i(0) = (\iota_{1*}\,\varepsilon_i, 0)$. This ensures that insertions of the form $\tau_k\,\varepsilon$ must go to the Y_1 side in the degeneration formula.

For a general cohomology insertion $\alpha \in H(\tilde{E})$, by Lemma 3.1, the lifting can be chosen to be $\alpha(0) = (a, \alpha)$ for some a. From $\alpha(0) = (a, \alpha)$ and $\mathscr{F}\alpha(0) = (a', \mathscr{F}\alpha)$, Lemma 3.2 implies that $a = a'$.

As before the relative invariants on (Y_1, E) can be regarded as constants under $\mathscr{F}$. Then

$$\langle a_1, \cdots, a_n, \tau_{\mu_1-1}\mathbf{e}_{i_1}, \cdots, \tau_{\mu_\rho-1}\mathbf{e}_{i_\rho}\rangle^{\bullet\tilde{E}}_{g,\beta_S,d_2} = \sum_{\mu'} m(\mu')\times$$

$$\sum_{I'}\langle \tau_{\mu_1-1}\mathbf{e}_{i_1}, \cdots, \tau_{\mu_\rho-1}\mathbf{e}_{i_\rho} \mid \mathbf{e}^{I'}, \mu'\rangle^{\bullet(Y_1,E)}_{0,0}\langle a_1, \cdots, a_n \mid \mathbf{e}_{I'}, \mu'\rangle^{(\tilde{E},E)}_{g,\beta_S} + R,$$

where the main terms contain invariants whose $(\tilde{E}, E)$ components admit the highest order with respect to the first four induction parameters

$$(g, \beta_S, |\mu| = d_2, n).$$

In fact, the potentially highest order term $\langle a_1, \cdots, a_n \mid \mathbf{e}_I, \mu\rangle^{(\tilde{E},E)}_{g,\beta_S}$ occurs by the dimension count at the beginning of the proof. Yet it is not clear a priori whether it is also the highest one in ρ.

For the the remaining terms R, a term is in it if each connected component of its relative invariants on $(\tilde{E}, E)$ has either smaller genus or has β_S^2 strictly smaller than β_S or has smaller contact order or has fewer insertions than n. Notice that disconnected invariants on $(\tilde{E}, E)$ must lie in R.

For the main terms, by the genus constraint and the fact that the invariants on $(\tilde{E}, E)$ are connected, the invariants on (Y_1, E) must be of genus zero and the connected components are indexed by the contact points. Also each connected invariant contains fiber integrals with total fiber class $\beta^1 = d_2\bar{\gamma}$.

To get constraints about $(\mathbf{e}_{I'}, \mu')$ and ρ' on the main terms, we recall the dimension count on $\tilde{E}$ and $(\tilde{E}, E)$. Let $D = (c_1(\tilde{E}).\beta) + (\dim \tilde{E} - 3)(1 - g)$. For the absolute invariant on $\tilde{E}$,

$$\sum_{j=1}^{n} \deg a_j + |\mu| - \rho + \sum_{j=1}^{\rho}(\deg \mathbf{e}_{i_j} + 1) = D + n + \rho,$$

while on $(\tilde{E}, E)$ (notice that now $(c_1(\tilde{E}).\beta^2) = (c_1(\tilde{E}).\beta)$),

$$\sum_{j=1}^{n} \deg a_j + \sum_{j=1}^{\rho'} \deg \mathbf{e}_{i'_j} = D + n + \rho' - |\mu'|.$$

Hence $(\mathbf{e}_I, \mu)$ occurs in $(\mathbf{e}_{I'}, \mu')$'s and in particular, R is $\mathscr{F}$-invariant by induction. Moreover,

$$\deg \mathbf{e}_I - \deg \mathbf{e}_{I'} = \rho - \rho'.$$

We will show that the highest order term in the main terms, with respect to all five parameters, consists of the single one

$$C(\mu)\langle a_1, \cdots, a_n \mid \mathbf{e}_I, \mu\rangle^{(\tilde{E},E)}_{g,\beta_S}$$

with $C(\mu) \neq 0$.

For any $(\mathbf{e}_{I'}, \mu')$ in the main terms, consider the splitting of weighted partitions

$$(\mathbf{e}_I, \mu) = \coprod_{k=1}^{\rho'} (\mathbf{e}_{I^k}, \mu^k)$$

according to the connected components of the relative moduli of (Y_1, E), which are indexed by the contact points of μ'.

Since fiber class relative invariants on P^1 bundles over E can be computed by pairing cohomology classes in E with certain Gromov–Witten invariants in the fiber P^1 (c.f. [20], §1.2), we must have $\deg \mathbf{e}_{I^k} + \deg \mathbf{e}^{i'_k} \le \dim E$ to get non-trivial invariants. That is

$$\deg \mathbf{e}_{I^k} = \sum_j \deg \mathbf{e}_{i^k_j} \le \dim E - \deg \mathbf{e}^{i'_k} \equiv \deg \mathbf{e}_{i'_k}$$

for each k. In particular, $\deg \mathbf{e}_I \le \deg \mathbf{e}_{I'}$, hence also $\rho \le \rho'$.

The case $\rho < \rho'$ is handled by the induction hypothesis, so we assume that $\rho = \rho'$ and then $\deg \mathbf{e}_{I^k} = \deg \mathbf{e}_{i'_k}$ for each $k = 1, \ldots, \rho'$. In particular $I^k \ne \emptyset$ for each k. This implies that I^k consists of a single element. By reordering we may assume that $I^k = \{i_k\}$ and $(\mathbf{e}_{I^k}, \mu^k) = \{(\mathbf{e}_{i_k}, \mu_k)\}$.

Since the relative invariants on Y_1 contain genus zero fiber integrals, the virtual dimension for each k (connected component of the relative virtual moduli) is

$$\begin{aligned} &2\mu'_k + (\dim Y_1 - 3) + 1 + (1 - \mu'_k) \\ &\qquad = (\mu_k - 1) + (\deg \mathbf{e}_{i_k} + 1) + (\dim E - \deg \mathbf{e}_{i'_k}). \end{aligned}$$

Together with $\deg \mathbf{e}_{i_k} = \deg \mathbf{e}_{i'_k}$, this implies that

$$\mu'_k = \mu_k, \quad k = 1, \ldots, \rho.$$

From the fiber class invariants consideration and

$$\deg \mathbf{e}_{i_k} + \deg \mathbf{e}^{i'_k} = \dim E,$$

$\mathbf{e}_{i_k}$ and $\mathbf{e}^{i'_k}$ must be Poincaré dual to get non-trivial integral over E. That is, $\mathbf{e}_{i'_k} = \mathbf{e}_{i_k}$ for all k and $(\mathbf{e}_{I'}, \mu') = (\mathbf{e}_I, \mu)$. This gives the term we expect where $C(\mu)$ is a product of nontrivial fiber class invariants

$$\prod_{k=1}^{\rho} \left(\langle \tau_{\mu_k - 1} \mathbf{e}_{i_k} \mid \mathbf{e}^{i_k}, \mu_k \rangle^{(Y_1, E)}_{0, \mu_k \bar{\gamma}} \, q^{\mu_k \bar{\gamma}} \right) = c_\mu q^{d_2 \bar{\gamma}}$$

with $c_\mu \ne 0$.

In order to compare with the series $\langle a_1, \cdots, a_n, \tau_{\mu_1 - 1} \mathbf{e}_{i_1}, \cdots, \tau_{\mu_\rho - 1} \mathbf{e}_{i_\rho} \rangle^{\tilde{E}}_{g, \beta_S, d_2}$, which satisfies the functional equation under $\mathscr{F}$ by assumption, we need only to match the formal variables involved. Under $\phi : Y_1 \to \tilde{E}$ we set $q^{\bar{\gamma}} \mapsto q^{\gamma}$ and under $p : Y_2 \cong \tilde{E} \to \tilde{E}$ we set $q^{\gamma} \mapsto q^0 = 1$. Similarly we identify formal variables in the $\tilde{E}'$ side. It is clear that these identifications commute with $\mathscr{F}$. Hence

$$\mathscr{F}\langle a_1, \cdots, a_n \mid \mathbf{e}_I, \mu\rangle^{\tilde{E}}_{g,\beta_S} \cong \langle \mathscr{F}a_1, \cdots, \mathscr{F}a_n \mid \mathbf{e}_I, \mu\rangle^{\tilde{E}'}_{g,\beta_S},$$

and the proof of Proposition 3.7 is complete. □

4. Reconstructions on local models

In this section, X and X' are the projective local models (double projective bundles over S) of the flop

$$f : X = \tilde{E} = P_Z(N_{Z/X} \oplus \mathscr{O}) \dashrightarrow X' = \tilde{E}' = P_{Z'}(N_{Z'/X'} \oplus \mathscr{O}).$$

Since we consider only genus zero invariants for the discussion on big quantum rings, the subscript on genus will be omitted. One special feature for genus zero GW theory is that there exists several reconstruction theorems which allow us to deal with only some initial GW invariants.

By Leray–Hirsch,

$$H(X) = H(S)[h, \xi]/(f_F(h), f_{N\oplus\mathscr{O}}(\xi)).$$

So every $a \in H(X)$ admits a canonical presentation $a = \bar{t}h^i\xi^j$ with $0 \le i \le r$, $0 \le j \le r+1$ and $\bar{t} \in H(S)$. (In this case $\mathscr{F}a = \bar{t}(\mathscr{F}h)^i(\mathscr{F}\xi)^j = \bar{t}(\xi' - h')^i\xi'^j$ for $i \le r$ and for any j.) We abuse notations by writing $\xi|a$ if $j \ge 1$.

Definition 4.1 (f-special invariants). An insertion $\tau_k a$ is called *special* if $k \neq 0$ implies that $\xi|a$. A (possibly) descendent invariant is f-special it is not extremal (i.e. $(\beta_S, d_2) \neq (0,0)$) and if all of its insertions are special. An f-special invariant is of type I if ξ divides some insertion, otherwise it is called of type II.

4.1. **Topological recursion relation and divisor axiom.**

Theorem 4.2. *The $\mathscr{F}$-invariance for descendent invariants of f-special type is equivalent to the $\mathscr{F}$-invariance of big quantum rings.*

Proof. We only need to prove "⇐":

Consider the generating series $\langle \tau_{k_1}a_1, \cdots, \tau_{k_n}a_n\rangle_{\beta_S,d_2}$ of f-special type with $(\beta_S, d_2) \neq (0,0)$. Let $k = \sum_i k_i$ be the total descendent degree. We will prove the theorem by induction on k.

If $k = 0$, we may assume that $n \ge 3$ by adding divisors ξ or $D \in H^2(S)$ into the insertions. Since $(\xi.\ell) = 0 = (D.\ell)$, this only affects the series by a nonzero constant, hence the $\mathscr{F}$-invariance reduces to the case of big quantum ring.

Now let $k > 0$. Without loss of generality we assume that $k_1 \ge 1$. By induction the results holds for strictly smaller descendent degree and for any $n \ge 1$.

We first treat the case $n \ge 3$. By the *topological recursion relation*

$$\psi_1 = [D_{1|2,3}]^{virt},$$

we get

$$\begin{aligned}&\langle \tau_{k_1} a_1, \cdots, \tau_{k_n} a_n \rangle_{\beta_S, d_2} \\ &\qquad = \sum_\mu \langle \tau_{k_1 - 1} a_1, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \langle T^\mu, \tau_{k_2} a_2, \tau_{k_3} a_3, \cdots \rangle_{\beta''_S, d''_2},\end{aligned}$$

where the sum is over all splitting of curve classes such that $(\beta'_S, d'_2) + (\beta''_S, d''_2) = (\beta_S, d_2)$.

Notice that on the RHS, the case $(\beta'_S, d'_2) = (0, 0)$ is excluded since $\xi | a_1$ and it will lead to trivial invariants. The (β'_S, d'_2) series is then $\mathscr{F}$-invariant since it has strictly smaller descendent order $k_1 - 1 < k$. (Recall that on the X' side we may choose $\mathscr{F} T_\mu$ and $\mathscr{F} T^\mu$ for the splitting since $\mathscr{F}$ preserves the Poincaré pairing.)

The (β''_S, d''_2) series is also $\mathscr{F}$-invariant: It has strictly smaller descendent degree and it has at least 3 insertions. So even if $(\beta''_S, d''_2) = (0, 0)$ we still get the $\mathscr{F}$-invariance.

The case $n = 1$ can be reduced to the case $n = 2$ by the divisor equation for descendant invariants. Namely let b be a divisor coming from the base S or ξ such that $b.(\beta_S + d_2 \gamma) \neq 0$. Then $(b.\beta) \neq 0$ is independent of d and

$$\langle b, \tau_k a \rangle_{\beta_S, d_2} = (b.\beta) \langle \tau_k a \rangle_{\beta_S, d_2} + \langle \tau_{k-1} ab \rangle_{\beta_S, d_2}.$$

The case $n = 2$ can be similarly reduced to the case $n = 3$. If there is only one descendent insertion, say $\langle a_1, \tau_k a_2 \rangle_{\beta_S, d_2}$, then

$$\langle b, a_1, \tau_k a_2 \rangle_{\beta_S, d_2} = (b.\beta) \langle a_1, \tau_k a_2 \rangle_{\beta_S, d_2} + \langle a_1, \tau_{k-1} a_2 b \rangle_{\beta_S, d_2}.$$

If there are two descendent insertions, say $\langle \tau_l a_1, \tau_{k-l} a_2 \rangle_{\beta_S, d_2}$, then

$$\begin{aligned}\langle b, \tau_l a_1, \tau_{k-l} a_2 \rangle_{\beta_S, d_2} &= (b.\beta) \langle \tau_l a_1, \tau_{k-l} a_2 \rangle_{\beta_S, d_2} \\ &\quad + \langle \tau_{l-1} a_1 b, \tau_{k-l} a_2 \rangle_{\beta_S, d_2} + \langle \tau_l a_1, \tau_{k-l-1} a_2 b \rangle_{\beta_S, d_2}.\end{aligned}$$

All the other series are either 3-point functions or have descendent degree drops by one. Thus by induction the proof is complete. □

4.2. **Divisorial reconstruction and quasi-linearity.** Theorem 4.2 reduces the analytic continuation problem to the local models completely. However, in the actual determination of GW invariants (as will see in later sections), another natural set of initial GW invariants are those with at most one descendent insertion. This suggests another reconstruction procedure.

Definition 4.3 (Quasi-linearity). We say that the flop f is quasi-linear if for every special insertion $\alpha \in H(X) \cup \tau_\bullet H(E)$, $\bar{t}_i \in H(S)$ and $(\beta_S, d_2) \neq (0, 0)$, we have

$$\mathscr{F} \langle \bar{t}_1, \cdots, \bar{t}_{n-1}, \alpha \rangle^X_{\beta_S, d_2} \cong \langle \bar{t}_1, \cdots, \bar{t}_{n-1}, \mathscr{F} \alpha \rangle^{X'}_{\beta_S, d_2}.$$

We call invariants of the above type (with only one insertion not from the base) *elementary*. Quasi-linearity is the $\mathscr{F}$-invariance for elementary f-special invariants.

Notice that the similar statement for descendent invariants, even for simple flops, is generally wrong if $\alpha = \tau_k a$ with $k > 0$ but $a \notin H(E)$ (c.f. [11]).

Theorem 4.4. *Suppose that f is quasi-linear. Then all descendent invariants of f-special type are $\mathscr{F}$-invariant. Namely for $\alpha = (\alpha_1, \ldots, \alpha_n)$ ($n \geq 1$) with $\alpha_i \in H(X) \cup \tau_\bullet H(E)$ and for $(\beta_S, d_2) \neq (0,0)$, we have*

$$\mathscr{F}\langle\alpha\rangle^X_{\beta_S,d_2} \cong \langle\mathscr{F}\alpha\rangle^{X'}_{\beta_S,d_2}.$$

More precisely, any series of f-special type can be reconstructed, in an $\mathscr{F}$-compatible manner, from the extremal functions with $n \geq 3$ points and elementary f-special series.

We will prove the reconstruction by induction on $(\beta_S, d_2) \in W$, and then on m which is the number of insertions not coming from base classes. This is based on the following observations:

(1) Under *divisorial reconstruction*: $\psi_i + \psi_j = [D_{i|j}]^{virt}$, and for $L \in \mathrm{Pic}(X)$,

$$e_i^* L = e_j^* L + (\beta.L)\psi_j - \sum_{\beta_1+\beta_2=\beta} (\beta_1.L)[D_{i\beta_1|j\beta_2}]^{virt} \tag{4.1}$$

([13], c.f. also [11]), the degree β is either preserved or split into effective classes $\beta = \beta_1 + \beta_2$.

(2) When summing over $\beta \in (d_2\gamma + \bar{\psi}^*\beta_S.H_r + \mathbb{Z}\ell) \cap NE(X)$, the splitting terms can usually be written as the product of two generating series with no more marked points in a manner which will be clear in each context during the proof.

We also need to comment on the excluded cases $(\beta_S, d_2) = (0,0)$:

(3) Let $\alpha_i = \tau_{k_i} a_i$. If $k = \sum k_i \neq 0$, say $\xi | a_1$, then the extremal invariants survive only for the case $\beta = 0$. Since $\overline{M}_{0,n}(X, 0) \cong \overline{M}_{0,n} \times X$, we have

$$\langle \tau_{k_1} a_1, \cdots, \tau_{k_n} a_n \rangle_{n,\beta=0} = \int_{\overline{M}_{0,n}} \psi_1^k \times \int_X a_1 \cdots a_n. \tag{4.2}$$

It is non-trivial only if $k = \dim \overline{M}_{0,n} = n - 3$, and then

$$\int_X a_1 \cdots a_n = \int_{X'} \mathscr{F} a_1 \cdots \mathscr{F} a_n$$

since the flop f restricts to an isomorphism on E.

(4) For extremal invariants with $k = 0$, since $\xi|_Z = 0$ and the extremal curves will always stay in Z, we get trivial invariant if one of the insertions involves ξ. Hence by Theorem 2.9 the statement in the theorem still holds in this initial case except for the 2-point invariants $\langle \bar{t}_1 h^r, \bar{t}_2 h^r \rangle$. By the divisor axiom

$$\delta_h \langle \bar{t}_1 h^r, \bar{t}_2 h^r \rangle = \langle h, \bar{t}_1 h^r, \bar{t}_2 h^r \rangle_+,$$

the 2-point invariants will satisfy the $\mathscr{F}$-invariance functional equation up to analytic continuation only after incorporated with classical defect. Thus we may base our induction on $(\beta_S, d_2) = (0,0)$ with special care taken to handle this case.

Proof. Let $(\beta_S, d_2) \neq (0,0)$. If $m = 1$ then we are done, so let $m \geq 2$.

Step 1. First we handle the type I case, i.e. with the appearance of ξ in some α_i.

By reordering we may assume that $\alpha_n = \tau_s \xi a$, $s \geq 0$. Write

$$\alpha_1 = \bar{t}_1 \tau_k h^l \xi^j.$$

We will reduce m by moving divisors in α_1 into α_n in the order of ψ, h and ξ. This process is compatible with $\mathscr{F}$ since $\mathscr{F}a.\mathscr{F}\xi = \mathscr{F}(a.\xi)$.

For ψ, we use the equation

$$\psi_1 = -\psi_n + [D_{1|n}]^{virt}.$$

If $k \geq 1$ then $j \neq 0$ and we get

$$\begin{aligned}\langle \bar{t}_1 \tau_k h^l \xi^j, \cdots, \tau_s \xi a \rangle_{\beta_S, d_2} &= -\langle \bar{t}_1 \tau_{k-1} h^l \xi^j, \cdots, \tau_{s+1} \xi a \rangle_{\beta_S, d_2} \\ &\quad + \sum_\mu \langle \bar{t}_1 \tau_{k-1} h^l \xi^j, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \langle T^\mu, \cdots, \tau_s \xi a \rangle_{\beta''_S, d''_2}.\end{aligned}$$

For each i, if one of (β'_S, d'_2) and (β''_S, d''_2) is $(0,0)$ then since both terms contain ξ the splitting term must vanish. So we may assume that

$$(\beta'_S, d'_2) < (\beta_S, d_2) \quad \text{and} \quad (\beta''_S, d''_2) < (\beta_S, d_2)$$

and these terms are done by the induction hypothesis. (By performing this procedure to $\alpha_1, \ldots, \alpha_{n-1}$ we may assume that the only descendent insertion is α_n.)

For h, if $l \geq 1$ we use the divisor relation (4.1) for $L = h$ to get

$$\begin{aligned}&\langle \bar{t}_1 h^l \xi^j, \cdots, \tau_s \xi a \rangle_{\beta_S, d_2} \\ &\quad = \langle \bar{t}_1 h^{l-1} \xi^j, \cdots, \tau_s \xi a h \rangle_{\beta_S, d_2} + \delta_h \langle \bar{t}_1 h^{l-1} \xi^j, \cdots, \tau_{s+1} \xi a \rangle_{\beta_S, d_2} \\ &\quad\quad - \sum_\mu \delta_h \langle \bar{t}_1 h^{l-1} \xi^j, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \langle T^\mu, \cdots, \tau_s \xi a \rangle_{\beta''_S, d''_2}.\end{aligned}$$

The only cases for the splitting term to have one factor with the same (β_S, d_2) and m are of the form (denote by $\bar{t}_*$ some set of insertions $\alpha_j \in H(S)$)

$$\delta_h \langle \bar{t}_1 h^{l-1} \xi^j, \bar{t}_*, T_\mu \rangle_{0,0} \langle T^\mu, \cdots, \tau_s \xi a \rangle_{\beta_S, d_2},$$

where the LHS has n' points, or

$$\delta_h \langle \bar{t}_1 h^{l-1} \xi^j, \cdots, T_\mu \rangle_{\beta_S, d_2} \langle T^\mu, \bar{t}_*, \tau_s \xi a \rangle_{0,0}.$$

But $l - 1 < r$ forces the former LHS invariants to vanish: For $j \neq 0$ this is trivial. For $j = 0$, the codimension (c.f. §2)

$$\mu = |h| - (2r + 1 + n' - 3) < 2r - 2r = 0. \tag{4.3}$$

The latter RHS invariants also vanish since they contain ξ.

If $j = 0$, the case $(\beta'_S, d'_2) = (0,0)$ may still support nontrivial invariants with 3 or more points. In that case m decreases in the RHS. For the other terms, the only possible appearance of type II invariants (i.e. without ξ insertion) is

$$\delta_h \langle \bar{t}_1 h^{l-1}, \cdots, T_\mu \rangle_{\beta'_S, d'_2} = \langle h, \bar{t}_1 h^{l-1}, \cdots, T_\mu \rangle_{\beta'_S, d'_2}, \tag{4.4}$$

where $j = 0$, which has at least 3 points and $(0,0) < (\beta'_S, d'_2) < (\beta_S, d_2)$.

For ξ, the argument is entirely similar. For $j \geq 1$, the divisor relation says that

$$\begin{aligned}
&\langle \bar{t}_1 \xi^j, \cdots, \tau_s \xi a \rangle_{\beta_S, d_2} \\
&\quad = \langle \bar{t}_1 \xi^{j-1}, \cdots, \tau_s \xi^2 a \rangle_{\beta_S, d_2} + \delta_\xi \langle \bar{t}_1 \xi^{j-1}, \cdots, \tau_{s+1} \xi a \rangle_{\beta_S, d_2} \\
&\qquad - \sum_\mu \delta_\xi \langle \bar{t}_1 \xi^{j-1}, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \langle T^\mu, \cdots, \tau_s \xi a \rangle_{\beta''_S, d''_2}.
\end{aligned}$$

We then have $(\beta'_S, d'_2) < (\beta_S, d_2)$ and $(\beta''_S, d''_2) < (\beta_S, d_2)$ as before. Notice that only type I invariants appear in the reduction.

Step 2. Next we deal with the type II case: $\alpha_i = \bar{t}_i h^{l_i}$, $1 \leq i \leq n$. In case $\beta_S = 0$, we can add one ξ into the insertions and then go back to Step 1. From (4.4), (β_S, d_2) will be getting smaller when the possible type II invariants appear again, so it is done by induction. Thus we can allow $\beta_S \neq 0$ here. By adding base divisors into the insertions we may always assume that $n \geq 3$.

We can not apply (4.1) to move divisors since it will produce non f-special invariants. Instead, since $n \geq 3$ we may apply (2.1), the descendent-free form of the divisor relation, as we have used in the proof of Theorem 2.9.

Suppose that $l_1 > 0$ and $l_2 > 0$ and we move h from α_1 to α_2. We run induction on l_1. Namely we assume the $\mathscr{F}$-invariant reduction holds for $\alpha_1 = \bar{t}_1 h^j$ with $j \leq l_1 - 1$. The initial case $j = 0$ holds since m drops by 1. Then

$$\begin{aligned}
&\langle \bar{t}_1 h^{l_1}, \bar{t}_2 h^{l_2}, \alpha_3, \cdots \rangle_{\beta_S, d_2} \\
&\quad = \langle \bar{t}_1 h^{l_1 - 1}, \bar{t}_2 h^{l_2 + 1}, \alpha_3, \cdots \rangle_{\beta_S, d_2} \\
&\qquad + \sum_\mu \langle \bar{t}_1 h^{l_1 - 1}, \alpha_3, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \delta_h \langle \bar{t}_2 h^{l_2}, \cdots, T^\mu \rangle_{\beta''_S, d''_2} \\
&\qquad\quad - \delta_h \langle \bar{t}_1 h^{l_1 - 1}, \cdots, T_\mu \rangle_{\beta'_S, d'_2} \langle \bar{t}_2 h^{l_2}, \alpha_3, \cdots, T^\mu \rangle_{\beta''_S, d''_2}.
\end{aligned}$$

If $l_2 \leq r - 1$, the processes on X and X' are clearly $\mathscr{F}$-compatible and the splitting terms are all handled by induction. Indeed, if $(\beta'_S, d'_2) = (\beta_S, d_2)$ and $m' = m$ then $(\beta''_S, d''_2) = (0,0)$ which gives an extremal function with $m'' \leq 2$. The analogous codimension condition as in (4.3) forces the term to vanish. Similar consideration applies to the case $(\beta''_S, d''_2) = (\beta_S, d_2)$ as well.

If $l_2 = r$, the first term is no longer $\mathscr{F}$-compatible. The topological defect of the second insertion is given by Lemma 2.8: $\mathscr{F}(h^{r+1}) - (\mathscr{F} h)^{r+1} = (-1)^{r+1} \mathscr{F} \Theta_{r+1}$, where Θ_{r+1} is the dual class of pt.$h^r \xi^0$. Meanwhile, the splitting terms also contain one term not of lower order in (β_S, d_2) and m.

By the codimension consideration as in (4.3), we have $T^\mu = \check{\bar{t}}_2 h^r$ and the term is given by

$$\langle \bar{t}_1 h^{l_1-1}, \alpha_3, \cdots, \alpha_n, \bar{t}_2 \Theta_{r+1} \rangle_{\beta_S, d_2} \delta_h \langle \bar{t}_2 h^r, \check{\bar{t}}_2 h^r \rangle_{0,0}.$$

Comparing with its corresponding term on X'

$$\langle \bar{t}_1 \mathscr{F} h^{l_1-1}, \mathscr{F}\alpha_3, \cdots, \mathscr{F}\alpha_n, \bar{t}_2 \mathscr{F}\Theta_{r+1} \rangle_{\beta_S, d_2} \delta_{\mathscr{F}h} \langle \bar{t}_2 \mathscr{F} h^r, \check{\bar{t}}_2 \mathscr{F} h^r \rangle_{0,0}$$

and using the induction, we get the difference to be

$$\begin{aligned} &- \langle \bar{t}_1 \mathscr{F} h^{l_1-1}, \mathscr{F}\alpha_3, \cdots, \mathscr{F}\alpha_n, \bar{t}_2 \mathscr{F}\Theta_{r+1} \rangle_{\beta_S, d_2} \times (-1)^{r+1} \\ &= -\langle \bar{t}_1 \mathscr{F} h^{l_1-1}, \bar{t}_2 \mathscr{F}(h^{r+1}), \cdots \rangle_{\beta_S, d_2} + \langle \bar{t}_1 \mathscr{F} h^{l_1-1}, \bar{t}_2 (\mathscr{F}h)^{r+1}, \cdots \rangle_{\beta_S, d_2}. \end{aligned}$$

This cancels the defect of the non $\mathscr{F}$-compatible terms.

Thus the whole reduction is $\mathscr{F}$-invariant and the proof is complete. □

4.3. **WDVV equations.** We may strengthen Theorem 4.4 to

Theorem 4.5. *If the quasi-linearity holds for elementary type I series*

$$\langle \bar{t}_1, \cdots, \bar{t}_{n-1}, \tau_k a \xi \rangle,$$

then the $\mathscr{F}$-invariance holds for all series of f-special type.

The significance of this reduction will become clear after we introduce the practical method to calculate GW invariants. The proof is based on

Proposition 4.6. *Any type II series over (β_S, d_2) can be transformed into sum of products of (1) type I series over $(\beta'_S, d'_2) \le (\beta_S, d_2)$, (2) type II series over $\beta'_S < \beta_S$, and (3) extremal functions. Also, the processes can be done in a $\mathscr{F}$-compatible manner.*

Indeed, with Proposition 4.6, Theorem 4.5 then follows from the proof of Theorem 4.4: Simply replace Step 2 by the proposition and run the induction. All type II special series eventually disappear. (*Degenerate* type II series with $(\beta_S, d_2) = (0, 0)$ are simply extremal functions.)

The remaining of this subsection is devoted to the proof of Proposition 4.6. Notice that if $d_2 \neq 0$ then this is trivial: By the divisor axiom,

$$\langle a_1, \cdots, a_n \rangle_{\beta_S, d_2} = \langle a_1, \cdots, a_n, \xi \rangle_{\beta_S, d_2} / d_2.$$

Thus we consider $\langle a_1, \cdots, a_{n-1}, \bar{t}_i h^j \rangle_{\beta_S, 0}$ with $a_1, \ldots, a_{n-1} \in H(Z)$.

Let $\{\bar{T}_i\}$ be a basis for $H(S)$ and $\{\check{\bar{T}}_i\}$ be its dual basis. We start with the case of three-point functions $\langle a, b, \bar{T}_i h^j \rangle_{\beta_S, 0}$ for any $a, b \in H(Z)$. This certainly includes also the one-point and two-point cases by picking suitable $a, b \in H^2(S)$.

For any $c, d \in H(X)$, the WDVV equations

$$\sum_{m,n} \partial_{ijm} F_0 \, g^{mn} \, \partial_{nkl} F_0 = \sum_{m,n} \partial_{ikm} F_0 \, g^{mn} \, \partial_{njl} F_0$$

lead to the diagram

$$[a \vee b \mapsto \xi c \vee \xi d] = [a \vee \xi c \mapsto b \vee \xi d].$$

We apply it to split the curve classes over $(\beta_S, d_2 = 1)$ and get a linear equation

$$\sum_{i,j} \langle a, b, \bar{T}_i h^j \rangle_{\beta_S, 0} \langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \xi c, \xi d \rangle_{0,d_2} = I_{c,d}, \tag{4.5}$$

where all terms in the LHS of WDVV with either (1) $\beta'_S < \beta_S$, (2) $d'_2 \neq 0$, or (3) with basis class insertion $T_\mu = \bar{T}_i h^j \xi^k$ $(k > 0)$ from the diagonal splitting, have been moved into the RHS. Since the original RHS of WDVV are all type I series, any series in $I_{c,d}$ over (β'_S, d'_2) must satisfy $\beta'_S < \beta_S$ or $(\beta'_S, d'_2) = (\beta_S, 0)$.

Let $m = \sum_i h^i(S)$. We intend to form an $N \times N$ invertible system with $N = m(r+1)$. The virtual dimension of the second series is

$$d_2(r+2) + 2r + 1 + s.$$

Thus for $d_2 = 1$, we should require $|c| + |d| = r + |\bar{T}_i| + j$ to match the dimension.

Natural choices of $\{(c, d)\}$ are

$$c = c_{k,l} := \bar{T}_k \xi^l, \quad d = h^r. \tag{4.6}$$

The set $\{c_{k,l}\}$ is partially ordered by $|\bar{T}_k|$ and then by l.

We claim that the resulting system is upper triangular with non-zero diagonal. Indeed,

$$\langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \bar{T}_k \xi^{l+1}, \xi h^r \rangle_{0,1} \neq 0$$

only if $|\bar{T}_k| + l = |\bar{T}_i| + j$.

The key point is to use the fiber bundle structure $\overline{M}_{0,n}(X, \beta) \to S$ for $\beta = d\ell + d_2\gamma$ as in the extremal case (where $d_2 = 0$). The fiber is given by $\overline{M}_{0,n}$ of the toric local model for the simple flop case.

Thus if $|\bar{T}_k| > |\bar{T}_i|$ then $|\check{\bar{T}}_i| + |\bar{T}_k| > s$ and the invariant is zero. Even in the case $|\bar{T}_k| = |\bar{T}_i|$, and so $l = j$, we must have $\bar{T}_k = \bar{T}_i$ to avoid trivial invariants. The other cases $|\bar{T}_k| < |\bar{T}_i|$ belong to the strict upper triangular region which do not affect our concern.

It remains to calculate the diagonal fiber series (sum in $d \geq 0$)

$$\sum_i \langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \bar{T}_i \xi^{j+1}, \xi h^r \rangle_{0,1} = \langle h^{r-j}(\xi - h)^{r+1}, \xi^{j+1}, \xi h^r \rangle^{\text{simple}}_{d_2=1}.$$

We had done a similar calculation before for the extremal case in [11], Proposition 3.8. In the current case we have

Lemma 4.7. *For simple flops, the fiber series in d with $d_2 = 1$ are given by*

$$\langle h^{r-j}(\xi - h)^{r+1}, \xi^{j+1}, \xi h^r \rangle_{d_2=1} = \begin{cases} (-1)^j q^\ell q^\gamma, & 0 \leq j \leq r-1; \\ (1 - (-1)^{r+1} q^\ell) q^\gamma, & j = r. \end{cases}$$

Proof. By applying the divisor relation to move one ξ class with respect to $(i,j,k)=(2,1,3)$, we get (notice that $\xi(\xi-h)^{r+1}=0$)

$$
\begin{aligned}
&\langle h^{r-j}(\xi-h)^{r+1}, \xi^{j+1}, \xi h^r\rangle_{d_2=1}\\
&=\sum_\mu \langle \xi^j, \xi h^r, T_\mu\rangle_0 \delta_\xi \langle T^\mu, h^{r-j}(\xi-h)^{r+1}\rangle_1 - \delta_\xi\langle \xi^j, T_\mu\rangle_1 \langle T^\mu, h^{r-j}(\xi-h)^{r+1}, \xi h^r\rangle_0\\
&=\langle h^{r-j}(\xi-h)^{r+1}, \xi^{j+1}h^r\rangle_1.
\end{aligned}
$$

By another divisor relation (4.1), we can keep track on the 2-point invariants as follows:

$$
\begin{aligned}
&\langle h^{r-j}(\xi-h)^{r+1}, \xi^{j+1}h^r\rangle_1\\
&=\langle \psi h^{r-j}(\xi-h)^{r+1}, \xi^j h^r\rangle_1 - \sum_\mu \delta_\xi \langle \xi^j h^r, T_\mu\rangle_1 \langle T^\mu, h^{r-j}(\xi-h)^{r+1}\rangle_0\\
&=\langle \psi h^{r-j}(\xi-h)^{r+1}, \xi^j h^r\rangle_1 = \cdots\\
&=\langle \psi^{j+1} h^{r-j}(\xi-h)^{r+1}, h^r\rangle_1.
\end{aligned}
$$

Here we use the fact that there is no extremal invariants with any insertion involving ξ (notice that $(\xi-h)^{r+1}=\xi(\cdots)$ since $h^{r+1}=0$).

Next we move the divisor class h in h^r to the left one by one:

$$
\begin{aligned}
&\langle \psi^{j+1} h^{r-j}(\xi-h)^{r+1}, h^r\rangle_1\\
&=\langle \psi^{j+1} h^{r-j+1}(\xi-h)^{r+1}, h^{r-1}\rangle_1 + \delta_h \langle \psi^{j+2} h^{r-j}(\xi-h)^{r+1}, h^{r-1}\rangle_1\\
&\qquad - \sum_\mu \delta_h \langle h^{r-1}, T_\mu\rangle_0 \langle T^\mu, \psi^{j+1} h^{r-j}(\xi-h)^{r+1}\rangle_1\\
&=\langle \psi^{j+1}(h+d\psi) h^{r-j}(\xi-h)^{r+1}, h^{r-1}\rangle_1 = \cdots\\
&=\langle \psi^{j+1}(h+d\psi)^{r-1} h^{r-j}(\xi-h)^{r+1}, h\rangle_1.
\end{aligned}
$$

Note that $\langle h^{r-1}, T_\mu\rangle_0 = 0$ since the power of h is less than r.

Finally, the divisor axiom helps us to obtain the result:

$$
\begin{aligned}
&\langle \psi^{j+1}(h+d\psi)^{r-1} h^{r-j}(\xi-h)^{r+1}, h\rangle_1\\
&=d\langle \psi^{j+1}(h+d\psi)^{r-1} h^{r-j}(\xi-h)^{r+1}\rangle_1 + \langle h\psi^j(h+d\psi)^{r-1} h^{r-j}(\xi-h)^{r+1}\rangle_1\\
&=\langle \psi^j(h+d\psi)^r h^{r-j}(\xi-h)^{r+1}\rangle_1,
\end{aligned}
$$

which is the constant term in the z expansion in

$$
\begin{aligned}
&\Big\langle \sum_{k\ge 0} \frac{\psi^k}{z^k} z^j (h+dz)^r h^{r-j}(\xi-h)^{r+1}\Big\rangle_1\\
&= z^{j+2} e_{1*}\Big(\frac{1}{z(z-\psi)} e_1^*(h+dz)^r h^{r-j}(\xi-h)^{r+1}\Big).
\end{aligned}
$$

According to the same discussion of quasi-linearity in [11], if $d_2-d<0$ then P_β vanishes after multiplication by ξ. Here $h^{r-j}(\xi-h)^{r+1}$ does contain at least one ξ. Hence we only need to consider $d_2\ge d$. Now $d_2=1$, thus $d=0$ or 1.

If $d = 0$, then $h^r h^{r-j}(\xi - h)^{r+1}$ is nontrivial only if $j = r$ and in this case we get $h^r(\xi - h)^{r+1} = h^r \xi^{r+1} = \mathrm{pt}$. It is clear that the constant term of z in

$$z^{r+2} J_\beta.\mathrm{pt} = z^{r+2} \frac{1}{(\xi - h + z)^{r+1}(\xi + z)}.\mathrm{pt}$$

is equal to 1.

If $d = 1$, then $J_\beta = 1/(h+z)^{r+1}(\xi + z)$. Thus

$$\begin{aligned} & z^{j+2} \frac{(h+z)^r h^{r-j}(\xi - h)^{r+1}}{(h+z)^{r+1}(\xi + z)} \\ &= \frac{z^{j+2}}{z^2} \frac{h^{r-j}(\xi - h)^{r+1}}{(1 + h/z)(1 + \xi/z)} \\ &= z^j h^{r-j}(\xi - h)^{r+1} \Big(1 - \frac{h}{z} + \frac{h^2}{z^2} - \cdots (-1)^j \frac{h^j}{z^j} + \cdots \Big) \Big(1 - \frac{\xi}{z} + \cdots \Big). \end{aligned}$$

Since $\xi(\xi - h)^{r+1} = 0$, the constant term is given by

$$(-1)^j h^r (\xi - h)^{r+1} = (-1)^j h^r \xi^{r+1} = (-1)^j.$$

The proof is complete. □

Now we consider n-point functions with $n \ge 3$. The WDVV equation is for triple derivatives of the $g = 0$ potential function. Let $t \in H^{>2}(X)$ be a general insertion without the fundamental class and divisors. Then we have

$$\sum_{i,j} \langle a, b, \bar{T}_i h^j \rangle_{\beta_S, 0}(t) \langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \bar{T}_k \xi^{l+1}, \xi h^r \rangle_{0,1}(t) = I_{k,l}(t) \tag{4.7}$$

where any series in $I_{c,d}$ over (β'_S, d'_2) must satisfy $\beta'_S < \beta_S$ or $(\beta'_S, d'_2) = (\beta_S, 0)$.

By dimension counting, one more marked point increases one virtual dimension while t has Chow degree more than one, so we find that

$$\langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \bar{T}_k \xi^{l+1}, \xi h^r \rangle_{0,1}(t) = \langle \check{\bar{T}}_i H_{r-j} \Theta_{r+1}, \bar{T}_k \xi^{l+1}, \xi h^r \rangle_{0,1}$$

is in fact independent of t when $|\bar{T}_i| + j = |\bar{T}_k| + l$. The linear system (4.7) is thus $\mathscr{F}$-compatible by the quantum invariance of simple flop case [11].

In any case, if $|\bar{T}_k| > |\bar{T}_i|$ then the invariants are still zero. In particular the $N \times N$ system is still upper triangular. Moreover the diagonal entries are still given by the original 3 point (finite) series. Thus the series

$$\langle a, b, \bar{T}_i h^j \rangle_{\beta_S, 0}(t)$$

are solvable in terms of the expected terms.

5. Birkhoff factorization

In this section, a general framework for calculating the J function for a split toric bundle is discussed. It relies on a given (partial) section I of the Lagrangian cone generated by J. The process to go from I to J is introduced in a constructive manner, and Theorem 0.4 will be proved (= Proposition 5.6 + Theorem 5.10).

5.1. **Lagrangian cone and the J function.** We start with Givental's symplectic space reformulation of Gromov–Witten theory arising from Witten's basic *dilaton, string, and topological recursion relation* in two-dimensional gravity [25]. The main references for this section are [6, 2], with supplements and clarification from [15, 10]. In the following, the underlying ground ring is the *Novikov ring*

$$R = \mathbb{C}[\widehat{NE(X)}].$$

All the complicated issues on completion are deferred to [15].

Let $H := H(X)$, $\mathcal{H} := H[z, z^{-1}]\!]$, $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[\![z^{-1}]\!]$. Let $1 \in H$ be the identity. One can identify $\mathcal{H}$ as $T^*\mathcal{H}_+$ and this gives a canonical symplectic structure and a vector bundle structure on $\mathcal{H}$.

Let

$$\mathbf{q}(z) = \sum_\mu \sum_{k=0}^\infty \mathbf{q}_k^\mu T_\mu z^k \in \mathcal{H}_+$$

be a general point, where $\{T_\mu\}$ form a basis of H. In the Gromov–Witten context, the natural coordinates on $\mathcal{H}_+$ are $\mathbf{t}(z) = \mathbf{q}(z) + 1z$ (dilaton shift), with $\mathbf{t}(\psi) = \sum_{\mu,k} t_k^\mu T_\mu \psi^k$ serving as the general descendent insertion. Let $F_0(\mathbf{t})$ be the generating function of genus zero *descendent* Gromov–Witten invariants on X. Since F_0 is a function on $\mathcal{H}_+$, the one form dF_0 gives a section of $\pi : \mathcal{H} \to \mathcal{H}_+$.

Givental's *Lagrangian cone* $\mathcal{L}$ is defined as the graph of dF_0, which is considered as a section of π. By construction it is a Lagrangian subspace. The existence of $\mathbb{C}^*$ action on $\mathcal{L}$ is due to the dilaton equation $\sum \mathbf{q}_k^\mu \partial/\partial \mathbf{q}_k^\mu F_0 = 2F_0$. Thus $\mathcal{L}$ is a cone with vertex $\mathbf{q} = 0$ (c.f. [6, 10]).

Let $\tau = \sum_\mu \tau^\mu T_\mu \in H$. Define the (big) J-function to be

(5.1)
$$\begin{aligned} J^X(\tau, z^{-1}) &= 1 + \frac{\tau}{z} + \sum_{\beta,n,\mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau, \cdots, \tau \right\rangle_{0,n+1,\beta} \\ &= e^{\frac{\tau}{z}} + \sum_{\beta \neq 0, n, \mu} \frac{q^\beta}{n!} e^{\frac{\tau_1}{z} + (\tau_1 . \beta)} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau_2, \cdots, \tau_2 \right\rangle_{0,n+1,\beta}, \end{aligned}$$

where in the second expression $\tau = \tau_1 + \tau_2$ with $\tau_1 \in H^2$. The equality follows from the divisor equation for descendent invariants. Furthermore, the string equation for J says that we can take out the fundamental class 1 from the variable τ to get an overall factor $e^{\tau^0/z}$ in front of (5.1).

The J function can be considered as a map from H to $z\mathcal{H}_-$. Let $L_{\mathbf{f}} = T_{\mathbf{f}}\mathcal{L}$ be the tangent space of $\mathcal{L}$ at $\mathbf{f} \in \mathcal{L}$. Let $\tau \in H$ be embedded into $\mathcal{H}_+$ via

$$H \cong -1z + H \subset \mathcal{H}_+.$$

Denote by $L_\tau = L_{(\tau, dF_0(\tau))}$. Here we list the basic structural results from [6]:

(i) $zL \subset L$ and so $L/zL \cong \mathcal{H}_+/z\mathcal{H}_+ \cong H$ has rank $N := \dim H$.
(ii) $L \cap \mathcal{L} = zL$, considered as subspaces inside $\mathcal{H}$.
(iii) The subspace L of $\mathcal{H}$ is the tangent space at every $\mathbf{f} \in zL \subset \mathcal{L}$. Moreover, $T_{\mathbf{f}} = L$ implies that $\mathbf{f} \in zL$. zL is considered as the *ruling* of the cone.
(iv) The intersection of $\mathcal{L}$ and the affine space $-1z + z\mathcal{H}_-$ is parameterized by its image $-1z + H \cong H \ni \tau$ via the projection by π.

$$-zJ(\tau, -z^{-1}) = -1z + \tau + O(1/z)$$

is the function of τ whose graph is the intersection.
(v) The set of all directional derivatives $z\partial_\mu J = T_\mu + O(1/z)$ spans an N dimensional subspace of L, namely $L \cap z\mathcal{H}_-$, such that its projection to L/zL is an isomorphism.

Note that we have used the convention of the J function which differs from that of some more recent papers [6, 2] by a factor z.

Lemma 5.1. $z\nabla J = (z\partial_\mu J^\nu)$ *forms a matrix whose column vectors* $z\partial_\mu J(\tau)$ *generates the tangent space* L_τ *of the Lagrangian cone* $\mathcal{L}$ *as an* $R\{z\}$*-module. Here* $a = \sum q^\beta a_\beta(z) \in R\{z\}$ *if* $a_\beta(z) \in \mathbb{C}[z]$.

Proof. Apply (v) to L/zL and multiply z^k to get $z^k L/z^{k+1}L$. $\square$

We see that the germ of $\mathcal{L}$ is determined by an N-dimensional submanifold. In this sense, *zJ generates $\mathcal{L}$*. Indeed, all discussions are applicable to the Gromov–Witten context only as *formal germs* around the neighborhood of $\mathbf{q} = -1z$.

5.2. **Generalized mirror transform for toric bundles.** Let $\bar{p} : X \to S$ be a smooth fiber bundle such that $H(X)$ is generated by $H(S)$ and *fiber divisors* D_i's as an algebra, such that there is no linear relation among D_i's and $H^2(S)$. An example of X is a toric bundle over S. Assume that $H(X)$ is a free module over $H(S)$ with finite generators $\{D^e := \prod_i D_i^{e_i}\}_{e\in\Lambda}$.

Let $\bar{t} := \sum_s \bar{t}^s \bar{T}_s$ be a general cohomology class in $H(S)$, which is identified with $\bar{p}^* H(S)$. Similarly denote $D = \sum t^i D_i$ the general fiber divisor. Elements in $H(X)$ can be written as linear combinations of $\{T_{(s,e)} = \bar{T}_s D^e\}$. Denote the $\bar{T}_s$ directional derivative on $H(S)$ by $\partial_{\bar{T}_s} \equiv \partial_{\bar{t}^s}$, and denote the multiple derivative

$$\partial^{(s,e)} := \partial_{\bar{t}^s} \prod_i \partial_{t^i}^{e_i}.$$

Note, however, most of the time z will appear with derivative. For the notational convenience, denote the index (s,e) by $\mathbf{e}$. We then denote

$$\partial^{z\mathbf{e}} \equiv \partial^{z(s,e)} := z\partial_{\bar{t}^s} \prod_i z\partial_{t^i}^{e_i} = z^{|e|+1}\partial^{(s,e)}. \tag{5.2}$$

As usual, the $T_{\mathbf{e}}$ directional derivative on $H(X)$ is denoted by $\partial_{\mathbf{e}} = \partial_{T_{\mathbf{e}}}$. This is a special choice of basis T_μ (and ∂_μ) of $H(X)$, which is denoted by

$$T_{\mathbf{e}} \equiv T_{(s,e)} \equiv \bar{T}_s D^e; \qquad \mathbf{e} \in \Lambda^+.$$

The two operators $\partial^{z\mathbf{e}}$ and $z\partial_{\mathbf{e}}$ are by definition very different, nevertheless they are closely related in the study of quantum cohomology as we will see below.

Assuming that $\bar{p} : X \to S$ is a toric bundle of the split type, i.e. toric quotient of a split vector bundle over S. Let $J^S(\bar{t}, z^{-1})$ be the J function on S. The hypergeometric modification of J^S by the $\bar{p}$-fibration takes the form

$$I^X(\bar{t}, D, z, z^{-1}) := \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S}(z, z^{-1}) J_{\beta_S}^S(\bar{t}, z^{-1}) \tag{5.3}$$

with the relative factor $I_\beta^{X/S}$, whose explicit form for $X = \tilde{E} \to S$ will be given in Section 6.2.

The major difficulty which makes I^X being deviated from J^X lies in the fact that in general positive z powers may occur in I^X. Nevertheless for each $\beta \in NE(X)$, the power of z in $I_\beta^{X/S}(z, z^{-1})$ is bounded above by a constant depending only on β. Thus we may study I^X in the space $\mathcal{H} := H[z, z^{-1}]\!]$ over R.

Notice that the I function is defined only in the subspace

$$\hat{t} := \bar{t} + D \in H(S) \oplus \bigoplus_i \mathbb{C}D_i \subset H(X). \tag{5.4}$$

J. Brown recently established the following result:

Theorem 5.2 ([1] Theorem 1)**.** *$(-z)I^X(\hat{t}, -z)$ lies in the Lagrangian cone $\mathcal{L}$ of X.*

Definition 5.3 (GMT)*.* For each $\hat{t}$, $zI(\hat{t})$ lies in L_τ of $\mathcal{L}$. The correspondence

$$\hat{t} \mapsto \tau(\hat{t}) \in H(X) \otimes R$$

is called the *generalized mirror transformation* (c.f. [2, 6]).

Remark 5.4*.* In general $\tau(\hat{t})$ may be outside the submodule of the Novikov ring R generated by $H(S) \oplus \bigoplus_i \mathbb{C}D_i$. This is in contrast to the (classical) mirror transformation where τ is a transformation within $(H^0(X) \oplus H^2(X))_R$ (small parameter space).

To make use of Theorem 5.2, we start by outlining the idea behind the following discussions. By the properties of $\mathcal{L}$, Theorem 5.2 implies that I can be obtained from J by applying certain differential operator in $z\partial_{\mathbf{e}}$'s to it, with coefficients being series in z. However, what we need is the reverse direction, namely to obtain J from I, which amounts to removing the positive z powers from I. Note that, the I function has variables only in the subspace $H(S) \oplus \bigoplus_i \mathbb{C}D_i$. Thus a priori the reverse direction does not seem to be possible.

The key idea below is to replace derivatives in the missing directions by higher order differentiations in the fiber divisor variables t^i's, a process similar to transforming a first order ODE system to higher order scaler equation. This is possible since $H(X)$ is generated by D_i's as an algebra over $H(S)$.

Lemma 5.5. *$z\partial_1 J = J$ and $z\partial_1 I = I$.*

Proof. The first one is the string equation. For the second one, by definition $I = \sum_\beta q^\beta e^{D/z+(D.\beta)} I_\beta^{X/S} J_{\beta_S}^S(\bar{t})$, where $I_\beta^{X/S}$ depends only on z. The differentiation with respect to t^0 (dual coordinate of 1) only applies to $J_{\beta_S}^S(\bar{t})$. Hence the string equation on $J_{\beta_S}^S(\bar{t})$ concludes the proof. □

Proposition 5.6. (1) *The GMT: $\tau = \tau(\hat{t})$ satisfies $\tau(\hat{t}, q = 0) = \hat{t}$.*

(2) *Under the basis $\{T_{\mathbf{e}}\}_{\mathbf{e}\in\Lambda^+}$, there exists an* invertible *$N \times N$ matrix-valued formal series $B(\tau, z)$, which is free from cohomology classes, such that*

$$\left(\partial^{z\mathbf{e}} I(\hat{t}, z, z^{-1})\right) = \left(z\nabla J(\tau, z^{-1})\right) B(\tau, z), \tag{5.5}$$

where $(\partial^{z\mathbf{e}} I)$ is the $N \times N$ matrix with $\partial^{z\mathbf{e}} I$ as column vectors.

Proof. By Theorem 5.2, $zI \in \mathcal{L}$, hence $z\partial I \in T\mathcal{L} = L$. Then $z(z\partial)I \in zL \subset \mathcal{L}$ and so $z\partial(z\partial)I$ lies again in L. Inductively, $\partial^{z\mathbf{e}} I$ lies in L. The factorization $(\partial^{z\mathbf{e}} I) = (z\nabla J)B(z)$ then follows from Lemma 5.1. Also Lemma 5.5 says that the I (resp. J) function appears as the first column vector of $(\partial^{z\mathbf{e}} I)$ (resp. $(z\nabla J)$). By the $R\{z\}$ module structure it is clear that B does not involve any cohomology classes.

By the definitions of J, I and $\partial^{z\mathbf{e}}$ (c.f. (5.1), (5.3), (5.2)), it is clear that

$$\partial^{z\mathbf{e}} e^{\hat{t}/z} = T_{\mathbf{e}} e^{\hat{t}/z}, \qquad z\partial_{\mathbf{e}} e^{t/z} = T_{\mathbf{e}} e^{t/z} \tag{5.6}$$

($t \in H(X)$). Hence modulo Novikov variables $\partial^{z\mathbf{e}} I(\hat{t}) \equiv T_{\mathbf{e}} e^{\hat{t}/z}$ and $z\partial_{\mathbf{e}} J(\tau) \equiv T_{\mathbf{e}} e^{\tau/z}$

To prove (1), modulo all q^β's we have

$$e^{\hat{t}/z} \equiv \sum_{\mathbf{e}\in\Lambda^+} B_{\mathbf{e},1}(z) T_{\mathbf{e}} e^{\tau(\hat{t})/z}.$$

Thus

$$e^{(\hat{t}-\tau(\hat{t}))/z} \equiv \sum_{\mathbf{e}} B_{\mathbf{e},1}(z) T_{\mathbf{e}},$$

which forces that $\tau(\hat{t}) \equiv \hat{t}$ (and $B_{\mathbf{e},1}(z) \equiv \delta_{T_{\mathbf{e}},1}$).

To prove (2), notice that by (1) and (5.6), $B(\tau, z) \equiv I_{N\times N}$ when modulo Novikov variables, so in particular B is invertible. Notice that in getting

(5.5) we do not need to worry about the sign on "$-z$" since it appears in both I and J. $\square$

Definition 5.7 (BF). The left-hand side of (5.5) involves z and z^{-1}, while the right-hand side is the product of a function of z and a function of z^{-1}. Such a matrix factorization process is termed the *Birkhoff factorization*.

Besides its existence and uniqueness, for actual computations it will be important to know how to calculate $\tau(\hat{t})$ directly or inductively.

Proposition 5.8. *There are scalar-valued formal series $C_{\mathbf{e}}(\hat{t}, z)$ such that*

$$J(\tau, z^{-1}) = \sum_{\mathbf{e}\in\Lambda^+} C_{\mathbf{e}}(\hat{t}, z)\, \partial^{z\mathbf{e}} I(\hat{t}, z, z^{-1}), \tag{5.7}$$

where $C_{\mathbf{e}} \equiv \delta_{T_{\mathbf{e}},1}$ modulo Novikov variables.

In particular $\tau(\hat{t}) = \hat{t} + \cdots$ is determined by the $1/z$ coefficients of the RHS.

Proof. Proposition 5.6 implies that

$$z\nabla J = (\partial^{z\mathbf{e}} I)\, B^{-1}.$$

Take the first column vector in the LHS, which is $z\nabla_1 J = J$ by Lemma 5.5, one gets expression (5.7) by defining $C_{\mathbf{e}}$ to be the corresponding $\mathbf{e}$-th entry of the first column vector of B^{-1}. Modulo q^β's, $B^{-1} \equiv I_{N\times N}$, hence $C_{\mathbf{e}} \equiv \delta_{T_{\mathbf{e}},1}$. $\square$

Definition 5.9. A differential operator P is of degree Λ^+ if $P = \sum_{\mathbf{e}\in\Lambda^+} C_{\mathbf{e}}\partial^{z\mathbf{e}}$ for some $C_{\mathbf{e}}$. Namely, its components are multi-derivatives indexed by Λ^+.

Theorem 5.10 (BF/GMT)**.** *There is a unique, recursively determined, scalar-valued degree Λ^+ differential operator*

$$P(z) = 1 + \sum_{\beta\in NE(X)\setminus\{0\}} q^\beta P_\beta(t^i, \bar{t}^s, z; z\partial_{t^i}, z\partial_{\bar{t}^s}),$$

with each P_β being polynomial in z, such that $P(z)I(\hat{t}, z, z^{-1}) = 1 + O(1/z)$.

Moreover,

$$J(\tau(\hat{t}), z^{-1}) = P(z)I(\hat{t}, z, z^{-1}),$$

with $\tau(\hat{t})$ being determined by the $1/z$ coefficient of the right-hand side.

Proof. The operator $P(z)$ is constructed by induction on $\beta \in NE(X)$. We set $P_\beta = 1$ for $\beta = 0$. Suppose that $P_{\beta'}$ has been constructed for all $\beta' < \beta$ in $NE(X)$. We set $P_{<\beta}(z) = \sum_{\beta'<\beta} q^{\beta'} P_{\beta'}$. Let

$$A_1 = z^{k_1} q^\beta \sum_{\mathbf{e}\in\Lambda^+} f^{\mathbf{e}}(t^i, \bar{t}^s) T_{\mathbf{e}} \tag{5.8}$$

be the top z-power term in $P_{<\beta}(z)I$. If $k_1 < 0$ then we are done. Otherwise we will remove it by introducing "certain P_β". Consider the "naive quantization"

$$\hat{A}_1 := z^{k_1} q^\beta \sum_{\mathbf{e}\in\Lambda^+} f^{\mathbf{e}}(t^i, \bar{t}^s)\partial^{z\mathbf{e}}. \tag{5.9}$$

In the expression

$$(P_{<\beta}(z) - \hat{A}_1)I = P_{<\beta}(z)I - \hat{A}_1 I,$$

the target term A_1 is removed since

$$\hat{A}_1 I(q=0) = \hat{A}_1 e^{\hat{t}/z} = A_1 e^{\hat{t}/z} = A_1 + A_1 O(1/z).$$

All the newly created terms either have smaller z-power or have curve degree $q^{\beta''}$ with $\beta'' > \beta$ in $NE(X)$. Thus we may keep on removing the new top z-power term A_2, which has $k_2 < k_1$. Since the process will stop in no more than k_1 steps, we simply define P_β by

$$q^\beta P_\beta = -\sum_{1 \le j \le k_1} \hat{A}_j.$$

By induction we get $P(z) = \sum_{\beta \in NE(X)} q^\beta P_\beta$, which is clearly of degree Λ^+.

Now we prove the uniqueness of $P(z)$. Suppose that $P_1(z)$ and $P_2(z)$ are two such operators. The difference $\delta(z) = P_1(z) - P_2(z)$ satisfies

$$\delta(z)I =: \sum_\beta q^\beta \delta_\beta I = O(1/z).$$

Clearly $\delta_0 = 0$. If $\delta_\beta \neq 0$ for some β, then β can be chosen so that $\delta_{\beta'} = 0$ for all $\beta' < \beta$. Let the highest non-zero z-power term of δ_β be $z^k \sum_{\mathbf{e}} \delta_{\beta,k,\mathbf{e}} \partial^{z\mathbf{e}}$. Then

$$q^\beta z^k \sum_{\mathbf{e}} \delta_{\beta,k,\mathbf{e}} \partial^{z\mathbf{e}} \Big(e^{\hat{t}/z} + \sum_{\beta_1 \neq 0} q^{\beta_1} I_{\beta_1} \Big) + RI = O(1/z).$$

Here R denotes the remaining terms in δ. Note that terms in RI either do not contribute to q^β or have z-power smaller than k. Thus the only q^β term is

$$q^\beta z^k \sum_{\mathbf{e}} \delta_{\beta,k,\mathbf{e}} T_{\mathbf{e}} e^{\hat{t}/z}.$$

This is impossible since $k \ge 0$ and $\{T_{\mathbf{e}}\}$ is a basis. Thus $\delta = 0$.

Finally, by Lemma 5.1 B, and so does B^{-1}, has entries in $R\{z\}$. Thus Proposition 5.8 provides an operator which satisfies the required properties. By the uniqueness it must coincide with the effectively constructed $P(z)$. □

5.3. **Reduction to special BF/GMT.**

Proposition 5.11. *Let $f : X \dashrightarrow X'$ be the projective local model of an ordinary flop with graph correspondence $\mathscr{F}$. Suppose there are formal liftings τ, τ' of $\hat{t}$ in $H(X) \otimes R$ and $H(X') \otimes R$ respectively, with $\tau(\hat{t}), \tau'(\hat{t}) \equiv \hat{t}$ when modulo Novikov variables in $NE(S)$, and with $\mathscr{F}\tau(\hat{t}) \cong \tau'(\hat{t})$. Then*

$$\mathscr{F} J(\tau(\hat{t})).\xi \cong J'(\tau'(\hat{t})).\xi' \Longrightarrow \mathscr{F} J(\hat{t}).\xi \cong J'(\hat{t}).\xi'$$

and consequently $QH(X)$ and $QH(X')$ are analytic continuations to each other under $\mathscr{F}$.

Proof. By induction on the weight $w := (\beta_S, d_2) \in W$, suppose that for all $w' < w$ we have invariance of any n-point function (except that if $\beta'_S = 0$ then $n \geq 3$). Here we would like to recall that $W := (NE(\tilde{E})/\sim) \subset NE(S) \oplus \mathbb{Z}$ is the quotient Mori cone.

By the definition of J in (5.1), for any $a \in H(X)$ we may pick up the fiber series over w from the $\xi a z^{-(k+2)}$ component of the assumed $\mathscr{F}$-invariance:

$$\mathscr{F}\langle \tau^n, \psi^k \xi a\rangle^X \cong \langle \tau'^n, \psi^k \xi' \mathscr{F} a\rangle^{X'}. \tag{5.10}$$

Write $\tau(\hat{t}) = \sum_{\bar{w} \in W} \tau_{\bar{w}}(\hat{t}) q^{\bar{w}}$. The fiber series is decomposed into sum of subseries in q^ℓ of the form

$$\langle \tau_{\bar{w}_1}(\hat{t}), \cdots, \tau_{\bar{w}_n}(\hat{t}), \psi^k \xi a\rangle^X_{w''} q^{\sum_{j=1}^n \bar{w}_j + w''}.$$

Since $\sum \bar{w}_j + w'' = w$, any $\bar{w}_j \neq 0$ term leads to $w'' < w$, whose fiber series is of the form $\sum_i g_i(q^\ell, \hat{t}) h_i(q^\ell)$ with g_i from $\prod \tau_{\bar{w}_j}(\hat{t})$ and h_i a fiber series over w''. The g_i is $\mathscr{F}$-invariant by assumption and h_i is $\mathscr{F}$-invariant by induction, thus the sum of products is also $\mathscr{F}$-invariant.

¿From (5.10) and $\tau_0(\hat{t}) = \hat{t}$, the remaining fiber series with all $\bar{w}_j = 0$ satisfies

$$\mathscr{F}\langle \hat{t}^n, \psi^k \xi a\rangle^X_w \cong \langle \hat{t}^n, \psi^k \xi \mathscr{F} a\rangle^{X'}_{w'},$$

which holds for any n, k and a.

Now by Theorem 4.5 (divisorial reconstruction and WDVV reduction) this implies the $\mathscr{F}$-invariance of all fiber series over w. □

Later we will see that for the GMT $\tau(\hat{t})$ and $\tau'(\hat{t})$, the lifting condition $\tau(\hat{t}) \equiv \hat{t}$ modulo $NE(S)\backslash\{0\}$ (instead of modulo $NE(X)\backslash\{0\}$) and the *identity* $\mathscr{F} J(\tau(\hat{t})).\xi \cong J'(\tau'(\hat{t})).\xi'$ holds for split ordinary flops.

6. Hypergeometric modification

From now on we work with a split local P^r flop $f : X \dashrightarrow X'$ with bundle data (S, F, F'), where

$$F = \bigoplus_{i=0}^r L_i \quad \text{and} \quad F' = \bigoplus_{i=0}^r L'_i.$$

We study the explicit formula of the hypergeometric modification I^X and $I^{X'}$ associated to the double projective bundles $X \to S$ and $X' \to S$, especially the symmetry property between them.

In order to get a better sense of the factor $I^{X/S}$ it is necessary to have a precise description of the Mori cone first. We then describe the Picard–Fuchs equations associated to the I function.

6.1. **The minimal lift of curve classes and $\mathscr{F}$-effective cone.** Let C be an irreducible projective curve with $\psi : V = \bigoplus_{i=0}^r \mathscr{O}(\mu_i) \to C$ a split bundle. Denote by $\mu = \max \mu_i$ and $\bar{\psi} : P(V) \to C$ the associated projective bundle. Let $h = c_1(\mathscr{O}_{P(V)}(1))$,

$$b = \bar{\psi}^*[C].H_r = H_r = h^r + c_1(V)h^{r-1}$$

be the canonical lift of the base curve, and ℓ be the fiber curve class.

Lemma 6.1. *$NE(P(V))$ is generated by ℓ and $b - \mu\ell$.*

Proof. Consider $V' = \mathscr{O}(-\mu) \otimes V = \mathscr{O} \oplus N$. Then N is a semi-negative bundle and $NE(P(V)) \cong NE(P(V'))$ is generated by ℓ and the zero section b' of $N \to P^1$. In this case b' is also the canonical lift $b' = h'^r + c_1(V')h'^{r-1}$. From the Euler sequence we know that $h' = h + \mu p$. Hence

$$\begin{aligned} b' &= (h+\mu p)^r + \sum_{i=1}^r (\mu_i - \mu) p (h+\mu p)^{r+1} \\ &= h^r + r\mu p h^{r-1} + \sum_{i=1}^r (\mu_i - \mu) p h^{r-1} \\ &= h^r + c_1(V)h^{r-1} - \mu p h^{r-1} \\ &= b - \mu\ell. \end{aligned}$$

□

Let $\psi : V = \bigoplus_{i=0}^r L_i \to S$ be a split bundle with $\bar{\psi} : P = P(V) \to S$. Since $\bar{\psi}_* : NE(P) \to NE(S)$ is surjective, for each $\beta_S \in NE(S)$ represented by a curve $C = \sum_j n_j C_j$, the determination of $\bar{\psi}_*^{-1}(\beta_S)$ corresponds to the determination of $NE(P(V_{C_j}))$ for all j. Therefore by Lemma 6.1, the minimal lift with respect to this curve decomposition is given by

$$\beta^P := \sum_j n_j (\bar{\psi}^*[C_j].H_r - \mu_{C_j}\ell) = \beta_S - \mu_{\beta_S}\ell,$$

with $\mu_{C_j} = \max_i (C_j.L_i)$ and $\mu = \mu_{\beta_S} := \sum_j n_j \mu_{C_j}$. As before we identify the canonical lift $\bar{\psi}^*\beta_S.H_r$ with β_S. Thus the crucial part is to determine the case of primitive classes. The general case follows from the primitive case by additivity. When there are more than one way to decompose into primitive classes, the minimal lift is obtained by taking the minimal one. Notice that further decomposition leads to smaller (or equal) lift. Also there could be more than one minimal lifts coming from different (non-comparable) primitive decompositions.

Now we apply the above results to study the effective and $\mathscr{F}$-effective curve classes under local split ordinary flop $f : X \dashrightarrow X'$ of type (S, F, F'). Fixing a *primitive* curve class $\beta_S \in NE(S)$, we define

$$\mu_i := (\beta_S.L_i), \quad \mu_i' := (\beta_S.L_i').$$

Let $\mu = \max \mu_i$ and $\mu' = \max \mu'_i$. Then by working on an irreducible representation curve C of β_S, we get by Lemma 6.1

$$NE(Z)_{\beta_S} = (\beta_S - \mu\ell) + \mathbb{Z}_{\geq 0}\ell \equiv \beta_Z + \mathbb{Z}_{\geq 0}\ell,$$
$$NE(Z')_{\beta_S} = (\beta_S - \mu'\ell') + \mathbb{Z}_{\geq 0}\ell' \equiv \beta_{Z'} + \mathbb{Z}_{\geq 0}\ell'.$$

Now we consider the further lift of the primitive element β_Z (resp. $\beta_{Z'}$) to X. The bundle $N \oplus \mathscr{O}$ is of splitting type with Chern roots $-h + L'_i$ and 0, $i = 0, \ldots, r$. On β_Z they take values

$$\mu + \mu'_i \quad (i = 0, \ldots, r) \quad \text{and} \quad 0. \tag{6.1}$$

To determine the minimal lift of β_Z in X, we separate it into two cases:

Case (1): $\mu + \mu' > 0$. The largest number in (6.1) is $\mu + \mu'$ and

$$NE(X)_{\beta_Z} = (\beta_Z - (\mu + \mu')\gamma) + \mathbb{Z}_{\geq 0}\gamma.$$

Case (2): $\mu + \mu' \leq 0$. The largest number in (6.1) is 0 and

$$NE(X)_{\beta_Z} = \beta_Z + \mathbb{Z}_{\geq 0}\gamma.$$

To summarize, we have

Lemma 6.2. *Given a primitive class $\beta_S \in NE(S)$, $\beta = \beta_S + d\ell + d_2\gamma \in NE(X)$ if and only if*

$$d \geq -\mu \quad \textit{and} \quad d_2 \geq -\nu, \tag{6.2}$$

where $\nu = \max\{\mu + \mu', 0\}$.

Remark 6.3. For the general case $\beta_S = \sum_j n_j[C_j]$, the constants μ, ν are replaced by

$$\mu = \mu_{\beta_S} := \sum_j n_j \mu_{C_j}, \qquad \nu = \nu_{\beta_S} := \sum_j n_j \max\{\mu_{C_j} + \mu_{C'_j}, 0\}.$$

Thus a *geometric minimal lift* $\beta_S^X \in NE(X)$ for $\beta_S \in NE(S)$, *with respect to the given primitive decomposition*, is

$$\beta_S^X = \beta_S - \mu\ell - \nu\gamma.$$

(If $\mu_{C_j} + \mu'_{C_j} \geq 0$ for all j, then $\nu = \mu + \mu'$.)

The geometric minimal lifts describe $NE(X)$. We will however only need a "generic lifting" (I-minimal lift in Definition 6.7) in the study of GW invariants.

Definition 6.4. A class $\beta \in N_1(X)$ is $\mathscr{F}$-effective if $\beta \in NE(X)$ and $\mathscr{F}\beta \in NE(X')$.

Proposition 6.5. *Let $\beta_S \in NE(S)$ be primitive. A class $\beta \in NE(X)$ over β_S is $\mathscr{F}$-effective if and only if*

$$d + \mu \geq 0 \quad \textit{and} \quad d_2 - d + \mu' \geq 0. \tag{6.3}$$

Proof. Let $\beta = \beta_S + d\ell + d_2\gamma$, then $\mathscr{F}\beta = \beta_S - d\ell' + d_2(\gamma' + \ell') = \beta_S + (d_2 - d)\ell' + d_2\gamma =: \beta_S + d'\ell' + d_2'\gamma'$. It is clear that β is $\mathscr{F}$-effective implies both inequalities. Conversely, the two inequalities imply that

$$d_2 \geq d - \mu' \geq -(\mu + \mu') \geq -\nu,$$

hence $\beta \in NE(X)$. Similarly $\mathscr{F}\beta \in NE(X')$. □

6.2. **Symmetry for** I**.** For $F = \bigoplus_{i=0}^r L_i$, $F' = \bigoplus_{i=0}^r L_i'$, the Chern polynomials for F and $N \oplus \mathscr{O}$ take the form

$$f_F = \prod a_i := \prod (h + L_i), \qquad f_{N\oplus\mathscr{O}} = b_{r+1} \prod b_i := \xi \prod (\xi - h + L_i').$$

For $\beta = \beta_S + d\ell + d_2\gamma$, we set $\mu_i := (L_i.\beta_S)$, $\mu_i' := (L_i'.\beta_S)$. Then for $i = 0, \ldots, r$, $(a_i.\beta) = d + \mu_i$, $(b_i.\beta) = d_2 - d + \mu_i'$, and $(b_{r+1}.\beta) = d_2$. Let

$$\lambda_\beta = (c_1(X/S).\beta) = (c_1(F) + c_1(F')).\beta_S + (r+2)d_2. \tag{6.4}$$

The relative I factor is given by
(6.5)

$$I_\beta^{X/S} := \frac{1}{z^{\lambda_\beta}} \frac{\Gamma(1 + \frac{\xi}{z})}{\Gamma(1 + \frac{\xi}{z} + d_2)} \prod_{i=0}^r \frac{\Gamma(1 + \frac{a_i}{z})}{\Gamma(1 + \frac{a_i}{z} + \mu_i + d)} \frac{\Gamma(1 + \frac{b_i}{z})}{\Gamma(1 + \frac{b_i}{z} + \mu_i' + d_2 - d)},$$

and the hypergeometric modification of $\bar{p} : X \to S$ is

$$I = I(D, \bar{t}; z, z^{-1}) = \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S} J_{\beta_S}^S(\bar{t}), \tag{6.6}$$

where $D = t^1 h + t^2 \xi$ is the fiber divisor and $\bar{t} \in H(S)$.

In more explicit terms, for a split projective bundle $\bar{\psi} : P = P(V) \to S$, the relative I factor is

$$I_\beta^{P/S} := \prod_{i=0}^r \frac{1}{\prod\limits_{m=0}^{\beta.(h+L_i)} (h + L_i + mz)}, \tag{6.7}$$

where the product in $m \in \mathbb{Z}$ is directed in the sense that

$$\prod_{m=0}^{s} \equiv \prod_{m=0+}^{s+} := \prod_{m=-\infty}^{s} \Big/ \prod_{m=-\infty}^{0}. \tag{6.8}$$

Thus for each i with $\beta.(h + L_i) \leq -1$, the corresponding subfactor is understood as in the numerator which must contain the factor $h + L_i$ corresponding to $m = 0$. In general I is viewed as a cohomology valued Laurent series in z^{-1}. By the dimension constraint it in fact has only finite terms.

Remark 6.6. The relative factor comes from the equivariant Euler class of $H^0(C, T_{P/S}|_C) - H^1(C, T_{P/S}|_C)$ at the moduli point $[C \cong P^1 \to X]$.

Definition 6.7 (I-minimal lift). Introduce

$$\mu_{\beta_S}^I := \max_i \{\beta_S.L_i\}, \quad \mu_{\beta_S}'^I := \max_i \{\beta_S.L_i'\}$$

and

$$\nu^I_{\beta_S} = \max\{\mu^I_{\beta_S} + \mu'^I_{\beta_S}, 0\} \ge 0.$$

Define the *I-minimal lift* of β_S to be

$$\beta^I_S := \beta_S - \mu^I_{\beta_S}\ell - \nu^I_{\beta_S}\gamma \in NE(X)$$

where $\beta_S \in NE(X)$ is the *canonical lift* such that $h.\beta_S = 0 = \xi.\beta_S$.

Clearly, β^I_S is an effective class in $NE(X)$, as $\mu^I_{\beta_S} \le \mu_{\beta_S}$ and $\nu^I_{\beta_S} \le \nu_{\beta_S}$. When the inequality is strict, the I-minimal lift is more effective than the geometric minimal lift. Nevertheless it is uniquely defined and we will show that it encodes the informations of the hypergeometric modification.

Definition 6.8. Define β to be *I-effective*, denoted $\beta \in NE^I(X)$, if

$$d \ge -\mu^I_{\beta_S} \quad \text{and} \quad d_2 \ge -\nu^I_{\beta_S}.$$

It is called $\mathscr{F}I$-effective if β is I-effective and $\mathscr{F}\beta$ is I'-effective. By the same proof of Proposition 6.5, this is equivalent to

$$d + \mu^I_{\beta_S} \ge 0 \quad \text{and} \quad d_2 - d + \mu'^I_{\beta_S} \ge 0.$$

Lemma 6.9 (Vanishing lemma). *If $\bar\psi_*\beta \in NE(S)$ but $\beta \notin NE(P)$ then $I^{P/S}_\beta = 0$. In fact the vanishing statement holds for any $\beta = \beta_S + d\ell$ with $d < -\mu^I_{\beta_S}$.*

Proof. We have $\beta.(h + L_i) = d + \mu_i \le d + \mu^I_{\beta_S} < 0$ for all i. This implies that $I^{P/S}_\beta = 0$ since it contains the Chern polynomial factor $\prod_i (h + L_i) = 0$ in the numerator. □

Now $I^{X/S}_\beta \equiv I^{Z/S}_\beta I^{X/Z}_\beta$ is given by

$$\prod_{i=0}^{r} \frac{1}{\prod_0^{\beta.a_i}(a_i + mz)} \prod_{i=0}^{r} \frac{1}{\prod_0^{\beta.b_i}(b_i + mz)} \frac{1}{\prod_0^{\beta.\xi}(\xi + mz)} =: A_\beta B_\beta C_\beta. \tag{6.9}$$

Although (6.9) makes sense for any $\beta \in N_1(X)$, we have

Lemma 6.10. *$I^{X/S}_\beta$ is non-trivial only if $\beta \in NE^I(X)$.*

Proof. Indeed, if $\beta_S \in NE(S)$ but $\beta \notin NE^I(X)$ then either $d < -\mu^I_{\beta_S}$ and $A_\beta = 0$ by Lemma 6.9, or $d \ge -\mu^I_{\beta_S}$ and we must have $d_2 < -\nu^I_{\beta_S} \le 0$ and all factors in B_β appear in the numerator:

$$d_2 - d + \mu'_i \le d_2 + \mu^I_{\beta_S} + \mu'^I_{\beta_S} \le d_2 + \nu^I_{\beta_S} < 0.$$

In particular $B_\beta C_\beta$ contains the Chern polynomial $f_{N\oplus\mathscr{O}} = 0$. □

Remark 6.11. In view of Lemma 6.2, $\beta \in NE^I(X)$ is the "effective condition for β as if it is a primitive class". One way to think about this is that the localization calculation of the I factor is performed on the main component of the stable map moduli where β is represented by a smooth rational curve.

As far as I is concerned, the I-effective class plays the role of effective classes. However one needs to be careful that the converse of Lemma 6.10 is not true: If β is I-effective, it is still possible to have $I^{X/S}_\beta = 0$.

The expression (6.9) agrees with (6.5) by taking out the z factor with m. The total factor is clearly

$$z^{-(\sum_{i=0}^r a_i + \sum_{i=0}^{r+1} b_i).\beta} = z^{-c_1(X/S).\beta}.$$

Similarly for $\beta' \in NE(X')$, $I^{X'/S}_{\beta'} \equiv I^{Z'/S}_{\beta'} I^{X'/Z'}_{\beta'}$ is given by

$$\prod_{i=0}^r \frac{1}{\prod_0^{\beta'.a'_i}(a'_i + mz)} \prod_{i=0}^r \frac{1}{\prod_0^{\beta'.b'_i}(b'_i + mz)} \frac{1}{\prod_0^{\beta'.\xi'}(\xi' + mz)} =: A'_{\beta'} B'_{\beta'} C'_{\beta'}. \tag{6.10}$$

Here $a'_i = h' + L'_i = \mathscr{F} b_i$ and $b'_i = \xi' - h' + L_i = \mathscr{F} a_i$.

By the invariance of the Poincaré pairing, $(\beta.a_i) = d + \mu_i = (\mathscr{F}\beta.b'_i)$ and $(\beta.b_i) = d_2 - d + \mu'_i = (\mathscr{F}\beta.a'_i)$, and it is clear that all the *linear subfactors* in $I^{X/S}_\beta$ and $I^{X'/S}_{\mathscr{F}\beta}$ correspond perfectly under $A_\beta \mapsto B'_{\mathscr{F}\beta}$, $B_\beta \mapsto A'_{\mathscr{F}\beta}$ and $C_\beta \mapsto C'_{\mathscr{F}\beta}$.

However, since the cup product is not preserved under $\mathscr{F}$, in general $\mathscr{F} I_\beta \ne I'_{\mathscr{F}\beta}$. Clearly, any direct comparison of I_β and $I'_{\mathscr{F}\beta}$ (without analytic continuations) can make sense only if β is $\mathscr{F}I$-effective. This is the case for $(\beta.a_i)$'s (resp. $(\beta.b_i)$'s) not all negative. Namely A_β and B_β both contain factors in the denominator.

Lemma 6.12 (Naive quasi-linearity)**.** (1) $\mathscr{F} I_\beta.\xi = I'_{\mathscr{F}\beta}.\xi'$.

(2) *If* $d_2 := \beta.\xi < 0$ *then* $\mathscr{F} I_\beta = I'_{\mathscr{F}\beta}$.

The expressions in (1) *or* (2) *are nontrivial only if* β *is* $\mathscr{F}I$*-effective.*

Proof. (1) follows from the facts that $f : X \dashrightarrow X'$ is an isomorphism over the infinity divisors $E \cong E$. For (2), notice that since $d_2 < 0$ the factor C_β contains ξ in the numerator corresponding to $m = 0$. Similarly $C'_{\mathscr{F}\beta}$ contains ξ' in the numerator. Hence (2) follows from the same reason as in (1). The last statement follows from Lemma 6.10. ☐

6.3. **Picard–Fuchs system.** Now we return to the BF/GMT constructed in Theorem 5.10 and multiply it by the infinity divisor ξ:

$$J^X(\tau(\hat{t})).\xi = P(z) I^X(\hat{t}).\xi.$$

By Proposition 5.11 and Lemma 6.12, we need to show the $\mathscr{F}$-invariance for $P(z)$ and $\tau(\hat{t})$ in order to establish the general analytic continuation.

The very first evidence for this is that, as in the case of classical hypergeometric series, I^X (resp. $I^{X'}$) is a solution to certain Picard–Fuchs system which turns out to be $\mathscr{F}$-compatible:

Proposition 6.13 (Picard–Fuchs system on X). *$\Box_\ell I^X = 0$ and $\Box_\gamma I^X = 0$, where*

$$\Box_\ell = \prod_{j=0}^r z\partial_{a_j} - q^\ell e^{t^1} \prod_{j=0}^r z\partial_{b_j}, \qquad \Box_\gamma = z\partial_\xi \prod_{j=0}^r z\partial_{b_j} - q^\gamma e^{t^2}.$$

Recall that t^1, t^2 are the dual coordinates of h, ξ respectively. Here we use ∂_v to denote the directional derivative in v. Thus if $v = \sum v^i T_i \in H^2$ then $\partial_v = \sum v^i \partial_{t^i}$.

Proof. By extracting all the divisor variables $D = t^1 h + t^2 \xi$ and $\bar{t}_1 \in H^2(S)$ from I^X (where $\bar{t} = \bar{t}_1 + \bar{t}_2$), we get

$$I^X = \sum_{\beta \in NE(X)} q^\beta e^{\frac{D+\bar{t}_1}{z} + (D+\bar{t}_1).\beta} I^{X/S}_\beta J^S_{\beta_S}(\bar{t}_2).$$

It is clear that $z\partial_v$ produce the factor $v + z(v.\beta)$ for $v \in H^2$. From (6.9), $\prod_j z\partial_{a_j}$ modifies the $A_\beta B_\beta C_\beta$ factor to

$$\prod_{j=0}^r \frac{1}{\prod_{m=0}^{\beta.a_j - 1} (a_j + mz)} B_\beta C_\beta = A_{\beta-\ell} B_{\beta-\ell} \prod_{j=0}^r (b_j + (\beta-\ell).b_j) C_{\beta-\ell}$$

(since $\beta.a_j - 1 = (\beta-\ell).a_j$, $(\beta-\ell).b_j = \beta.b_j + 1$ and $(\beta-\ell).\xi = \beta.\xi$).

Clearly it equals the corresponding term from $q^\ell e^{t^1} \prod_j z\partial_{b_j} I^X$ unless $\beta - \ell$ is not effective. But in that case the term is itself zero since $A_{\beta-\ell} = 0$ by Lemma 6.9.

The proof for $\Box_\gamma I^X = 0$ is similar and is thus omitted. $\Box$

Similarly $I^{X'}$ is a solution to

$$\Box_{\ell'} = \prod_{j=0}^r z\partial_{a'_j} - q^{\ell'} e^{-t^1} \prod_{j=0}^r z\partial_{b'_j}, \qquad \Box_{\gamma'} = z\partial_{\xi'} \prod_{j=0}^r z\partial_{b'_j} - q^{\gamma'} e^{t^2+t^1},$$

where the dual coordinates of h' and ξ' are $-t^1$ and $t^2 + t^1$ (since $\mathscr{F}(t^1 h + t^2 \xi) = t^1(\xi' - h') + t^2 \xi' = (-t^1)h' + (t^2 + t^1)\xi'$).

Proposition 6.14.

$$\mathscr{F}\langle \Box^X_\ell, \Box^X_\gamma \rangle \cong \langle \Box^{X'}_{\ell'}, \Box^{X'}_{\gamma'} \rangle.$$

Proof. It is clear that

$$\mathscr{F}\Box_\ell = -q^{-\ell'} e^{t^1} \Box_{\ell'},$$

and

$$\mathscr{F}\Box_\gamma = z\partial_{\xi'} \prod_{j=0}^r z\partial_{a'_j} - q^{\gamma'+\ell'} e^{t^2} = z\partial_{\xi'} \Box_{\ell'} + q^{\ell'} e^{-t^1} \Box_{\gamma'}.$$

□

Namely, the Picard–Fuchs system on X and X' are indeed equivalent under $\mathscr{F}$. Moreover, both $I = I^X$ and $I' = I^{X'}$ satisfy this system, but in different coordinate charts "$|q^\ell| < 1$" and "$|q^\ell| > 1$" (of the Kähler moduli) respectively.

We do not expect I and I' to be the same solution under analytic continuations. We know this is not true for J and J' since the general descendent invariants are not $\mathscr{F}$-invariant. Nevertheless it turns out that $P(z)$ and $\tau(\hat{t})$ are correct objects to admit $\mathscr{F}$-invariance.

Lemma 6.15. *Modulo q^{β_S}, $\beta_S \in NE(S)$ and γ, we have $P(z) \equiv 1$ and $\tau(\hat{t}) \equiv \hat{t}$.*

Proof. One simply notices that in the proof of Theorem 5.10 to construct $P(z)$, the induction can be performed on $[\beta] = (\beta_S, d_2) \in W$, as in section 3.2, by removing the whole series in q^ℓ with the same top non-negative z power once a time. For the initial step $[\beta] = 0$ and $J^S([\beta] = 0) = e^{\bar{t}/z}$, from (6.9) we have extremal ray contributions:

$$I_{[\beta]=0} = e^{\hat{t}/z}(1 + O(1/z^{r+1})).$$

As there is no non-negative z powers besides 1, also later inductive steps will create only higher order $q^{[\beta]}$'s with respect to W, hence the result follows. □

Remark 6.16. By the virtual dimension count and (5.1), J is weighted homogeneous of degree 0 in the following weights $|\cdot|$: We set $|T_\mu|$ to be its Chow degree, $|t^\mu| = 1 - |T_\mu|$, $|q^\beta| = (c_1(X).\beta)$ and $|\psi| = |z| = 1$. This is usually expressed as: The Frobenius manifold $(QH(X), *)$ is conformal with respect to the Euler vector field

$$E = \sum (1 - |T_\mu|) t^\mu \partial_\mu + c_1(X) \in \Gamma(TH).$$

For the hypergeometric modification I, the base J^S has degree 0 with $|q^{\beta_S}| = (c_1(S).\beta_S)$. But when β_S is viewed as an object in X the weight increases by $(c_1(X/S).\beta_S)$. This cancels with the weight of the factor $I^{X/S} q^{\beta - \beta_S}$, which is

$$\begin{aligned} &- c_1(X/S).\beta + c_1(X).\beta - c_1(X).\beta_S \\ &= c_1(S).\beta - c_1(X).\beta_S \\ &= -c_1(X/S).\beta_S. \end{aligned}$$

Hence *I is also homogeneous of degree 0.*

7. Extension of quantum $\mathscr{D}$ modules

In this section we will complete the proof of the main theorem (Theorem 0.6) on invariance of quantum rings under ordinary flops of splitting type. Proposition refp:6.11 guarantees the $\mathscr{F}$-invariance of the Picard–Fuchs systems (in the fiber directions). In order to construct the $\mathscr{D}$ module $\mathscr{M}_I = \mathscr{D} I$, we will need to find the derivatives in the general base directions. This will be accomplished by a lifting of the QDE on the base S. Put these together, we will show that they generate enough (correct) equations for $\mathscr{M}_I^X$. This is referred as the quantum Leray–Hirsch theorem, which is the content of Theorem 0.5 (= Theorem 7.6 + Theorem 7.8 + Theorem 7.10).

To obtain the (true) quantum $\mathscr{D}$-module $\mathscr{M}_J^X$ (on a sufficiently large Zariski closed subset given by the image of $\tau(\hat{t})$), we apply the Birkhoff factorization on $\mathscr{M}_I^X$. We specifically choose a way to perform BF such that the $\mathscr{F}$-invariance can be checked.

Before proceeding to the first step, let us lay out the notations and conventions for this section.

Notations 7.1. We use $\bar{\beta} \in NE(S)$, $\bar{t} \in H(S)$ etc. to denote objects in S. When they are viewed as objects in X, $\bar{\beta}$ means the canonical lift, $\bar{t}$ means the pullback $\bar{p}^* : H(S) \to H(X)$.

For a basis $\{\bar{T}_i\}$ of $H(S)$, denote $\bar{t} = \sum \bar{t}^i \bar{T}_i$ a general element in $H(S)$. When $\bar{T}_i$ is considered as an element in $H(X)$, we sometimes abuse the notation by setting $T_i := \bar{T}_i$.

Given a basis $\{\bar{T}_i\}$ of $H(S)$, we use the following *canonical basis* for $H(X)$:

$$\{T_{\mathbf{e}} = \bar{T}_i h^l \xi^m \mid 0 \le l \le r, 0 \le m \le r+1\}.$$

A general element in $H(X)$ is denoted $t = \sum t^{\mathbf{e}} T_{\mathbf{e}}$. The index set of the canonical basis is denoted Λ^+.

By abusing the notations, if $T_{\mathbf{e}} = \bar{T}_i$ (i.e. $l = m = 0$), we set $t^{\mathbf{e}} = t^i = \bar{t}^i$. Similarly we set $t^{\mathbf{e}} = t^1$ for $T_{\mathbf{e}} = h$, and $t^{\mathbf{e}} = t^2$ for $T_{\mathbf{e}} = \xi$. That is, we reserve the index 0, 1 and 2 for 1, h and ξ respectively.

On $H(X')$ the canonical basis is chosen to be

$$\{T'_{\mathbf{e}} := \mathscr{F} T_{\mathbf{e}} = \bar{T}_i (\xi' - h')^l \xi'^m\}$$

so that it shares the same coordinate system as $H(X)$:

$$t = \sum_{\mathbf{e}} t^{\mathbf{e}} T_{\mathbf{e}} \mapsto \mathscr{F} t = \sum_{\mathbf{e}} t^{\mathbf{e}} \mathscr{F} T_{\mathbf{e}} = \sum_{\mathbf{e}} t^{\mathbf{e}} T'_{\mathbf{e}}.$$

7.1. ***I*-lifting of the Dubrovin connection.** Let the quantum differential equation of $QH(S)$ be given by

$$z\partial_i z\partial_j J^S(\bar{t}) = \sum_k \bar{C}_{ij}^k(\bar{t}, \bar{q})\, z\partial_k J^S(\bar{t}).$$

If we write $\bar{C}_{ij}^k(\bar{t}, \bar{q}) = \sum \bar{C}_{ij,\bar{\beta}}^k(\bar{t})\, q^{\bar{\beta}}$, then the effect on the $\bar{\beta}$-components reads as

$$z\partial_i z\partial_j J^S_{\bar{\beta}} = \sum_{k, \bar{\beta}_1} \bar{C}_{ij,\bar{\beta}_1}^k\, z\partial_k J^S_{\bar{\beta} - \bar{\beta}_1}.$$

Now we lift the equation to X. In the following, for a curve class $\bar{\beta} \in NE(S)$, its I-minimal lift in $NE(X)$ is denoted by $\bar{\beta}^I$. We compute

$$\begin{aligned} z\partial_i z\partial_j I &= \sum_{\beta} q^{\beta} e^{\frac{D}{z}+(D.\beta)} I^{X/S}_{\beta} z\partial_i z\partial_j J^S_{\bar{\beta}} \\ &= \sum_{k,\beta,\bar{\beta}_1} q^{\beta} e^{\frac{D}{z}+(D.\beta)} I^{X/S}_{\beta} \bar{C}^k_{ij,\bar{\beta}_1} z\partial_k J^S_{\bar{\beta}-\bar{\beta}_1} \\ &= \sum_{k,\bar{\beta}_1} q^{\bar{\beta}^I_1} e^{D.\bar{\beta}^I_1} \bar{C}^k_{ij,\bar{\beta}_1} z\partial_k \sum_{\beta} q^{\beta-\bar{\beta}^I_1} e^{\frac{D}{z}+D.(\beta-\bar{\beta}^I_1)} I^{X/S}_{\beta} J^S_{\bar{\beta}-\bar{\beta}_1}. \end{aligned} \tag{7.1}$$

The terms in last sum are non-trivial only if $\bar{\beta} - \bar{\beta}_1 \in NE(S)$. However, in this presentation it is not a priori guaranteed that $\beta - \bar{\beta}^I_1$ is I-effective. (Hence, there might be some vanishing terms in the presentation.)

In order to obtain the RHS as an operator acting on I, we will seek to "transform" terms of the form $e^{\frac{D}{z}+D.(\beta-\bar{\beta}^I_1)} I^{X/S}_{\beta} J^S_{\bar{\beta}-\bar{\beta}_1}$ to those of the form $e^{\frac{D}{z}+D.(\beta-\bar{\beta}^I_1)} I^{X/S}_{\beta-\bar{\beta}^I_1} J^S_{\bar{\beta}-\bar{\beta}_1}$. This can be achieved by differentiation the RHS judiciously and will be explained below.

As a first step, we will show that $I^{X/S}_{\beta} = 0$ if $\beta - \bar{\beta}^I_1 \notin NE^I(X)$ and $\bar{\beta} - \bar{\beta}_1 \in NE(S)$.

Definition 7.2. For any one cycle $\beta \in A_1(X)$, effective or not, we define

$$\begin{aligned} n_i(\beta) &:= -\beta.(h + L_i), \\ n'_i(\beta) &:= -\beta.(\xi - h + L'_i), \\ n'_{r+1}(\beta) &:= -\beta.\xi, \end{aligned}$$

where $0 \le i \le r$.

Lemma 7.3. *For $\bar{\beta} \in NE(S)$, the I-minimal lift $\bar{\beta}^I \in NE(X)$ satisfies $n_i(\bar{\beta}^I) \ge 0$, $n'_i(\bar{\beta}^I) \ge 0$ for all i.*

Proof. During the proof, the *superscript I* is omitted for simplicity.

By definition,

$$n_i = -\bar{\beta}^I.(h + L_i) = -\mu + \mu_i \ge 0.$$

Similarly for $0 \le i \le r$,

$$n'_i = -\bar{\beta}^I.(\xi - h + L'_i) = \max\{\mu + \mu', 0\} - \mu - \mu'_i.$$

If $\mu + \mu' \ge 0$, we have

$$n'_i = \mu' - \mu'_i \ge 0.$$

Otherwise if $\mu + \mu' < 0$, then we get

$$n'_i = 0 - (\mu + \mu'_i) \ge -(\mu + \mu') > 0. \tag{7.2}$$

Finally for the compactification factor $\mathscr{O}$, we get

$$n'_{r+1} = -\bar{\beta}^I.\xi = \max\{\mu + \mu', 0\} \ge 0.$$

□

Let $\beta, \beta' \in A_1(X)$ be (not necessarily effective) one cycles. By definition of I-function, the β factor corresponding to $h + L_i$ is

$$A_{i,\beta} = \frac{1}{\prod_{m=0}^{\beta.(h+L_i)} (h + L_i + mz)},$$

which depends only on the intersection number. Suppose that

$$l_i := \beta'.(h + L_i) - \beta.(h + L_i) \geq 0,$$

we have

$$A_{i,\beta} = A_{i,\beta'} \prod_{m=\beta.(h+L_i)+1}^{\beta'.(h+L_i)} (h + L_i + mz). \tag{7.3}$$

We say that $A_{i,\beta}$ is a product of $A_{i,\beta'}$ with a (cohomology-valued) factor of length l_i. The factors corresponding to $\xi - h + L'_i$ and ξ behave similarly.

Lemma 7.4. *Let $\beta \in NE(X)$ and $\beta - \bar{\beta}^I_1$ be an I-effective class. $I^{X/S}_\beta$ is the product of $I^{X/S}_{\beta-\bar{\beta}^I_1}$ with a factor which is a product of length $n_i(\bar{\beta}^I_1)$, $n'_i(\bar{\beta}^I_1)$, and $n'_{r+1}(\bar{\beta}^I_1)$ corresponding to $h + L_i$, $\xi - h + L'_i$, and ξ respectively.*

If $\beta - \bar{\beta}^I_1$ is not I-effective, the conclusion holds in the sense that $I^{X/S}_\beta = 0$.

Proof. Set $\beta' = \beta - \bar{\beta}^I_1$ in (7.3), the length is

$$(\beta' - \beta).(h + L_i) = -\bar{\beta}^I_1.(h + L_i) = n_i(\bar{\beta}^I_1).$$

The argument for $\xi - h + L'_i$ and ξ are similar.

If $\beta - \bar{\beta}^I_1$ is not I-effective, formally $I^{X/S}_{\beta-\bar{\beta}^I_1} = 0$ contains either the Chern polynomial f_F or $f_{N\oplus\mathscr{O}}$ in its numerator. Notice that (7.3) holds formally.

This proves the lemma. □

our next step is to show that the factors in (7.3) can be obtained by introducing certain differential operators acting on I.

Definition 7.5. An one cycle $\beta \in A_1(X)$ is called *admissible* if $n_i(\beta) \geq 0$, $n'_i(\beta) \geq 0$, and $n'_{r+1}(\beta) \geq 0$. For admissible β we define differential operators

$$\begin{aligned}
D^A_\beta &:= \prod_{i=0}^{r} \prod_{m=0}^{n_i(\beta)-1} (z\partial_{h+L_i} - mz),\\
D^B_\beta &:= \prod_{i=0}^{r} \prod_{m=0}^{n'_i(\beta)-1} (z\partial_{\xi-h+L'_i} - mz),\\
D^C_\beta &:= \prod_{m=0}^{n'_{r+1}(\beta)-1} (z\partial_\xi - mz),\\
D_\beta(z) &:= D^A_\beta D^B_\beta D^C_\beta.
\end{aligned}$$

Now we are ready to lift the quantum differential equations for J^S to equations for I^X.

Theorem 7.6 (I-lifting of QDE)**.** *The Dubrovin connection on $QH(S)$ can be lifted to $H(X)$ as*

$$z\partial_i z\partial_j I = \sum_{k,\bar\beta} q^{\bar\beta^*} e^{D.\bar\beta^*} \bar C^k_{ij,\bar\beta}(\bar t)\, z\partial_k D_{\bar\beta^*}(z) I \tag{7.4}$$

where $\bar\beta^* \in A_1(X)$ is any admissible lift of $\bar\beta$, which in particular implies the well-definedness of the operators $D_{\bar\beta^*}(z)$.

Furthermore, one can always choose $\bar\beta^*$ to be effective. An example of an effective lift is the I-minimal lift $\bar\beta^* = \bar\beta^I$, which is the only admissible lift if and only if $\mu + \mu' \ge 0$.

In general, all liftings are related to each other modulo the Picard–Fuchs system generated by $\Box_\ell$ and $\Box_\gamma$.

Proof. We apply the calculation in (7.1) with $\bar\beta_1^I$ being replaced by a general admissible lift $\bar\beta_1^*$. For $\bar t = \bar t_1 + \bar t_2$ with $\bar t_1$ being the divisor part,

$$\begin{aligned}&\sum_\beta q^{\beta-\bar\beta_1^*} e^{\frac{D}{z}+D.(\beta-\bar\beta_1^*)} I^{X/S}_\beta J^S_{\bar\beta-\bar\beta_1}(\bar t)\\ &\qquad = \sum_\beta D_{\bar\beta_1^*}(z) q^{\beta-\bar\beta_1^*} e^{\frac{D+\bar t_1}{z}+(D+\bar t_1).(\beta-\bar\beta_1^*)} I^{X/S}_{\beta-\bar\beta_1^*} J^S_{\bar\beta-\bar\beta_1}(\bar t_2) = D_{\bar\beta_1^*}(z) I.\end{aligned}$$

Now we prove the last statement. Any two (admissible) lifts differ by some $a\ell + b\gamma$. Say, $\beta'' = \beta' + a\ell + b\gamma$. Then we have

$$\begin{aligned} n_i(\beta'') &= n_i(\beta') - a,\\ n_i'(\beta'') &= n_i'(\beta') + (a-b),\\ n'_{r+1}(\beta'') &= n'_{r+1}(\beta') - b.\end{aligned} \tag{7.5}$$

Then it is elementary to see that we may connect β' to β'' by adding or subtracting ℓ or γ once a time, with all the intermediate steps β_j' being admissible. For example, if $a > 0$, $b > 0$ and $a - b > 0$, then we start by adding ℓ up to $j = a - b$ times. Then we iterate the process: Adding γ followed by adding ℓ, up to b times. Thus we only have to consider the two cases (1) $\beta'' = \beta' + \ell$ or (2) $\beta'' = \beta' + \gamma$.

For case (1), we get from (7.5) with $(a, b) = (1, 0)$ that $n_i(\beta') \ge 1$ for all i. This implies that $D^A_{\beta'} = D^{A+}_{\beta'} D^A_0$ where $D^A_0 = \prod_{j=0}^r z\partial_{a_j}$ comes from the product of $m = 0$ terms. Since $\Box_\ell I = 0$, we compute

$$D_{\beta'}(z) I = D^B_{\beta'} D^C_{\beta'} D^{A+}_{\beta'} q^\ell e^{t^1} \prod_{j=0}^r z\partial_{b_j} I.$$

Now we move $q^\ell e^{t^1}$ to the left hand side of all operators by noticing

$$z\partial_h e^{t^1} = e^{t^1}(z\partial_h + z)$$

in the operator sense. Then (notice that $D^C_{\beta'} = D^C_{\beta'+\ell}$)

$$D_{\beta'}(z)I = q^\ell e^{t^1} D^{B+}_{\beta'+\ell} D^C_{\beta'} D^A_{\beta'+\ell} \prod_{j=0}^r z\partial_{b_j} I = q^\ell e^{t^1} D_{\beta'+\ell}(z)I,$$

which is the desired factor for β''.

The proof for case (2) is entirely similar, with $\Box_\gamma I = 0$ being used instead, and is thus omitted.

The uniqueness statement for $\mu + \mu' \geq 0$ follows from (7.5) and the observation: $n_i(\bar{\beta}^I) = \mu - \mu_i$ and $n'_i(\bar{\beta}^I) = \mu' - \mu'_i$, both attain 0 somewhere and there is no room to move around. The proof is complete. $\Box$

Notice that the liftings of QDE may not be unique. We will see the importance of such a freedom when we discuss the $\mathscr{F}$-invariance property.

7.2. **Quantum Leray–Hirsch.**

Definition 7.7. Let $T_{\mathbf{e}} = \bar{T}_i h^l \xi^m$ be an element in the canonical basis of $H(X)$. The *naive quantization* of $T_{\mathbf{e}}$ is defined as (c.f. (5.2) and (5.9))

$$\hat{T}_{\mathbf{e}} := \partial^{z\mathbf{e}} = z\partial_{\bar{t}^i}(z\partial_{t^1})^l(z\partial_{t^2})^m.$$

Theorem 7.8 (Quantum Leray–Hirsch)**.** *The I-lifting (7.4) of quantum differential equations on S and the Picard–Fuchs equations determine a first order matrix system under the naive quantization $\partial^{z\mathbf{e}}$ of canonical basis $T_{\mathbf{e}}$'s of $H(X)$:*

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I)C_a(z,q), \qquad t^a \in \{t^1, t^2, \bar{t}^i\}.$$

This system has the property that for any fixed $\bar{\beta} \in NE(S)$, the coefficients are formal functions in $\bar{t}$ and polynomial functions in $q^\gamma e^{t^2}$, $q^\ell e^{t^1}$ and the basic rational function $\mathbf{f}(q^\ell e^{t^1})$, defined in (2.2)*.*

We start with an overview of the general ideas involved in the proof. The Picard–Fuchs system generated by $\Box_\ell$ and $\Box_\gamma$ is a perturbation of the Picard–Fuchs (hypergeometric) system associated to the (toric) fiber by operators in base divisors. The fiberwise toric case is a GKZ system, which by the theorem of Gelfand–Kapranov–Zelevinsky is a holonomic system of rank $(r+1)(r+2)$, the dimension of cohomology space of a fiber. It is also known that the GKZ system admits a Gröbner basis reduction to the holonomic system.

We apply this result in the following manner: We will construct a $\mathscr{D}$ module with basis $\partial^{z\mathbf{e}}$, $\mathbf{e} \in \Lambda^+$. We apply operators $z\partial_{t^1}$, $z\partial_{t^2}$ and first order operators $z\partial_i$'s to this selected basis. Notice that

$$\begin{aligned} \Box_\ell &= (1 - (-1)^{r+1} q^\ell e^{t^1})(z\partial_{t^1})^{r+1} + \cdots, \\ \Box_\gamma &= (z\partial_{t^2})^{r+2} + \cdots. \end{aligned}$$

The Gröbner basis reduction allows one to reduce the differentiation order in $z\partial_{t^1}$ and $z\partial_{t^2}$ to smaller one. In the process higher order differentiation in $z\partial_i$'s will be introduced. Using the I-lifting, the differentiation in the base direction with order higher than one can be reduced to one by introducing more terms with strictly larger effective classes in $NE(S)$. A refinement of these observations will lead to a proof, which is presented below.

Remark 7.9. In fact, neither the Gröbner basis nor the GKZ theorem will be needed, due to the simple feature of the Picard–Fuchs system we have for split ordinary flops.

Proof. Consider first the case of simple P^r flops ($S = \mathrm{pt}$). In this special case the Gröbner basis is already at hand. The naive quantization of canonical cohomology basis gives

$$\partial^{z(i,j)} := (z\partial_{t^1})^i (z\partial_{t^2})^j, \qquad 0 \le i \le r, \quad 0 \le j \le r+1.$$

Then further differentiation in the t^1 direction leads to

$$z\partial_{t^1}\partial^{z(i,j)} = \partial^{z(i+1,j)}.$$

It is clear that we only need to deal with the boundary case $i = r$, when the RHS goes beyond the standard basis.

Case $(i,j) = (r,0)$. The equation $\Box_\ell = (z\partial_{t^1})^{r+1} - q^\ell e^{t^1}(z\partial_{t^2} - z\partial_{t^1})^{r+1} \equiv 0$ modulo I leads to

$$(z\partial_{t^1})^{r+1} \equiv \frac{q^\ell e^{t^1}}{1-(-1)^{r+1}q^\ell e^{t^1}} \sum_{k=1}^{r+1} C_k^{r+1} (z\partial_{t^2})^k (-z\partial_{t^1})^{r+1-k}, \tag{7.6}$$

which solves the case.

Case $(i,j) = (r, \ge 1)$. For $j \ge 1$, notice that $\Box_\gamma = z\partial_{t^2}(z\partial_{t^2} - z\partial_{t^1})^{r+1} - q^\gamma e^{t^2} \equiv 0$ modulo I. Hence

$$\begin{aligned} (z\partial_{t^1})^{r+1}(z\partial_{t^2})^j &= q^\ell e^{t^1}(z\partial_{t^2})^j (z\partial_{t^2} - z\partial_{t^1})^{r+1} \\ &\equiv q^\ell e^{t^1}(z\partial_{t^2})^{j-1} q^\gamma e^{t^2} \\ &= q^\ell e^{t^1} q^\gamma e^{t^2}(z\partial_{t^2} + z)^{j-1}. \end{aligned} \tag{7.7}$$

This in particular solves the other cases $1 \le j \le r+1$.

Similarly differentiation in the t^2 direction:

$$z\partial_{t^2}\partial^{z(i,j)} = \partial^{z(i,j+1)}.$$

And we only need to deal with the boundary case $j = r+1$.

Case $(i,j) = (0, r+1)$. First of all, $\Box_\gamma I = 0$ leads to

(7.8)

$$\begin{aligned} (z\partial_{t^2})^{r+2} &\equiv -(-1)^{r+1}(z\partial_{t^1})^{r+1} z\partial_{t^2} - \sum_{k=1}^{r} C_k^{r+1}(z\partial_{t^2})^{k+1}(-z\partial_{t^1})^{r+1-k} + q^\gamma e^{t^2} \\ &= (1-(-1)^{r+1}q^\ell e^{t^1})q^\gamma e^{t^2} - \sum_{k=1}^{r}(-1)^{r+1-k} C_k^{r+1}\partial^{z(r+1-k,k+1)}, \end{aligned}$$

which solves the case.

Case $(i,j) = (\geq 1, r+1)$. By further differentiating t^1 on (7.8) and on (7.7) with $j = r+2$ we get

(7.9)
$$
\begin{aligned}
(z\partial_{t^1})^i (z\partial_{t^2})^{r+2} &\equiv (z\partial_{t^1})^i q^\gamma e^{t^2} - (-1)^{r+1} (z\partial_{t^1})^i q^\ell e^{t^1} q^\gamma e^{t^2} \\
&\quad - \sum_{k=1}^{r} (-1)^{r+1-k} C_k^{r+1} (z\partial_{t^1})^{r+1+(i-k)} (z\partial_{t^2})^{k+1} \\
&= q^\gamma e^{t^2} (z\partial_{t^1})^i - (-1)^{r+1} q^\ell e^{t^1} q^\gamma e^{t^2} (z\partial_{t^1} + z)^i \\
&\quad - \sum_{k=i+1}^{r} (-1)^{r+1-k} C_k^{r+1} \partial^{z(r+i+1-k, k+1)} \\
&\quad - q^\ell e^{t^1} q^\gamma e^{t^2} \sum_{k=1}^{i} (-1)^{r+1-k} C_k^{r+1} (z\partial_{t^1} + z)^{i-k} (z\partial_{t^2} + z)^k.
\end{aligned}
$$

This in particular solves the remaining cases $1 \leq i \leq r$.

An important observation of the above calculation of the matrix $C_1(z,q)$, $C_2(z,q)$ is that C_i is constant in z when modulo q^γ. Moreover $q^{d_2\gamma}$ appears only in $d_2 = 1$.

Now we consider the case with base S. The Picard–Fuchs equations are

$$
\begin{aligned}
\Box_\ell &= \prod_{j=0}^{r} z\partial_{h+L_j} - q^\ell e^{t^1} \prod_{j=0}^{r} z\partial_{\xi - h + L'_j}, \\
\Box_\gamma &= z\partial_\xi \prod_{j=0}^{r} z\partial_{\xi - h + L'_j} - q^\gamma e^{t^2}.
\end{aligned}
\tag{7.10}
$$

Recall that for a basis element $T_{\mathbf{e}} = \bar{T}_s h^i \xi^j$ in its canonical presentation $(0 \leq i \leq r, 0 \leq j \leq r+1)$, we associated its naive quantization

$$
\hat{T}_{\mathbf{e}} = \partial^{z\mathbf{e}} = z\partial_{\bar{t}^s} (z\partial_{t^1})^i (z\partial_{t^2})^j.
\tag{7.11}
$$

The above calculations (7.6) — (7.9) need to be corrected by adding more differential symbols which may consist of higher derivatives in base divisors $z\partial_{L_j}$'s and $z\partial_{L'_j}$'s instead of a single $z\partial_{\bar{t}^s}$. Thus they are not yet in the desired form (7.11). The I-lifting (7.4) helps to reduce higher derivatives in base to the first order ones. Although new derivatives $D_{\bar{\beta}}$'s may appear during this reduction, it is crucial to notice that they all come with non-trivial classes $q^{\bar{\beta}^I}$'s.

With these preparations, we will prove the theorem by constructing

$$
C_{a,\bar{\beta}}(z) = \sum_{\beta \mapsto \bar{\beta}} C_{a;\beta}(z)\, q^\beta
$$

for any fixed $\bar{\beta} \in NE(S)$.

For $\bar{\beta} = 0$, the I-lifting (7.4) introduces no further derivatives: $D_{\bar{\beta}=0}(z) = \mathrm{Id}$. Thus higher order differentiations on $\bar{t}^s$'s can all be reduced to the first order. Notice that in (7.10) all the corrected terms have $(z\partial_{t^1})^i (z\partial_{t^2})^j$ in the canonical range, hence (7.6) — (7.9) plus (7.4) lead to the desired matrix $C_{a;\bar{\beta}=0}(z)$.

Given $\bar{\beta} \in NE(S)$, to determine the coefficient $C_{a,\bar{\beta}}$ from calculating $z\partial_a(\partial^{z\mathbf{e}})$, it is enough to consider the restriction of (7.4) to the finite sum over $\bar{\beta}' \leq \bar{\beta}$. We repeatedly apply the following two constructions:

(i) The double derivative in base can be reduced to single derivative by (7.4). If new non-trivial derivative $D_{\bar{\beta}_1}(z)$ is introduced then the order $q^{\bar{\beta}_1^I}$ is added, thus such processes will produce classes with image outside $NE_{\leq\bar{\beta}}(S)$ in finite steps. In fact the only term in (7.4) not increasing the order in $NE(S)$ is given by

$$\bar{C}^k_{aj;\bar{\beta}=0}\, z\partial_k.$$

This is precisely the structural constant of cup product on $H(S)$, which is non-zero only if

$$\deg \bar{T}_a + \deg \bar{T}_j = \deg \bar{T}_k.$$

Hence $\deg \bar{T}_k \geq \deg \bar{T}_a$, with equality holds only if $\bar{T}_j = 1$, which may occur only for the first step. Any further reduction of base double derivatives $z\partial_k z\partial_l$ into a single derivative $z\partial_m$ must then increase the cohomology degree $\deg \bar{T}_m > \deg \bar{T}_k$, if the order in $NE(S)$ is not increased. It is clear the process stops in finite steps.

(ii) Each time we have terms not in the reduced form (7.11) we perform the Picard–Fuchs reduction (7.6) — (7.9) with correction terms. After the first step in simplifying $z\partial_{t^1}(\partial^{z\mathbf{e}})$ and $z\partial_{t^2}(\partial^{z\mathbf{e}})$, in all the remaining steps we face such a situation only when we have non-trivial terms $D_{\bar{\beta}_1}(z)$ from construction (i). As before this produces classes with image outside $NE_{\leq\bar{\beta}}(S)$ in finite steps.

Combining (i) and (ii) we obtained $C_{a;\bar{\beta}}$ in finite steps. It is clearly polynomial in z, $q^{\gamma}e^{t^2}$, $q^{\ell}e^{t^1}$ and $\mathbf{f}(q^{\ell}e^{t^1})$ since this holds for each steps. $\square$

Theorem 7.10 (Naturality). *The system is $\mathscr{F}$-invariant. That is, $\mathscr{F}C_a(\hat{t}) \cong C'_a(\mathscr{F}\hat{t})$.*

Proof. We have seen the $\mathscr{F}$-invariance of the Picard–Fuchs systems. It remains to show the $\mathscr{F}$-invariance of the I-lifting of the base Dubrovin connection, up to modifications by $\Box_\ell$ and $\Box_\gamma$.

By (7.4), the simplest situation to achieve such an invariance is the case that $\mathscr{F}\bar{\beta}^I = \bar{\beta}^{I'}$, since then $\mathscr{F}D_{\bar{\beta}^I}(z) = D'_{\bar{\beta}^I}(z)$ as well.

Indeed, when $\mu + \mu' \geq 0$ for a curve class $\bar{\beta}$, we do have

$$\begin{aligned}\mathscr{F}\bar{\beta}^I &= \mathscr{F}(\bar{\beta} - \mu\ell - (\mu+\mu')\gamma)\\ &= \bar{\beta} + \mu\ell' - (\mu+\mu')(\ell'+\gamma')\\ &= \bar{\beta} - \mu'\ell' - (\mu+\mu')\gamma' = \bar{\beta}^{I'}.\end{aligned}$$

It remains to analyze the case $\mu + \mu' < 0$ for $\bar\beta$. In this case,

$$\mathscr{F}\bar\beta^I - \bar\beta^{I'} = \bar\beta + \mu\ell' - (\bar\beta - \mu'\ell') = (\mu + \mu')\ell' = -\delta\ell',$$

where $\delta := -(\mu + \mu') > 0$ is the finite gap. Thus

$$\mathscr{F} q^{\bar\beta^I - \delta\ell} = q^{\bar\beta^{I'}}$$

and this suggests that we should try to decrease $\bar\beta^I$ by ℓ for δ times.

In other words, we should expect to have another valid lifting:

$$z\partial_i z\partial_j I = \sum_{k,\bar\beta} q^{\bar\beta^I - \delta\ell} e^{D.\bar\beta^I} \bar C^k_{ij,\bar\beta}(\bar t)\, z\partial_k D_{\bar\beta^I - \delta\ell}(z) I. \tag{7.12}$$

This is easy to check: Notice that $n_i(\bar\beta^I - \delta\ell) = n_i(\bar\beta^I) + \delta > 0$. $n'_i(\bar\beta^I - \delta) = n'_i(\bar\beta^I) - \delta$, which is also $n'_i(\bar\beta + \mu'\ell) = \mu' - \mu'_i \ge 0$ (c.f. the gap in (7.2)). $n'_{r+1} \ge 0$ is unchanged. Thus, the operator $D_{\bar\beta^I - \delta\ell}$ is well defined, though $\bar\beta^I - \delta\ell$ may not be effective. By Theorem 7.6, (7.12) is also a lift and the theorem is proved. □

7.3. **Reduction to the canonical form: The final proof.** We will construct a gauge transformation B to eliminate all the z dependence of C_a. The final system is then equivalent to the Dubrovin connection on $QH(X)$. Such a procedure is well known in small quantum cohomology of Fano type examples or in the context of abstract quantum cohomology. (See e.g. [7] and references therein.) Here we will also need to take into account the role played by the generalized mirror transformation (GMT) $\tau(\hat t)$.

In fact B is nothing more than the Birkhoff factorization introduced before:

$$\partial^{z\mathbf{e}} I(\hat t) = (z\nabla J)(\tau) B(\tau) \tag{7.13}$$

valid at the generalized mirror point $\tau = \tau(\hat t)$. Thus B exists uniquely via an inductive procedure. However the analytic (formal) dependence of B is not manifest if one proceeds in this direction, as the procedure involves J and I, for neither the analytic dependence holds. Therefore, it is not clear how to prove $\mathscr{F} B \cong B'$ up to analytic continuations.

In this subsection we will proceed in a slightly different, but ultimately equivalent, way. Namely we study instead the gauge transformation B directly from the differential system

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a. \tag{7.14}$$

Even though the solutions I are not $\mathscr{F}$-invariant, the system is by Theorem 7.10. This system can be analyzed inductively with respect to the partial ordering of the Mori cone on the base $NE(S)$.

Substituting (7.13) into (7.14), we get $z\partial_a(\nabla J)B + z(\nabla J)\partial_a B = (\nabla J)BC_a$, hence

$$z\partial_a(\nabla J) = (\nabla J)(-z\partial_a B + BC_a)B^{-1} =: (\nabla J)\tilde C_a. \tag{7.15}$$

We note the subtlety in the meaning of $\tilde{C}_a(\hat{t})$. Let $\tau = \sum \tau^\mu T_\mu$. Then the QDE reads as

$$z\partial_\mu(\nabla J)(\tau) = (\nabla J)(\tau)\tilde{C}_\mu(\tau),$$

where $\tilde{C}_\mu(\tau)$ is the structure matrix of quantum multiplication at the point $\tau \in H(X)$. Then

$$z\partial_a(\nabla J) = \sum_\mu \frac{\partial \tau^\mu}{\partial t^a} z\partial_\mu(\nabla J) = (\nabla J)\sum_\mu \tilde{C}_\mu \frac{\partial \tau^\mu}{\partial t^a},$$

hence

$$\tilde{C}_a(\hat{t}) \equiv \sum_\mu \tilde{C}_\mu(\tau(\hat{t}))\frac{\partial \tau^\mu}{\partial t^a}(\hat{t}). \tag{7.16}$$

In particular $\tilde{C}_a$ is independent of z.

With this understood, (7.15) in fact is equivalent to

$$\tilde{C}_a = B_0 C_{a;0} B_0^{-1} \tag{7.17}$$

and the cancellation equation

$$z\partial_a B = BC_a - B_0 C_{a;0} B_0^{-1} B, \tag{7.18}$$

where the subscript 0 stands for the coefficients of z^0 in the z expansion.

Our plan is to analyze $B = B(z)$ with respect to the weight $w := (\bar{\beta}, d_2) \in W$, which carries a natural partial ordering. The initial condition is $B_{w=(0,0)} = \mathrm{Id}$: Since we have seen that for $w = (0,0)$, C_a has only z constant terms $C_{a;(0,0),0}\, z^0$. The total z constant terms in (7.18) are trivially compatible. They are $-B_0 C_{a,0}$ on both sides.

Now perform the induction on W. Suppose that $B_{w'}$ satisfies $\mathscr{F} B_{w'} = B'_{w'}$ for all $w' < w$. Then

$$z\partial_a B_w = \sum_{w_1+w_2=b} B_{w_1} C_{a;w_2} - \sum_{w_1+w_2+w_3+w_4=w} B_{w_1,0} C_{a;w_2,0} B^{-1}_{w_3,0} B_{w_4}. \tag{7.19}$$

Write $C_{a;w} = \sum_{j=0}^{m(w)} C_{a;w,j}\, z^j$ and $B_w = \sum_{j=0}^{n(b)} B_{w,j}\, z^j$. Then (7.19) implies that

$$n(w) = \max_{w'<w}(n(w') + m(w - w')) - 1.$$

Notice that on the RHS all the B terms have strictly smaller degree than w except

$$B_w C_{a;(0,0)} - C_{a;(0,0)} B_w + B_{w,0} C_{a;(0,0)} - C_{a;(0,0)} B^{-1}_{w,0}$$

which has maximal z degree $\leq n(w)$. Moreover, by descending induction on the z degree, to determine $B_{w,j}$ we need only $B_{w'}$ with $w' < w$ or $B_{w,j'}$ with $j' > j$, which are all $\mathscr{F}$-invariant by induction. Hence the difference satisfies

$$\partial_a(\mathscr{F} B_{w,j} - B'_{w,j}) = 0.$$

The functions involved are all formal in $\bar{t}$ and analytic in t^1, t^2, and without constant term ($B_{w=(0,0)} = \mathrm{Id}$). Hence $\mathscr{F} B_{w,j} = B'_{w,j}$.

To summarize, we have proved that for any $\hat{t} = \bar{t} + D \in H(S) \oplus \mathbb{C}h \oplus \mathbb{C}\xi$,

$$\mathscr{F}B(\tau(\hat{t})) \cong B'(\tau'(\hat{t})).$$

In particular, by (7.17) this implies that the $\mathscr{F}$-invariance of $\tilde{C}_a(\hat{t})$. In more explicit terms, we have the $\mathscr{F}$-invariance of

$$\tilde{C}^{\kappa}_{a\nu} = \sum_{n \geq 0,\, \mu} \frac{q^{\beta}}{n!} \frac{\partial \tau^{\mu}(\hat{t})}{\partial t^{a}} \langle T_{\mu}, T_{\nu}, T^{\kappa}, \tau(\hat{t})^{n} \rangle_{\beta} \tag{7.20}$$

for arbitrary (basis elements) T_ν, $T^\kappa \in H(X)$.

The very special case $T_\nu = 1$ leads to non-trivial invariants only for 3-point classical invariant ($n = 0$) and $\beta = 0$, and also $\mu = \kappa$. Since κ is arbitrary, we have thus proved the $\mathscr{F}$-invariance of $\partial_a \tau$. Then

$$\partial_a(\mathscr{F}\tau - \tau') = \mathscr{F}\partial_a\tau - \partial_a\tau' = 0.$$

Again since $\tau(\hat{t}) = \hat{t}$ for $(\bar{\beta}, d_2) = (0, 0)$, this proves

$$\mathscr{F}\tau = \tau'.$$

Remark 7.11. $\tilde{C}_a$ is the derivative of the 2-point (Green) function at $\tau(\hat{t})$:

$$\tilde{C}^{\kappa}_{a\nu} = \frac{\partial}{\partial t^{a}} \langle\langle T_\nu, T^\kappa \rangle\rangle(\tau).$$

Now we may finish the proof of the quantum invariance (Theorem 0.6).

Proof. Since we have established the analytic continuation of B (hence P) and τ, by Proposition 5.11 (reduction to special BF/GMT with ξ class) and Lemma 6.12 (naive quasi-linearity with ξ class) the invariance of quantum ring is proved. □

Remark 7.12. We sketch an alternative shortcut to the proof to minimize the usage of extremal functions and completely get rid of the quasi-linearity reduction.

Indeed, by degeneration reduction (§3), the quantum invariance problem is reduced to local models for descendent invariants of special type. Theorem 4.2 then eliminates the necessity of using ψ classes and we only need to prove the invariance of quantum ring for local models.

Now for split flops, the Birkhoff factorization matrix $B(z)$ exists uniquely. Then quantum Leray–Hirsch theorem (Theorem 7.8) produces the matrix $C_a(z)$ which satisfies the analytic continuation property. The analytic continuation of $B(z)$ is then deduced from it. In particular, (7.17) gives the analytic continuation of $\tilde{C}_a(\hat{t})$, namely (7.20), and then of $\tau(\hat{t})$.

Now we apply the *reduction method* used in the proof of Proposition 5.11, with the role of *special insertion $\tau_k a\xi$* being replaced by *3 primary insertions* T_a, T_ν, T^κ with $T_a \in H(S)$ and T_ν, $T^\kappa \in H(X)$ being arbitrary. We can do so because $\partial\tau/\partial t^a = T_a + \cdots$. Notice that since $n \geq 3$, the divisor reconstruction we need can all be performed within primary invariants.

Namely, using (2.1) for h and ξ, we may reconstruct any $n \geq 3$ point primary invariants by the the one with only two general insertions not from

$H(S)$. As in Step 2 of the proof of Theorem 4.4, the moving of ξ class will always be $\mathscr{F}$-compatible, while the moving of h class to an insertion $t_i h^r$ may generate topological defect. The key point is that this defect can be canceled out by the extremal quantum corrections from some diagonal splitting term. (In fact this is the building block of our determination of the extremal invariants in §2.)

This leads to a logically shorter and more conceptual proof of the quantum invariance theorem.

We present the complete argument for at least two reasons. Firstly, the quantum correction part (extremal case) works for non-split flops as well. Secondly, the BF/GMT algorithm, together with the divisorial reconstruction, provides an effective method to determine all genus zero descendent (not just primary) invariants for any split toric bundles.

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Y.-P. LEE: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112-0090, U.S.A.
E-mail address: `yplee@math.utah.edu`

H.-W. LIN: DEPARTMENT OF MATHEMATICS AND TAIDA INSTITUTE OF MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: `linhw@math.ntu.edu.tw`

C.-L. WANG: DEPARTMENT OF MATHEMATICS, CENTER FOR ADVANCED STUDIES IN THEORETICAL SCIENCES (CASTS), AND TAIDA INSTITUTE OF MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: `dragon@math.ntu.edu.tw`