K-EQUIVALENCE IN BIRATIONAL GEOMETRY

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In this article we survey the background and recent development on the K-equivalence relation among birational manifolds. The content is based on the author's talk at ICCM-2001 at Taipei. I would like to dedicate this article to Professor Chern, Shiing-shen to celebrate his 90th birthday. For manifolds, K-equivalence is the same as c_1-equivalence. In this sense, a major part of birational geometry is really to understand the geometry of the first Chern class.

After a brief historical sketch of birational geometry in §1, we define in §2 the K-partial ordering and K-equivalence in a birational class and discuss geometric situations that will lead to these notions. One application to the filling-in problem for threefolds is given. In §3 we discuss motivic aspect of K-equivalence relation. We believe that K-equivalent manifolds have the same Chow motive though we are unable to prove it at this moment. Instead we discuss various approaches toward the corresponding statements in different cohomological realizations. §4 is devoted to the *Main Conjectures* and the proof of a weak version of it. Namely, up to complex cobordism, K-equivalence can be decomposed into composite of classical flops. Finally in §5 we review some other current researches that are related to the study of K-equivalence relation.

1. A Brief History of Birational Geometry

If not specifically stated, the ground field is assumed to be the complex numbers $\mathbb{C}$. Two algebraic varieties are called birational if they have an isomorphic Zariski open subset. This is equivalent to say that they have isomorphic rational function fields over the ground field. One of the main goals in birational geometry is to find a good geometric model that is convenient for the study of the given algebraic variety or its function field.

1.1. **Minimal Models for Surfaces.** (c.f. [5])

Already at the beginning of the 20th century, Italian algebraic geometers had adapted the above point of view and successfully applied it to the classification theory of algebraic surfaces. This led to the famous *Enrique classification*. It started with the *Castelnuovo's contraction theorem*: if a smooth surface X contains at least one (-1) rational curve C, that is $C \cong \mathbb{P}^1$ with $C^2 = -1$, then C can be contracted to a smooth point under a projective birational morphism $\phi_C : X \to X'$ and one obtains a simplified smooth surface X'. By repeating this process in finite steps, one ends up with a smooth surface called a *minimal model*.

When $\kappa(X) = -\infty$, that is $\Gamma(X, K_X^m) = 0$ for all $m \in \mathbb{N}$, *Enrique's theorem* says that X is birational to a ruled surface $C \times \mathbb{P}^1$.

Date: December 18th, 2001. Revised September 30th, 2002.
Supported by NSC 89-2115-M-007-045 and NSC 90-2115-M-007-025.

When $\kappa(X) \geq 0$, that is $\Gamma(X, K_X^m) \neq 0$ for some $m \in \mathbb{N}$, the minimal model is unique and it admits semi-ample canonical bundle. Namely, the minimality of X implies the *abundance theorem*: the pluri-canonical system $\Gamma(X, K_X^m) : X \to \mathbb{P}^N$ is a morphism for m large. Based on this, a detailed classification of algebraic surfaces according to the Kodaira dimension is then achieved.

Later on, Enrique's classification was extended to compact complex surfaces by Kodaira and to algebraic surfaces over fields of characteristic $p > 0$ by Bombierie and Mumford.

1.2. **Minimal Models for Threefolds.** (c.f. [31] [34]) In 1982, Mori proved the three dimensional generalization of Castelnuovo's contraction theorem [36]. Mori's theorem opened the way to continue the *minimal model program* (MMP for short). He showed that if a three dimensional projective manifold X admits a curve $\tilde{C}$ such that $K_X.\tilde{C} < 0$ (K_X is not nef), then by degenerating $\tilde{C}$ inside X if necessary, one is able to find a rational curve C and a projective morphism $\phi_R : X \to X'$ associated to the *extremal ray* $R = \mathbb{R}[C]$ such that a curve C' is contracted to a point by ϕ_R if and only if $[C'] \in R$. He also classified all the possible types of contractions $\phi_R : X \to X'$. The situation that differs from the surface case is, the resulting threefold X' is usually mildly singular, with the so called *terminal singularities*.

A priori this seems to be an obstacle to continue the process. Fortunately the corresponding contraction theorem for terminal threefolds, as well as its higher dimensional generalization, was soon proved by Kawamata and Shokurov. However, worse singularities may sometimes occur so that $K_{X'}$ is not even rationally defined as a line bundle (X is not $\mathbb{Q}$-Gorenstein). This is the case precisely when the contraction ϕ_R is *small*, that is, the exceptional loci of ϕ_R has codimension at least two in X. In dimension three, this means that the extremal curve C is isolated. The most striking idea, which is also the most difficult step in the three dimensional MMP, is to develop a special type of algebraic surgeries called *flips*. A flip will correct such a pair $C \subset X$ in codimension two along C into another pair $C' \subset X'$ so that $K_{X'}.C' > 0$, hence it will avoid small extremal contractions that leads to uncontrollable singularities.

The existence of flips in dimension three was proved by Mori in 1988 [38] and the three dimensional MMP was thus established. It states that, starting with a smooth threefold X, after a finite number of divisorial extremal contractions and/or flips, one ends up with a $\mathbb{Q}$-factorial terminal model X' which is either (in case $\kappa(X) = -\infty$) a Mori fiber space $\phi : X' \to S$ under a further extremal contraction of fiber type (with each fiber X'_s a $\mathbb{Q}$-Fano variety of Picard number 1) or (in case $\kappa(X) \geq 0$) a minimal model in Mori's sense, namely $K_{X'}$ is nef. In the later case, the abundance conjecture that $K_{X'}$ is indeed semi-ample was subsequently proved by Miyaoka and Kawamata [28]. Detailed classifications of threefolds based on the MMP is now of current research interest.

1.3. **Birational Minimal Threefolds and Flops.** In higher dimensions, there are some recent approaches toward the existence of flips (e.g. Shokurov's work for fourfold log-flips), but these have not yet been completely justified at the time of this writing. Besides the existence problem, even in the three dimensional case, minimal models are in general not unique. Based on Reid and Mori's classification theory of three dimensional singularities [37], Kollár and Mori in 1989 [32] and then in 1992

[33] completely understood the precise relation between two ($\mathbb{Q}$-factorial terminal) birational minimal models. The birational map can always be decomposed into a finite sequence of algebraic surgeries called *flops*. Roughly speaking, each flop is obtained by removing one chain of rational curves C (corresponding to certain Dynkin diagram) in X with $K_X|_C = 0$ then gluing back C into the open space $X \backslash C$ in a different manner.

This statement was first shown by Kawamata in 1986 [27]. Kollár and Mori's method has the advantage to complete the classification of three dimensional flops (and also flips in 1992), and hence showed that although three dimensional birational minimal models are in general not homotopically equivalent, they do have naturally isomorphic ordinary (resp. intersection) cohomology groups, mixed (resp. intersection pure) Hodge structures, set of germs of isolated singularities and local moduli spaces (they actually showed that flops can be performed simultaneously in flat families). These results have put the three dimensional minimal model theory into a solid and useful stage.

2. K-partial Ordering in a Birational Class

2.1. **K-partial Ordering.** (c.f. [48]) We start with a simple observation of the MMP from the point of view of canonical divisors. For a birational map $f : X \dashrightarrow X'$ between two $\mathbb{Q}$-Gorenstein varieties, we say that $X \le_K X'$ (resp. $X <_K X'$) if there is a birational correspondence $(\phi, \phi') : X \leftarrow Y \rightarrow X'$ extending f with Y smooth, such that $\phi^* K_X \le_{\mathbb{Q}} \phi'^* K_{X'}$ (resp. $<_{\mathbb{Q}}$) as divisors. These relations are easily seen to be independent of the choice of Y. Notice that $X \le_K X'$ and $X \ge_K X'$ imply $X =_K X'$, that is $\phi^* K_X =_{\mathbb{Q}} \phi'^* K_{X'}$. In this case, we say that X and X' are K-equivalent. In this K-partial ordering, divisorial contractions and flips will decrease its K-level while flops inducing K-equivalence. It is easy to see that K-equivalent terminal varieties are isomorphic in codimension one. In fact, more is true in general (Theorem 1.4 in [48]):

Let $f : X \dashrightarrow X'$ be a birational map between two varieties with canonical singularities. Suppose that the exceptional locus $Z \subset X$ is proper and that K_X is nef along Z, then $X \le_K X'$. Moreover, If X' is terminal then $\operatorname{codim}_X Z \ge 2$.

Let us recall the proof briefly. Let $(\phi, \phi') : Y \to X \times X'$ be a resolution of f so that the union of the exceptional set of ϕ and ϕ' is a normal crossing divisor of Y. Let $K_Y =_{\mathbb{Q}} \phi^* K_X + E =_{\mathbb{Q}} \phi'^* K_{X'} + E'$. So

$$\phi'^* K_{X'} =_{\mathbb{Q}} \phi^* K_X + F, \quad \text{with } F := E - E'.$$

It suffices to show that $F \ge 0$. Let $F = \sum_{j=0}^{n-1} F_j$ with $\dim \phi'(\operatorname{Supp} F_j) = j$. We will show that $F_j \ge 0$ for $j = n-1, n-2, \cdots, 1, 0$ inductively. As E' is ϕ'-exceptional, $F_{n-1} \ge 0$ is clear. Suppose that we have already shown that $F_j \ge 0$ for $j \ge k+1$.

Consider the surface $S_k := H^{n-2-k}.\phi'^* L^k$ on Y where H is very ample on Y and L is very ample on X'. We get a relations of divisors on S_k:

$$\phi'^* K_{X'}|_{S_k} =_{\mathbb{Q}} \phi^* K_X|_{S_k} + a - b,$$

where $H^{n-2-k}.\phi'^* L^k.F = a - b$ with both a and b effective. Notice that b can only come from F_k since $\sum_{j \ge k+1} F_j \ge 0$ and $L^k \cap \phi'(F_j) = \emptyset$ for $j < k$. Now we look at

$$b.\phi'^* K_{X'} =_{\mathbb{Q}} b.\phi^* K_X + b.a - b^2.$$

The left hand side is always zero since $\phi'(b) \subset L^k \cap \phi'(F_k)$ is zero dimensional. Moreover, since $\phi'^* K_{X'} =_{\mathbb{Q}} \phi^* K_X$ on $\phi^{-1}(X \backslash Z)$, we must have that $\phi(\operatorname{Supp} F) \subset Z$. In particular, $b.\phi^* K_X \geq 0$. It is also clear that $b.a \geq 0$. However, if $b \neq 0$ then it is a nontrivial combination of ϕ' exceptional curves in S_k. By the Hodge index theorem for surfaces we then have that $b^2 < 0$, a contradiction. So $b = 0$ and $F_k \geq 0$. The codimension statement is easy and we omit its proof.

As a corollary, birational minimal models, if they exist, are K-equivalent and reach the lowest K-level among terminal (or even canonical) varieties within their birational class. It is suggestive to make use of this K-equivalence (quasi-uniqueness) and the minimum property to study minimal models.

2.2. **Filling-in Problem in Dimension Three.** (c.f. [47]) Among applications of the *Mori theory*, we mention only one example which also makes use of K-equivalence relation for fourfolds and an extra technique called *symplectic deformations* that will be important in 4.3 in formulating the Main Conjectures.

Let $\mathcal{X} \to \Delta$ be a projective smoothing of a nontrivial Gorenstein minimal threefold $\mathcal{X}_0$ over the unit disk. Then, up to any finite base change, $\mathcal{X} \to \Delta$ is not Δ-birational to a projective smooth family $\mathcal{X}' \to \Delta$ of minimal threefolds.

Proof. By an application of the Shokurov-Kollár connectedness theorem one may show that $\mathcal{X}$ has at most terminal singularities. Then by 2.1, any Δ-birational map f will induce K-equivalence of $\mathcal{X}$ and $\mathcal{X}'$. Hence they are isomorphic in codimension one and f induces a birational map $f_0 : \mathcal{X}_0 \dashrightarrow \mathcal{X}'_0$ between minimal models. By Kollár's result on birational minimal threefolds in 1.3, $\mathcal{X}_0$ can not be $\mathbb{Q}$-factorial since $\mathcal{X}'_0$ is smooth. Now by a result of Kawamata [27] (or by the three dimensional MMP), there is a $\mathbb{Q}$-factorialization $\phi : Y \to \mathcal{X}_0$ with ϕ a small morphism. Again this implies that Y is smooth and there is a birational map of smooth minimal threefolds $f_0 \circ \phi : Y \dashrightarrow \mathcal{X}'_0$. In particular,

$$H^k(Y) \cong H^k(\mathcal{X}'_0) \cong H^k(\mathcal{X}'_t) \cong H^k(\mathcal{X}_t) \quad \text{for all } k \geq 0 \text{ and } t \neq 0.$$

Now we are in a *small contraction/smoothing* diagram:

$$\begin{tikzcd} Y \arrow[d, "\phi"'] & & \\ \mathcal{X}_0 \arrow[r, hook] & \mathcal{X} & \mathcal{X}_t \arrow[l, hook'] \end{tikzcd}$$

In case that $\mathcal{X}_0$ has only ODP's, a simple Mayer-Vietoris argument shows that this is impossible. In fact, consider a diagram as above in the C^∞ category such that near each singular point of $\mathcal{X}_0$ it is a small contraction/smoothing diagram of a germ of ODP. Let C_i's be the rational curves contracted to those ODP's and let $e : \bigoplus_i \mathbb{Z}[C_i] \to H_2(Y, \mathbb{Z})$ be the class map, then $H_2(\mathcal{X}_t) = \operatorname{coker} e$. So, $H_2(\mathcal{X}_t) \cong H_2(Y) \Rightarrow \operatorname{Image} e = 0$, which is impossible since Y is projective.

In general, by Reid's classification [43], three dimensional Gorenstein terminal singularities are exactly isolated cDV points (one parameter deformations of surface ADE singularities). By Friedman's result [19], if $p \in V$ is a germ of an isolated cDV point and $C \subset U$ is the corresponding germ of the (possibly reducible) exceptional curve contracted to p, then the versal deformation spaces $\operatorname{Def}(p, V)$ and $\operatorname{Def}(C, U)$ are both smooth and there is an inclusion map of complex spaces $\operatorname{Def}(C, U) \hookrightarrow \operatorname{Def}(p, V)$. Moreover, one can deform the complex structure of a small neighborhood of C so that in this new complex structure C deforms into several $\mathbb{P}^1$'s and the contraction map deforms to a nontrivial contraction of these $\mathbb{P}^1$'s down to ODP's, while keeping a neighborhood of these ODP's to remain in the versal deformations of the germ $p \in V$.

We can preform this analytic process for all C's and p's simultaneously in each corresponding small neighborhoods and then patch them together smoothly. (In fact, one may keep the overlapped region admitting nearby almost complex structure which is tamed by the original symplectic form [53]. We call this *locally holomorphic* symplectic deformations.) As a result, we obtain a deformed C^∞ diagram which satisfies the conditions stated above, which again leads to a contradiction to the equality of H^2. □

This allows us to construct counterexamples to the so called three dimensional *filling-in problem*. In fact, Clemens had constructed A_2 degenerations of (simply connected) quintic Calabi-Yau hypersurfaces in $\mathbb{P}^4$ over Δ such that the algebraic sub-family over the punctured disk is C^∞ trivial (so that one may replace the special fiber, which is a singular Calabi-Yau with an A_2 singularity, by a real six dimensional smooth manifold to obtain a smooth family over Δ). However, we just show that this smooth replacement can not be achieved in the algebraic category.

One of the key points in the above proof is that birational smooth minimal threefolds have the same Betti numbers. This motivated the author to consider the validity of *equivalence of Betti numbers* in higher dimensions. It is clear that one should find methods independent of the MMP to study relations between K-equivalent varieties or manifolds.

2.3. **Integration Formalism/A Meta Theorem.** (c.f. [48]) Starting with a birational correspondence with smooth Y:

$$\begin{array}{ccc} & Y & \\ {}^{\phi}\swarrow & & \searrow^{\phi'} \\ X & & X' \end{array}$$

From $K_Y = \phi^* K_X + E$ and $K_Y = \phi'^* K_{X'} + E'$, we see that $X =_K X'$ is the same as saying that ϕ and ϕ' have the same *holomorphic Jacobian factor* $E = E'$. If for some geometric/topological invariant $I(X)$ that can be computed from certain integration theory whose *change of variable formula* respects the holomorphic Jacobian factor, then one may conclude that $X =_K X' \implies I(X) = I(X')$ via

$$I(X) = \int_X d\mu_X = \int_Y J(E) d\mu_Y = \int_{X'} d\mu_{X'} = I(X').$$

We are going to discuss several examples of this *Meta Theorem* in the following sections.

3. K-equivalence Relation and Motives

Grothendieck's theory of *motives* is in principle *the* universal cohomology theory which admits various realizations as usual cohomologies (Betti, de Rham, Hodge, ℓ-adic étale and others). The category of motives is supposed to be a homomorphic image of the category of varieties and should have many expected linear structures (like Hodge filtration and Galois actions). Unfortunately such a category has not yet been constructed. The closest one seems to be the Chow motives, or classical motives. Roughly, this category has all varieties as its objects and the morphisms $\mathrm{Hom}_{\mathrm{motive}}(X, X')$ are given by correspondences (cycles) $\Gamma \in A^*(X \times X')$ modulo an *adequate equivalence relation* (e.g. rational equivalence, homological equivalence or numerical equivalence). See e.g. [20] for some basic properties.

We would like to convince the reader that for K-equivalent manifolds under birational map $f : X \dashrightarrow X'$, there is a *naturally attached* correspondence $T \in A^{\dim X}(X \times X')$ of the form $T = \bar{\Gamma}_f + \sum_i T_i$ with $\bar{\Gamma}_f \subset X \times X'$ the cycle of graph closure of f and with T_i's being certain degenerate correspondences (i.e. T_i has positive dimensional fibers when projecting to X or X') such that T is an isomorphism of Chow motives. Currently we do not know how to prove it but some statements in various realizations do admit proofs along the line of our integration formalism.

3.1. Classical Integration and L^2 Cohomology.

The first clue for the author to believe in a close relationship between K-equivalent manifolds is a somewhat näive yet exciting idea which involves degenerate Kähler metrics and L^2 cohomology.

Let X and X be smooth projective (be Kähler is enough) and let $(\phi, \phi') : Y \to X \times X'$ be the birational correspondence which leads to $X =_K X'$. We may select arbitrary Kähler metrics ω and ω' with volume 1 on X and X' respectively. Then we pull backs them to Y to get two degenerate Kähler metrics $\phi^*\omega$ and ${\phi'}^*\omega'$. From c_1-equivalence we see that (let $\dim X = n$)

$$(-\phi^*\partial\bar{\partial}\log\omega^n) - (-{\phi'}^*\partial\bar{\partial}\log{\omega'}^n) = \partial\bar{\partial}f$$

for some C^∞ function f up to a constant. This simplifies to $({\phi'}^*\omega')^n = e^f(\phi^*\omega)^n$. That is, the two degenerate metrics $\phi^*\omega$ and ${\phi'}^*\omega'$ have quasi-equivalent volume forms (both volume forms have the same rate of degeneracy along the common degenerate loci $E \subset Y$).

By using cohomology of L_2 smooth differential forms with respect to a possibly degenerate smooth Kähler metric, $H^k(X) \cong L_2^k(X, \omega) = L_2^k(Y, \phi^*\omega)$. If we may *rotate* $\phi^*\omega$ to ${\phi'}^*\omega'$ through (not necessarily Kähler) degenerate metrics g_t, $t \in [0, 1]$ while keeping the volume degeneracy unchanged, then the theory of L^2 cohomology will lead to a proof of the equivalence of cohomology groups.

One candidate for this rotation is to solve a family of complex Monge-Amperè equations via Yau's solution to the Calabi conjecture [55]:

$$(\tilde{\omega} + \partial\bar{\partial}\varphi_t)^n = e^{t(f + c(t))}(\phi^*\omega)^n,$$

where $\tilde{\omega}$ is an arbitrary Kähler metric with volume 1 on Y and $c(t)$ is a normalizing constant at time t to make the right hand side has total integral 1 over Y. At this moment there are still analytical difficulties of this differential geometric approach that need to be overcome.

3.2. p-adic Integration and Étale Cohomology.

In [48], the author applied the idea of quasi-equivalent volume elements in the theory of p-adic integrals. This extended an earlier result of Batyrev [2] on the equivalence of Betti numbers for birational Calabi-Yau manifolds to general K-equivalent manifolds. In particular this applies to birational smooth minimal models (c.f. §2.1).

We will assume that X and X' are smooth projective. Take an integral model of the K-equivalence diagram, e.g. $X \to \operatorname{Spec} S$ etc. with S a finitely generated $\mathbb{Z}$-algebra. For *almost all* maximal ideals P in S, in fact Zariski open dense in the maximal spectrum of S, we have good reductions of X, X', Y, ϕ and ϕ'. In such cases, let $R = \hat{S}_P$, the completion of S at P with residue field $R/P \cong \mathbb{F}_q$, $q = p^r$ for some r. For ease of notation, we use the same symbol to denote the corresponding object over $\operatorname{Spec} R$. Let U_i's be a Zariski open cover of X such that $K_X|_{U_i}$ is trivial for each i. Then for a compact open subset $S \subset U_i(R) \subset X(R)$, we define its p-adic measure by

$$m_X(S) \equiv \int_S |\Omega_i|_p.$$

This is independent of the choice of Ω. The p-adic measure of $X(R)$ and $X'(R)$ are the same by the *change of variable formula* and $X =_K X'$. By a direct extension of Weil's formula [51], we see that (let $\dim X = n$ and $\bar{X}$ be the special fiber)

$$m_X(X(R)) = \frac{|\bar{X}(\mathbb{F}_q)|}{q^n}.$$

By applying this to finite extensions of $\mathbb{F}_q$, we conclude that X and X' have the same local zeta functions for almost all maximal ideals P.

Knowing this for one P already allows us to apply Grothendieck-Deligne's solution to the celebrated Weil conjecture [13] to conclude that K-equivalent manifolds have the same Betti numbers.

In fact more is true [50]. For simplicity let us assume that X, X' and $X =_K X'$ are defined over a $\mathbb{Z}$-algebra S such that the quotient field F of S is a number field. The general case can be reduced to the number field case by standard tricks, e.g. by taking an F-valued point $\eta : \operatorname{Spec} F \to \operatorname{Spec} S$ and considering the fiber diagram over η. We then know that X and X' have the same local zeta functions for almost all, hence all but finite, $P \in \operatorname{Spec} S$.

By the Cĕbotarev density theorem [44], this implies that, for suitable prime p, the two rational Galois representations $H^k_{et}(X_{\bar{F}}, \mathbb{Q}_p)$ and $H^k_{et}(X'_{\bar{F}}, \mathbb{Q}_p)$ have isomorphic semi-simplifications as $\operatorname{Gal}(\bar{F}/F)$ modules. In other words, X and X' have the same "motives" in the sense of L functions.

By the *Hodge-Tate decomposition theorem* proved by Fontaine and Messing [18] under certain restrictions or by Faltings' complete version of *p-adic Hodge theory* [17], this then implies the equivalence of $\mathbb{Q}_p$ (and hence $\mathbb{Q}$) Hodge numbers.

More precisely, select a prime P so that X and X' have good reductions (this is in fact unnecessary) and let $K = F_P$ be the completion with residue field k of characteristic p. Let $G = \operatorname{Gal}(\bar{K}/K)$ and $\mathbb{C}_p$ be the completion of $\bar{K}$. Then there exists a natural G-equivariant isomorphism [17]:

$$\bigoplus\nolimits_i \left(\mathbb{C}_p \otimes_K H^{m-i}(X_K, \Omega^i)(-i)\right) \cong \mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p),$$

where G acts on $H^{m-i}(X_K, \Omega^i)$ trivially and on the right hand side diagonally, and (j) is the Tate twist by j-th power of cyclotomic character. Since $\mathbb{C}_p^G = K$ and $\mathbb{C}_p(i)^G = 0$ for $i \neq 0$, elementary manipulation shows that

$$h^{i,m-i} = \dim_K \left(\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)^{ss}(i)\right)^G.$$

Finally we plug in $H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)^{ss} \cong H^m_{et}(X'_{\bar{K}}, \mathbb{Q}_p)^{ss}$, which holds by base change theorem, to conclude the non-canonical equivalence of $\mathbb{Q}_p$-Hodge structures. [1]

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[1] I am grateful to C.-L. Chai and J.-K. Yu for discussions on the sufficiency for determining Hodge numbers from semi-simplifications.

In the Algebraic Geometry Conference for Iitaka's 60 at Tokyo, February 2002, T. Ito informed the author that he also obtained the same proof independently [26].

3.3. **Grothendieck Group of Varieties and the Hodge Realization.** Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex varieties with reduced structures. That is, we modulo out the *motivic relation* $[X\backslash Z] = [X] - [Z]$ whenever $Z \subset X$ is a closed subvariety. (For X a variety, we use $[X]$ to denote its class in $K_0(\mathrm{Var}_{\mathbb{C}})$.) To see that $K_0(\mathrm{Var}_{\mathbb{C}})$ contains enough information, notice that the Hodge realization functor $h^{p,q}$ defined on smooth projective varieties has an unique extension to $\chi_c^{p,q} := \sum_i (-1)^i h^{p,q} H_c^i$, the (p,q)-th Euler functor of Deligne's mixed Hodge structures for compactly supported cohomology [14], such that it factors through the ring $K_0(\mathrm{Var}_{\mathbb{C}})$:

$$\begin{tikzcd} \mathrm{Var}_{\mathbb{C}} \arrow[r, "\chi_c^{p,q}"] \arrow[d] & K_0(\mathrm{Hodge}) \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[ur, dashed] & \end{tikzcd}$$

Let $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$ be the Lefschetz class. Notice that $\chi_c^{p,q}$ extends to $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ since $\mathbb{L}$ corresponds to degree shifting operator of Hodge structures.

Now assume that X is smooth and let $\phi : Y \to X$ be the blowing-up of X along a smooth center $Z \subset X$ of codimension r, with exceptional divisor $E \subset Y$. Then we have the well-known motivic equation for projective bundles $E = \mathbb{P}_Z(N_{Z/X}) \to Z$:

$$[E] = [Z](1 + \mathbb{L} + \cdots \mathbb{L}^{r-1}) = [Z][\mathbb{P}^{r-1}].$$

Since $[X] - [Z] = [Y] - [E]$, by formally localizing at $[\mathbb{P}^{r-1}]$ one gets

$$[X] = [Y] - [E] + [Z] = ([Y] - [E]) + [E][\mathbb{P}^{r-1}]^{-1}.$$

We call this a *nice change of variable formula* since the *Jacobian factor* here depends only on the class $[E]$ instead of the precise structure of the normal bundle $N_{Z/X}$.

The above computation can be performed inductively to show that, for $\phi : Y \to X$ a composite of blowing-ups along smooth centers with $K_Y = \phi^* K_X + \sum_{i=1}^n e_i E_i$ and $E := \bigcup_i E_i$ a normal crossing divisor, the change of variable from X to Y reads

$$[X] = \sum\nolimits_{I \subset \{1,\ldots,n\}} [E_I^\circ] \prod\nolimits_{i \in I} [\mathbb{P}^{e_i+1}]^{-1},$$

where $[E_I^\circ] := \bigcap_{i \in I} E_i \backslash \bigcup_{j \notin I} E_j$.

According to the Meta Theorem, if we can prove such a change of variable formula for any birational morphism $\phi : Y \to X$, we will then be able to deduce that two K-equivalent smooth varieties X, X' have $[X] = [X']$ in $S^{-1}K_0(\mathrm{Var}_{\mathbb{C}})$, where S is the multiplicative set generated by the classes of projective spaces. [2] Or equivalently, $[P][X] = [P][X']$ for P a product of projective spaces. Since $\chi_c(V) := \sum_{p,q} \chi_c^{p,q}(V)$ is not a zero divisor for smooth projective V, by applying the functor χ_c we conclude that X and X' have (non-canonically) isomorphic $\mathbb{Q}$-Hodge structures.

The first proof of a weak form of this in certain completion of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is due to Kontsevich and Denef-Loeser by constructing motivic integration (see §3.4 below). A numerical form of this was also proved by Batyrev [3] using his version of motivic integration. A full proof now is a consequence of the *Weak Factorization Theorem* of Wlodarsczyk [54] and Abramovich, Karu, Matsuki and Wlodarsczyk [1]. It states that any birational map can be factorized into sequences of blowing-ups and blowing-downs along smooth centers.

[2] I am grateful to W. Veys for suggesting the formulation of localizations on projective spaces.

3.4. **Nash's Arc Spaces and the Motivic Integration.** During the same time the p-adic proof appeared, based on Kontsevich's idea, Denef and Loeser [15] had constructed the motivic integration over Nash's formal arc spaces [41]. In a correspondence with Loeser (c.f. [48]), we realized that instead of using p-adic integral, if we use motivic integral then their *change of variable formula* (together with Deligne's theory of mixed Hodge structures on arbitrary complex varieties as above) will also lead to a proof of the above formula for arbitrary proper birational morphism ϕ hence a proof of the strengthened result that the Hodge numbers are the same. (See also [3] for an alternative version of motivic integration.)

We give a brief sketch of their construction here. It starts with a measure theory on the semi-algebraic subsets in Nash's arc spaces:

$$\mu_X : \mathbb{B}(\mathcal{L}(X)) \longrightarrow K_0(\widehat{\mathrm{Var}_{\mathbb{C}})}[\mathbb{L}^{-1}].$$

Here $\mathcal{L}(X)$ as a set is simply the power series points of X which also has the structure as a pro-variety $\mathrm{ind.lim}\,\mathcal{L}_m(X)$ with $\mathcal{L}_m(X) = \mathrm{Hom}\,(\mathrm{Spec}\,\mathbb{C}[t]/(t^{m+1}), X)$. There are obvious natural morphisms $\pi_m : \mathcal{L}(X) \to \mathcal{L}_m(X)$ and $\theta_m : \mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$. When X is smooth of dimension n, π_m is surjective and θ_m defines a piecewise affine $\mathbb{A}^n_{\mathbb{C}}$ bundle structure on $\mathcal{L}_m(X)$. In general, when X is singular, π_m, θ_m are not even surjective. But we will restrict to the smooth case here.

A set $S \subset \mathcal{L}(X)$ is stable (or cylindrical) if it is semi-algebraic and is of the form $S = \pi_m^{-1}(A)$ for some (necessarily constructible) set $A \subset \mathcal{L}_m(X)$. In this case we define its $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$-valued measure by $\mu_X(S) = [\pi_m(A)]\mathbb{L}^{-mn}$. This has an unique extension to all semi-algebraic sets $\mathbb{B}(\mathcal{L}(X))$ by partitioning them into certain countable union of stable sets modulo some *measure zero sets.* The measure then takes value in the completion of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ with respect to the filtration $F^p := \{[S]\mathbb{L}^{-i} \mid \dim S - i \le -p\}$. In particular $\mu_X(\mathcal{L}(X)) = [X]$ for smooth X. (Notice that the motivic measure defined here differs from the one in [15] by a factor $\mathbb{L}^{-n}$.)

Given a semi-algebraic set S and a simple function $f : S \to \mathbb{Z} \cup \{\infty\}$ such that $f^{-1}(k)$ is semi-algebraic all k, the motivic integration is defined by

$$\int_S \mathbb{L}^{-f} d\mu_X = \sum_{k \in \mathbb{Z}} \mathbb{L}^{-k} \mu_X(f^{-1}(k)).$$

Now for a birational morphism $\phi : Y \to X$ with Y smooth, ϕ naturally induces a map $\phi_* : \mathcal{L}(Y) \to \mathcal{L}(X)$. If $K_Y = \phi^* K_X + E$ with E a normal crossing divisor, the change of variable formula of Denef and Loeser states that

$$\int_S \mathbb{L}^{-f} d\mu_X = \int_{\phi^{-1}(S)} \mathbb{L}^{-f\circ\phi_* - \mathrm{ord}_t J\phi} d\mu_Y.$$

Here $J\phi := \mathcal{O}_Y(-E)$ is the ideal sheaf generated by the holomorphic Jacobian factor, $\mathrm{ord_t}\,\mathcal{I} : \mathcal{L}(X) \to \mathbb{N} \cup \{0\}$ for any ideal sheaf $\mathcal{I}$ is the function of minimal degree in t. Namely for $\gamma \in \mathcal{L}(X)$, $\mathrm{ord_t}\,\mathcal{I}(\gamma) := \min_{g \in \mathcal{I}} \deg_t g \circ \gamma(t)$.

The proof for general X is technical. However for smooth X the main idea of the proof is not hard to explain. Indeed it is an application of the inverse function theorem over power series rings which traces carefully the orders in t. Since we only need the smooth case here, we will give an outline of the proof in this case. Let $\phi : Y \to X$ be the birational morphism with $E_{\mathrm{red}} \subset Y$ and $Z \subset X$ be the exceptional loci in Y and X respectively.

For each $k \in \mathbb{N} \cup \{0\}$ let $S_k \subset \mathcal{L}(Y)$ be the subset $\gamma \in \mathcal{L}(Y)$ such that $\operatorname{ord}_t J\phi(\gamma) = k$. By the inverse function theorem (see below), the map $\phi_* : \mathcal{L}(Y) \to \mathcal{L}(X)$ is a bijection between $\mathcal{L}(Y)^\times := \mathcal{L}(Y) \backslash \mathcal{L}(E_{\rm red})$ and $\mathcal{L}(X)^\times := \mathcal{L}(X) \backslash \mathcal{L}(Z)$, thus there is no interesting geometry on the map $\phi_*|_{S_k} : S_k \to \mathcal{L}(X)^\times$. However, the important observation by Denef and Loeser is that when one takes finite truncations in

$$\begin{tikzcd} \mathcal{L}(Y) \arrow[r, "\pi_m"] \arrow[d, "\phi"'] & \mathcal{L}_m(Y) \arrow[d, "\phi_m"] \\ \mathcal{L}(X) \arrow[r, "\pi_m"'] & \mathcal{L}_m(X) \end{tikzcd}$$

for all large enough m the induced map $\phi_m|_{\pi_m(S_k)} : \pi_m(S_k) \to \mathcal{L}_m(X)$ is indeed a piece-wise trivial $\mathbb{C}^k$ fibration over its image. Together with the fact that $\mathcal{L}(Z)$ is *measure zero* in $\mathcal{L}(X)$, this will imply the change of variable formula.

To investigate the fibration structure near one arc $\gamma \in \mathcal{L}(Y)$, it is enough to restrict the map to formal neighborhoods $\phi : \hat{\mathbb{C}}^n_{(0)} \to \hat{\mathbb{C}}^n_{(0)}$. Or equivalently to represent ϕ by an algebraic map (still called ϕ) on power series $\phi : \mathbb{C}[[t]]^n \to \mathbb{C}[[t]]^n$ with $\phi(0) = 0$. Let $\phi(y(t)) = x(t)$ with $y(t) \in S_k$ and let $\ell \geq 2k+1$. We first notice that for each $v \in \mathbb{C}[[t]]^n$, there is a unique solution $u \in \mathbb{C}[[t]]^n$ of the equation

$$\phi(y(t) + t^{\ell-k}u) = x(t) + t^\ell v.$$

Indeed by Taylor's expansion

$$\phi(y(t) + t^{\ell-k}u) = \phi(y(t)) + D\phi(y(t))t^{\ell-k}u + t^{2(\ell-k)}R(t,u).$$

Let $A = D\phi(y(t))$. The equation becomes $Au + R(t,u)t^{\ell-k} = t^k v$. That is,

$$u = (\det A)^{-1} t^k A^*(v - R(t,u)t^{\ell-2k}).$$

Here A^* is the adjoint matrix of A. Since $\operatorname{ord}_t \det A = \operatorname{ord}_t J\phi(y(t)) = k$, the term $(\det A)^{-1}t^k$ has order zero. Also since $\ell - 2k \geq 1$, by repeated substitutions this relation solves u as a vector in formal power series.

Now let $m \geq 2k$ and let $\ell = m+1$. The above discussion shows that in order to find all solutions of $\phi(\tilde{y}(t) \mod t^{m+1}) = x(t) \mod t^{m+1}$, we may assume that $\tilde{y}(t) = y(t) + t^{m+1-k}u$. Notice that the residue classes $\bar{u} = u \mod t^k$ form a linear space isomorphic to $\mathbb{C}^{nk}$. By Hensel's lemma, in order to count the solutions we may simply consider the equation $At^{m+1-k}\bar{u} = 0 \mod t^{m+1}$. That is, $A\bar{u} = 0 \mod t^k$. Since $\operatorname{ord}_t \det(A^*) = (n-1)k$, the solution space of $\bar{u}$ has dimension $nk - (n-1)k = k$ as expected.

This verifies that $\phi_m^{-1}\bar{x}(t) \cong \mathbb{C}^k$. The piece-wise triviality needs other tools to prove it, which will not be reported here. For the complete details the readers are referred to the original paper [15].

We remark that for $S = \mathcal{L}(X)$ and $E = \sum_{i=1}^n e_i E_i$ a normal crossing, the change of variable formula gives

$$[X] = \int_{\mathcal{L}(X)} \mathbb{L}^0 d\mu_X = \sum\nolimits_{I \subset \{1,\ldots,n\}} [E_I^\circ] \prod\nolimits_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{e_i+1}-1}.$$

Since $\mathbb{L}^{e+1} - 1 = (\mathbb{L}-1)[\mathbb{P}^e]$, this coincides with the formula in §3.3.

4. K-equivalence Relation and Complex Elliptic Genera: Weak Deformation/Decomposition Theorem

4.1. **Some Background.** There is a build-in problem in all integration-theoretic approaches to the K-equivalence relation. Namely we arrive at only K-theoretic or in practice simply numerical conclusions. It is usually hard to get results of geometric nature just from numerical data. In dimension three, the result of Kollár and Mori [32] on the flop decomposition of birational minimal models can be easily generalized to any two K-equivalent threefolds. So the results mentioned in §1: naturally isomorphic cohomology groups, equivalent Hodge structures and local moduli spaces are all still true. Moreover, these canonical isomorphisms are all induced from the graph closure cycle of the given birational map. It is clear that we can not achieve these statements from integration theory only.

In the higher dimensional cases, due to the fact that it is (at least currently) impossible to classify (terminal) singularities, the existence problem of flops seems to be completely out of reach. This suggests that we should not restrict the study of K-equivalence relation inside the category of algebraic geometry only. We should allow (locally holomorphic) symplectic deformations. That is, small deformations of almost complex structures which are (integrable in a neighborhood of the exceptional loci and are) tamed by the original symplectic form. In dimension three, with the help of classification theory of singularities we may show that: if allowing symplectic deformations, then any birational map between three dimensional K-equivalent manifolds can be decomposed into composite of *classical flops* (see 4.2 below). All the natural isomorphisms that we are interested in are then just simple corollaries. The unsatisfactory fact is that we DO NOT know how to prove this deformation/decomposition theorem directly without using the classification theory. Such a proof should shed important light toward the higher dimensional cases.

In fact, this symplectic deformation/decomposition theorem is even more useful than the original flop decomposition theorem for certain problems. For example, Li and Ruan [35] had shown in 1998 that it can be used to prove that birational smooth minimal threefolds have equivalent quantum cohomology rings. Notice that the ring structure of ordinary cohomology groups are not preserved under flops $X \dashrightarrow X'$, in general X and X' are not even homotopically equivalent.

4.2. **Some Well Known Flops.** For the reader's convenience, we recall the definition of certain known flops. The simplest type of flops are called *ordinary flops.* An ordinary $\mathbb{P}^r$-flop (or simply $\mathbb{P}^r$-flop) $f : X \dashrightarrow X'$ is a birational map such that the exceptional set $Z \subset X$ has a $\mathbb{P}^r$-bundle structure $\psi : Z \to S$ over some smooth variety S and the normal bundle $N_{Z/X}$ is isomorphic to $\mathcal{O}(-1)^{r+1}$ when restricting to any fiber of ψ. The map f and the space X' are then obtained by first blowing up X along Z to get Y, with exceptional divisor E a $\mathbb{P}^r \times \mathbb{P}^r$-bundle over S, then blowing down E along another fiber direction. Ordinary $\mathbb{P}^1$-flops are also called classical flops. Three dimensional classical flops are the most well-known Atiyah flops over $(-1, -1)$ rational curves.

Another important example is the *Mukai flops* $f : X \dashrightarrow X'$. In this case it is required that the exceptional set $Z \subset X$ is of codimension r and has a $\mathbb{P}^r$-bundle structure $\psi : Z = \mathbb{P}_S(F) \to S$ (for some rank $r + 1$ vector bundle F) over a smooth base S, moreover the normal bundle $N_{Z/X} \cong T^*_{Z/S}$, the relative cotangent bundle of ψ. To get f, one first blows up X along Z to get $\phi : Y \to X$ with exceptional divisor $E = \mathbb{P}_Z(T^*_{Z/S}) \subset \mathbb{P}_S(F) \times_S \mathbb{P}_S(F^*)$ as the incidence variety. The first projection corresponds to ϕ and one may contract E through the second projection to get $\phi' : Y \to X'$. Mukai flops naturally occur in hyperkähler manifolds [24].

4.3. **Main Conjectures.** In 2000, the author made a series of conjectures on K-equivalent manifolds X and X' under birational map $f : X \dashrightarrow X'$:

I. The morphism $T : H^k(X, \mathbb{Q}) \to H^k(X', \mathbb{Q})$ induced from the graph closure $\bar{\Gamma}_f \subset X \times X'$ is an isomorphism which preserves the rational Hodge structures. There also exists a canonical correspondence $\bar{\Gamma}_f + \sum_i T_i \subset A^n(X \times X')$, with T_i being certain degenerate correspondences, which defines an isomorphism on integral cohomology groups modulo torsion. [3]

II. The local deformation spaces $\mathrm{Def}(X)$ and $\mathrm{Def}(X')$ are canonically isomorphic in the sense that the local universal families are K-equivalent over the base. Moreover, suitable compactifications of their polarized moduli spaces should also be K-equivalent.

III. X and X' have canonically isomorphic quantum cohomology rings over the extended Kähler moduli spaces. In other words, their quantum cohomology rings can be analytically continued to each other.

IV. Deformation/Decomposition Theorem: under generic symplectic perturbations which respect f, the deformed f can be decomposed into finite copies of ordinary $\mathbb{P}^r$-flops for various r's.

It is also expected that IV would be the key step toward resolving conjectures I, II and III.

The main progress made in [49] is to prove a weak form of conjecture IV: if we further modulo complex cobordism, then any birational map between K-equivalent manifolds can be decomposed into the composite of finite number of ordinary $\mathbb{P}^1$-flops. Notice that since in a flat family of algebraic cycles the dimension can not go down under specialization, the r's appear in conjecture IV can not take the value 1 only. This explains that the weak form we proved is still far away from the original conjecture. Another important remark is related to the Mukai flops. These flops are not included in conjecture IV since Huybrechts [24] had shown in 1996 that Mukai flops in hyperkähler manifolds will disappear (become isomorphisms) under generic deformations. Recently he completed the discussion by showing that birational hyperkähler manifolds become isomorphic under generic deformations of complex structures [25]. Huybrechts' results can be regarded as one of the most important evidences of the above conjectures (c.f. 5.3).

Also it should be remarked that for Calabi-Yau manifolds, the equivalence of Hodge numbers gives numerical evidence for Main Conjecture II since the relevant groups in the Kodaira-Spencer theory are all Hodge groups:

$$H^i(X, T_X) \cong H^i(X, \Omega_X^{n-1}) \cong H^{n-1,i}(X).$$

However, in order to proceed, we really need the validity of Conjecture I.

4.4. **Complex Elliptic Genera under $\mathbb{P}^1$-flops.** As for the proof of the weak deformation/decomposition theorem, we notice that according to a result of Milnor [39] and Novikov [42], the complex cobordism class of a compact stably almost

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[3] I am grateful to D. Huybrechts and Y. Namikawa for pointing out the necessity to modify the graph closure in order to get the conjectural isomorphisms on integral cohomologies.

complex manifold (X such that $T_X \oplus \xi$ has a complex vector bundle structure for some trivial bundle ξ) is determined precisely by all its Chern numbers. The complex cobordism ring Ω^U is defined to be the ring of compact stably almost complex manifolds modulo cobordism by such manifolds with boundaries.

Recently Totaro [45] showed that the most general Chern numbers that are invariant under ordinary $\mathbb{P}^1$-flops consists of the so-called *complex elliptic genera*. Recall that an R-genus is nothing but a ring homomorphism $\varphi : \Omega^U \to R$. Equivalently it can be defined through Hirzebruch's power series recipe [21]. Let $Q(x) \in R[\![x]\!]$ and X be an almost complex manifold with formal Chern roots decomposition $c(T_X) = \prod_{i=1}^n (1 + x_i)$, then

$$\varphi_Q(X) := \prod\nolimits_{i=1}^n Q(x_i)[X] =: \int_X K_Q(c(T_X))$$

defines an R-genus. As usual write $Q(x) = x/f(x)$ then the complex elliptic genera is defined by the three parameter ($k \in \mathbb{C}$, τ and a marked point z) power series

$$f(x) = e^{(k+\zeta(z))x}\,\frac{\sigma(x)\sigma(z)}{\sigma(x+z)}.$$

Hirzebruch [22] has reproved Totaro's theorem using Atiyah-Bott localization theorem. He showed that φ_Q is invariant under $\mathbb{P}^1$-flops if and only if $F(x) := 1/f(x)$ satisfies the functional equation

$$F(x+y)(F(x)F(-x) - F(y)F(-y)) = F'(x)F(y) - F'(y)F(x).$$

Moreover, the solutions is given by the above f exactly.

4.5. **Complex Elliptic Genera under K-equivalence.** The main contribution in [49] is to show that complex elliptic genera are also invariant among general K-equivalent manifolds. Hence in Ω^U, the ideal I_1 generated by $[X] - [X']$ with X and X' related by a $\mathbb{P}^1$-flop is equal to the seemingly much lager ideal I_K generated by K-equivalent pairs $[X] - [X']$.

Following the meta theorem, the most important step in the proof is to develop a change of variable formula for genera (or Chern numbers) under blowing-ups. First, using standard intersection theory and Hirzebruch's theory of virtual genus [23], we proved a residue formula for a single blowing-up $\phi : Y \to X$ along smooth center Z of codimension r. Namely, for any power series $A(t) \in R[\![t]\!]$:

$$\begin{aligned}\int_Y A(E)\,K_Q(c(T_Y)) \;&=\; \int_X A(0)\,K_Q(c(T_X)) \\ &\quad + \int_Z \operatorname{Res}_{t=0}\Big(\frac{A(t)}{f(t)\prod_{i=1}^r f(n_i - t)}\Big)\,K_Q(c(T_Z)).\end{aligned}$$

Here n_i's denote the formal Chern roots of the normal bundle $N_{Z/X}$ and the residue stands for the coefficient of the degree -1 term of a Laurent power series with coefficients in the cohomology ring or the Chow ring of X. In order to have a change of variable formula, we need the residue term to vanish. If we already know the expression of f as above, than for z not an r-torsion point it is not hard to find

$$A(t, r) = e^{-(r-1)(k+\zeta(z))t}\,\frac{\sigma(t+rz)\sigma(z)}{\sigma(t+z)\sigma(rz)}.$$

In fact the $r = 2$ case corresponds to a functional equation

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}$$

which also has solutions given by f (and A is determined by f), but with z not a 2-torsion points. [4] Notice that a classical theorem of Weierstrass states that solutions of functional equations involving only $f(x)$, $f(y)$ and $f(x+y)$ are constructed from the Weierstrass elliptic functions. However, the functional equations appeared here (and also the one considered by Hirzebruch in §4.4) are not of this type.

The general change of variable reads: let $\varphi = \varphi_{k,\tau,z}$ be the complex elliptic genera. Then for any algebraic cycle D in X and birational morphism $\phi : Y \to X$ with $K_Y = \phi^* K_X + \sum e_i E_i$, we have (write $K_Q = K_\varphi$)

$$\int_D K_\varphi(c(T_X)) = \int_{\phi^* D} \prod\nolimits_i A(E_i, e_i + 1)\, K_\varphi(c(T_Y)).$$

Or equivalently, $\phi_* \prod_i A(E_i, e_i + 1)\, K_\varphi(c(T_Y)) = K_\varphi(c(T_X))$. This is first proved by induction for ϕ a composite of blowing-ups. The general cases can be reduced to the blowing-up case by applying the weak factorization theorem [54] [1]. (A similar result for the case $k = 0$ (elliptic genera) was also obtained recently by Borisov and Libgober [7].)

The formula implies that $X =_K X' \Rightarrow \varphi_{k,\tau,z}(X) = \varphi_{k,\tau,z}(X')$ with z not a torsion point. But then we also get $\varphi(X) = \varphi(X')$ in all cases by continuity.

Notice that it is symbolically convenient to denote $K_\varphi(c(T_X))$ by $d\mu_X$ and regard it as an *elliptic measure*, though we do not really construct a measure theory as in §3. When we specialize to Todd genus (rational measure), the Jacobian factor reduces to $A = 1$ and the change of variable formula is a simple corollary of the Grothendieck Riemann-Roch theorem.

It is worth mentioning that except for the last step, the proof works both in the category of (stably almost) complex manifolds and in the category of algebraic manifolds in arbitrary characteristic. While the weak factorization theorem has been proved for both the complex analytic category [5] and also the algebraic category, the later is proved under the restriction over fields of characteristic zero. It is expected that one should find a Grothendieck Riemann-Roch type argument to replace the weak factorization theorem to get a more satisfactory conclusion.

4.6. **Chern Numbers versus Hodge Numbers.** Hodge numbers and Hodge structures determine a substantial part of the complex elliptic genera and also give information to the complex moduli. Recall that [45]:

$$\varphi(X) = \chi\Big(X, K_X^{\otimes(-k)} \otimes \prod\nolimits_{m\geq 1} (\Lambda_{-y^{-1}q^m} T \otimes \Lambda_{-y^{-1}q^{m-1}} T^* \otimes S_{q^m} T \otimes S_{q^m} T^*)\Big)$$

for $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$ and $T = T_X - n$ the rank zero virtual tangent bundle. The twisted χ_y-genus corresponds to the two parameter genera

$$\chi_y(X) := \chi\Big(X, K_X^{\otimes(-k)} \otimes \Lambda_y T_X^*\Big),$$

[4]It is expected that complex genera which admit the change of variable formula for codimension r blowing-ups consists of precisely the complex elliptic genera with z not an r-torsion point.

[5]I am grateful to B. Totaro for pointing out this to me.

which is equivalent to knowing all $\chi(X, K_X^{\otimes(-k)} \otimes \Omega_X^p)$ for $p \geq 0$. If $n = \dim X \leq 11$, the twisted χ_y genus contains the same Chern numbers as the complex elliptic genera. So in this range, twisted χ_y genus contains precisely all Chern numbers that are invariant under the K-equivalence relation. If $n \leq 4$, the twisted χ_y genus contains all Chern numbers, so all Chern numbers are invariant under K-equivalence for dimensions up to 4.

It is clear that if K_X is trivial, that is, X is a Calabi-Yau manifold, then the twisted χ_y genus becomes Hirzebruch's χ_y genus $\sum_{p\geq 0} \chi(X, \Omega_X^p)\, y^p$. In particular, it is determined by the Hodge numbers. So the equivalence of elliptic genera (that is, $k = 0$) follows from the equivalence of Hodge numbers when $n \leq 11$. But when $n \geq 12$, the elliptic genera and Hodge numbers contain quite a different type of information.

5. Other Aspects of K-equivalence Relation

For completeness, we add a few topics that are closely related to the study of K-equivalence relation but not directly related to the author's current approaches. The interested reader should consult the original papers for more details.

5.1. **K-equivalence for Singular Varieties.** The notion of K-partial ordering makes sense for general $\mathbb{Q}$-Gorenstein varieties. For log-terminal varieties, the integration formalism works well as in the smooth case (the measure is finite $\Leftrightarrow$ X is log-terminal) and K-equivalence still implies measure-equivalence, both the p-adic and motivic ones. The major problem here is to understand the geometric meaning of the total measure. For p-adic measure, it is a weighted counting of rational points over finite fields, but we do not know how to make it precise.

In [3] Batyrev defined the stringy E function for log-terminal varieties and also the *stringy Hodge numbers* when this E function is a polynomial. In terms of motivic measure, it can be defined by taking $h^{p,q}_{\rm st}(X) = \chi^{p,q}(\mu_X(\mathcal{L}(X))$. It is clear that K-equivalent varieties have the same stringy Hodge numbers, but its meaning still needs to be further clarified. Does there exist corresponding cohomology theories (spaces)? Noticed that $h^{p,q}_{\rm st}(X)$ is in general only a rational number and may as well be negative. Veys has recently investigated the situation for normal surface singularities [46].

A more manageable case is the crepant resolutions of Gorenstein quotient singularities $\phi : Y \to X := \mathbb{C}^n/G$ with $G \subset \mathrm{SL}(n, \mathbb{C})$ a finite subgroup such that $K_Y = \phi^* K_X$. The *McKay correspondence* asserts that, among other things, a natural basis of $H^*(Y)$ is in one to one correspondence with conjugacy classes of G. The numerical version of it has been proved by Batyrev [4] and also by Denef and Loeser [16] using motivic integration. The geometric correspondence has not been treated except in dimensions ≤ 3 [9]. Moreover, it is not known in general when does X admit crepant resolutions. A possible approach is to look at the p-adic measure of X. As it must be the measure of Y when Y exists, the corresponding counting function must behave like counting points on smooth varieties. This should put certain restrictions on G.

A recent attempt toward constructing the expected cohomology theory was given by Chen and Ruan's *orbifold cohomology* for Gorenstein orbifolds [12]. But the naturality problem has not been solved yet. It seems to be a difficult problem for the construction for general $\mathbb{Q}$-Gorenstein varieties. The author do not know even conceptually how to extend the Main Conjectures to the singular case.

5.2. **Derived Categories and Fourier-Mukai Transform.** In 1995, Bondal and Orlov [6] showed that for special cases of $\mathbb{P}^r$-flops (namely S is a point in the notation of 4.2), The natural transform $\Phi := \mathbf{R}\phi'_* \circ \mathbf{L}\phi^*$ gives an equivalence of triangulated categories $D^b(X) \cong D^b(X')$ (here $D^b(X)$ denotes the bounded derived category of coherent sheaves). Later on Bridgeland [8] had made significant progress on this approach by extending their result to all smooth threefold flops. The important issue here is that the flopped variety X^+ may be constructed as certain *fine* moduli spaces. More precisely, he showed that, for $\psi : X \to \bar{X}$ a flopping contraction (that is, K_X is ψ-trivial) from a smooth threefold X, let $X^+ := \mathrm{Per}(X/\bar{X})$ be the distinguished component of the moduli space of *perverse point sheaves*, then X^+ is smooth and $f : X \dashrightarrow X^+$ is the flop. Also the Fourier-Mukai transform

$$\mathbf{R}p_{2*}(\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-)) : D^b(X) \to D^b(X^+)$$

(here p_1, p_2 are the projections from $X \times X^+$ to X and X^+ respectively) induced by the *universal perverse point sheave* $\mathcal{E} \in D^b(X \times X^+)$ is an equivalence of categories.

Bridgeland's theorem is recently generalized by Chen [11] to 3-folds with Gorenstein terminal singularities and by Kawamata [29] to three dimensional orbifolds. By combining with Chen's result, Kawamata also proved the equivalence of derived categories for all three dimensional terminal flops [30]. There seems to be of some hope to deal with certain higher dimensional flopping contractions $\psi : X \to \bar{X}$ with relative dimension ≤ 1 through their methods.

In [30], Kawamata conjectured that for birational projective manifolds, the notion of K-equivalence should be equivalent to D-equivalence, namely varieties with equivalent derived categories of coherent sheaves. This is clearly closely related to our main conjectures, but a precise relation between derived categories and cohomologies does not seem to be well studied yet.

5.3. **Flop Decomposition for Hyperkähler Manifolds.** Hyperkähler manifolds (or holomorphically symplectic manifolds) have been extensively studied lately. All our Main Conjectures follow from Huybrechts' fundamental works [24] [25] mentioned in §4.3. For Conjecture I the correspondence cycle $\Gamma \subset X \times X'$ used by him is the limiting cycle $\lim_{t\to 0} \bar{\Gamma}_{f_t}$ induced from nearby isomorphisms $f_t : \mathcal{X}_t \cong \mathcal{X}'_t$ with $t \neq 0$. This cycle in general contains more than one irreducible components. In fact for a Mukai flop, the map T induced from the graph closure will in general preserve only the *rational* cohomologies. The statement in Conjecture I is still unknown for birational hyperkähler manifolds under the map T.

On the other direction, Burns, Hu and Luo [10] had shown that birational maps between hyperkähler fourfolds can be decomposed into composite of Mukai flops, if all the irreducible components of the exceptional loci are normal. Very recently this normality assumption was justified by Wierzba and Wiśniewski [52], hence the four dimensional case was settled completely. Notice that in this case the Mukai flop is of a particularly simple type (in the notation of §4.2, $Z \cong \mathbb{P}^2$ and S is a single point). Since the equivalence of derived categories for Mukai flops with S being a point is proved by Namikawa [40] and Kawamata [30], we see that birational hyperkähler fourfolds are indeed D-equivalent.

As a final question, can one prove the above results on D-equivalence without making use of the explicit flops decomposition? Notice that this is the main theme of our approach toward cohomologies and complex genera in higher dimensions.

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